Integrals of Stieltjes Type.

By

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The integrals of Stieltjes type hitherto considered are classified in three types as follows:

Type	Point function as integrand		Set function as integrator	
1	Real or complex	valued	Real or complex	valued
2	Vector	valued	Real or complex	valued
3	Real or complex	valued	Vector	valued

I shall in this paper apply the Kolmogoroffian(1) discussion of the absolutely convergent integral uniformly applicable to any type of integral above mentioned, which shall be equivalent to the usual integral for the 1st type, and to Bochner's(2) and Dunford's(3) for the 2nd type. In the case of the 3rd type Maeda's integral(4) contains ours owing to the speciality(5) of the integrator. I shall also give a definition equivalent to his. Most of the properties which hold for the Lebesgue integral also hold for ours; hence I shall give certain theorems with which the integrator is concerned.

1. Preliminaries.

Let $\mathfrak{X}$ be a complete linear vector space, and $\mathfrak{N}$ the real or complex number system. Let V be an abstract set, $\mathfrak{K}$ an additive family(6) of point sets in V, and E a set of $\mathfrak{K}$. A set function $\alpha(E)$ on $\mathfrak{K}$ to $\mathfrak{X}$ or $\mathfrak{N}$ is said to be completely additive, if for every sequence $\{E_n\}$ of disjoint sets in $\mathfrak{K}$, $\alpha(\sum_n E_n) = \sum_n \alpha(E_n)$. Let $\beta(E)$(7) be the total varia-

(1) A. Kolmogoroff, Math. Ann. **103** (1930), 654–696.
(2) S. Bochner, Fund. Math. **20** (1933), 262–276.
(3) N. Dunford, Transactions A. M. S. **37** (1935), 441–453.
(4) F. Maeda, this journal, **4** (1934), 60–69.
(5) F. Maeda, ibid.
(6) S. Saks, *Théorie de L'intégrale*, (1933), 247.
(7) That is, the maximum of $|\alpha(E_1)| + |\alpha(E_2)| + \dots$, where $E = E_1 + E_2 + \dots$

tion of α on E, and let $\beta(V)$ be finite. If for any set E' there exist two sets $E_1, E_2 \varepsilon \mathfrak{K}$ such that

$$(1) \qquad E_1 \subset E' \subset E_2 \qquad \text{and} \qquad \beta(E_1) = \beta(E_2),$$

then I shall call E' an α-normal set. α-normal sets form a complete[1] additive family over $\mathfrak{K}$, which we shall denote by $\mathfrak{K}_\alpha$. If we put $\alpha(E') = \alpha(E)$, $\alpha(E')$ is completely additive on $\mathfrak{K}_\alpha$. I shall say that an additive family $\mathfrak{K}_1$ is α-equivalent to $\mathfrak{K}$ if for any set $E \varepsilon \mathfrak{K}_1$ there exist two sets $E_1, E_2 \varepsilon \mathfrak{K}$ satisfying (1) and conversely. And I shall call E_1 an α-minorant set of E. A function $f(\lambda)$ on V to $\mathfrak{R}$ or $\mathfrak{N}$ is said to be α-measurable[2] if $\underset{\lambda}{E}[f(\lambda) \varepsilon$ any open set in $\mathfrak{R}$ or $\mathfrak{N}] \varepsilon \mathfrak{K}_\alpha$.

2. Definition of Integral.

Let $f(\lambda)$ be any point function on V to $\mathfrak{R}$ or $\mathfrak{N}$.

Let $\mathfrak{K}_1$ be α-equivalent to $\mathfrak{K}$ and $\mathfrak{D}$ a division of V into sets $E_i \varepsilon \mathfrak{K}_1$, or symbolically $\mathfrak{D} = (E_1, E_2, \ldots .)$. Put $\sum_{\mathfrak{D}} f(\lambda)\alpha(E) = \sum_i f(\lambda_i)\alpha(E_i)$, where $\lambda_i \varepsilon E_i$. $f(\lambda)$ is said to be integrable on V with respect to α, if

$$(1) \quad \varepsilon, \mathfrak{D}, \quad \sum_{\mathfrak{D}} |f(\lambda)|\beta(E) \quad \text{conv.}^{(3)} \text{ and } \quad \sum_{\mathfrak{D}} Osc(f \cdot E)\beta(E) < \varepsilon, ^{(4)}$$

where $Osc(f \cdot E)$ is the oscillation of $f(\lambda)$ on E.

From (1) we have

$$(2) \qquad \varepsilon, \mathfrak{D}, \qquad \mathfrak{D}' \geqq \mathfrak{D}^{(5)} \qquad \Big|\sum_{\mathfrak{D}'} - \sum_{\mathfrak{D}} f(\lambda)\alpha(E)\Big| < \varepsilon,$$

hence there exists an element $F \varepsilon \mathfrak{R}$ or $\varepsilon \mathfrak{N}$ such that

$$(3) \qquad \varepsilon, \mathfrak{D}, \qquad \mathfrak{D}' \geqq \mathfrak{D}, \qquad \Big|F - \sum_{\mathfrak{D}'} f(\lambda)\alpha(E)\Big| < \varepsilon, ^{(6)}$$

and we call F the integral of $f(\lambda)$ on V, and write it as

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(1) That is, $\mathfrak{K}_\alpha$ contains all the subsets of E for which $\beta(E) = 0$.

(2) This definition is equivalent to the ordinary one if $f(\lambda)$ is to $\mathfrak{N}$.

(3) If $|f(\lambda)|$ is bounded, this condition is superfluous.

(4) This is the abbreviated form of the fact that for any given positive number ε there exists a division $\mathfrak{D}$ such that $\sum_{\mathfrak{D}} |f(\lambda)|\beta(E)$ converges and $\sum_{\mathfrak{D}} Osc(f, E)\beta(E) < \varepsilon$. and I shall use similar abbreviations for analogous cases.

(5) That is $\mathfrak{D}'$ is any subdivision of $\mathfrak{D}$.

(6) If $f(\lambda)$ and $\alpha(E)$ are real, (1) follows from (3) and $\sum_{\mathfrak{D}'} f(\lambda)\alpha(E)$ abs. conv.

$$\int_V f(\lambda)d\alpha(E).$$

In this definition the choice of an α-equivalent family is not essential. For if (1) holds for $\mathfrak{K}_1$, it does so also for $\mathfrak{K}_\alpha$. Conversely if (1) holds for $\mathfrak{K}_\alpha$, it does so also for $\mathfrak{K}_1$ by considering α-minorant sets.

THEOREM. The conditions of integrability of $f(\lambda)$ on V with respect to α are

1° $|f(\lambda)|$ is integrable on V with respect to β,

2° $f(\lambda)$ is α-measurable,

and 3° the set of values of $f(\lambda)$ on V, with the possible exception of the β-null set, is separable.

Proof. Let $f(\lambda)$ be integrable.

Since $Osc(|f|, E) \leqq Osc(f, E)$, $|f(\lambda)|$ is integrable on V with respect to $\beta(E)$.

From (1)

$$(4) \qquad \frac{1}{n}, \quad \mathfrak{D}_{\frac{1}{n}}, \quad \sum_{\mathfrak{D}_{\frac{1}{n}}} Osc(f\cdot E)\beta(E) < \frac{1}{n^3},$$

where we may suppose that $\mathfrak{D}_{\frac{1}{n}} \leqq \mathfrak{D}_{\frac{1}{n+1}}$.

If we put $f_n(\lambda) = f(\lambda_i)$ on $E_i^{(n)}$, $\lambda_i \varepsilon E_i^{(n)}$, where λ_i is fixed, and $\mathfrak{D}_{\frac{1}{n}} = (E_1^{(n)}, E_2^{(n)}, \dots)$, then $f_n(\lambda)$ converges asymptotically to $f(\lambda)$,[1] and the set of the values of $f_n(\lambda)$ $(n = 1, 2, \dots)$ on V is enumerable, and dense in that of $f(\lambda)$ on V, with the possible exception of the values of $f(\lambda)$ on β-null set, that is, the set of values of $f(\lambda)$ on V, with the possible exception of those on β-null set, is separable. Let O be any open set in $\mathfrak{R}$ or $\mathfrak{N}$. If we put for $D_{\frac{1}{n}} = (E_1^{(n)}, E_2^{(n)}, \dots)$

$$E_{\mathfrak{D}_{\frac{1}{n}}, \frac{1}{n}} = \sum_{\mathfrak{D}_{\frac{1}{n}}}' E_i^{(n)},$$

where $\sum'$ means the summation of sets of $\mathfrak{D}_{\frac{1}{n}}$ for which $Osc(f\cdot E_i^{(n)}) \geqq \frac{1}{n}$,

(1) Let $E_{\mathfrak{D}_{\frac{1}{n}}, \varepsilon}$ be the sum of the sets of $\mathfrak{D}_{\frac{1}{n}}$ for which $Osc(f, E) \geqq \varepsilon$, then from (4) $\beta(E_{\mathfrak{D}_{\frac{1}{n}}, \varepsilon}) < \frac{1}{n^3\varepsilon}$. Since $\underset{\lambda}{E}(|f-f_n| \geqq \varepsilon) \leqq E_{\mathfrak{D}_{\frac{1}{n}}, \varepsilon}$, the outer β-measure of $\underset{\lambda}{E}(|f-f_n| \geqq \varepsilon)$ converges to zero when $n \to \infty$.

and
$$E_{\mathfrak{D}_{\frac{1}{n}}} = \sum_{\mathfrak{D}_{\frac{1}{n}}}' E_i^{(n)},$$

where $\sum'$ means the summation of sets of $\mathfrak{D}_{\frac{1}{n}}$ for which values of $f(\lambda)$ on $E_i^{(n)}$ are entirely contained in O.

Then we have

$$\sum_{n=1}^{\infty} E_{\mathfrak{D}_{\frac{1}{n}}} \subset \underset{\lambda}{E}[f(\lambda)\varepsilon O] \subset \sum_{n=1}^{\infty} E_{\mathfrak{D}_{\frac{1}{n}}} + \sum_{n=k}^{\infty} E_{\mathfrak{D}_{\frac{1}{n}},\frac{1}{n}},$$

k being any integer.

Since $\beta(E_{\mathfrak{D}_{\frac{1}{n}},\frac{1}{n}}) < \frac{1}{n^2}$ from (4), w have

$$\beta(\sum_{n=1}^{\infty} E_{\mathfrak{D}_{\frac{1}{n}}}) = \beta\Big(\prod_{k=1}^{\infty}(\sum_{n=1}^{\infty} E_{\mathfrak{D}_{\frac{1}{n}}} + \sum_{n=k}^{\infty} E_{\mathfrak{D}_{\frac{1}{n}},\frac{1}{n}})\Big);$$

thus $f(\lambda)$ is α-measurable by section 1 (1).

Conversely, if these conditions 1°, 2°, und 3° are satisfied, then we shall have a division satisfying (1) by Lindelöf's covering theorem. Thus we have the theorem.

For the 1st type of integral, if $f(\lambda)$ is integrable in the usual sense, there exists a division satisfying (1); thus $f(\lambda)$ is integrable in our sense. Since both integrals have the same value for any step function, they are equivalent.

For the 2nd type of integral, if $f(\lambda)$ is integrable in Bochner's sense,[1] then $|f(\lambda)|$ is integrable with respect to β, and $f(\lambda)$ is almost everywhere the limit of a sequence of step functions, and is hence integrable in our sense. The converse will hold also by the same reasoning. Thus the two integrals are equivalent. When V is a compact metric space, and $\mathfrak{K}$ is α-equivalent to the family of Borel sets, any continuous function is uniformly continuous, and hence integrable in Dunford's[2] and in our sense. Therefore the two integrals will be equivalent.

3. Maeda's Integral.

Where Maeda's integral is concerned, I shall write U and $\mathfrak{q}(U)$ instead of E and $\alpha(E)$ respectively, where $\mathfrak{q}(U)$ is on $\mathfrak{K}$ to a Hilbert

(1) S. Bochner, loc. cit.
(2) N. Dunford, loc. cit.

space, and completely additive in his sense.[1] Let $\sigma(U) = \|\mathfrak{q}(U)\|^2$, then $\sigma(U)$ is completely additive.

F is said to be the integral of $f(\lambda)$ on V, if

$$(1) \quad \varepsilon, \mathfrak{D}, \quad \mathfrak{D}' \geqq \mathfrak{D} \quad \sum_{\mathfrak{D}'} f(\lambda)\mathfrak{q}(U) \text{ conv. and } \| F - \sum_{\mathfrak{D}'} f(\lambda)\mathfrak{q}(U) \| < \varepsilon .$$

Then the condition of integrability is

$$(2) \quad \varepsilon, \mathfrak{D} \quad \sum_{\mathfrak{D}} |f(\lambda)|^2 \sigma(U) \quad \text{conv. and} \quad \sum_{\mathfrak{D}} Osc(f \cdot U)^2 \sigma(U) < \varepsilon .$$

As in section 2, Maeda's integral and ours are equivalent and we have the following theorem:

THEOREM. $f(\lambda)$ is integrable on V with respect to $\mathfrak{q}(U)$ when and only when $f(\lambda)$ is σ-measurable and $|f(\lambda)|^2$ is integrable with respect to σ.

4. Extension of Definition of Integral.

If $f(\lambda)$ is integrable on V, then it is also so on any set $E \varepsilon \mathfrak{K}$. Let

$$F(E) = \int_E f(\lambda) d\alpha(E) ;$$

then $F(E)$ is completely additive.

But when $\alpha(E)$ is not defined for all sets belonging to $\mathfrak{K}$, or $\beta(E)$ is not finite, we cannot apply our definition. We consider the family of sets of $\mathfrak{K}$ for which $\alpha(E)$ is defined and $\beta(E)$ is finite, and we assume that this family $\mathfrak{S}$ satisfies the conditions:

(1) when $E \varepsilon \mathfrak{S}$, any subset of E which belongs to $\mathfrak{K}$ also belongs to γ,

(2) when $E = \sum_n E_n$, $E_n \varepsilon \mathfrak{S}$, E belongs to $\mathfrak{S}$ when, and only when, $\sum_n \beta(E_n)$ converges,

and (3) $V = \sum V_n$, where $V_n \varepsilon \mathfrak{S}$.

If $f(\lambda)$ is integrable on any set of $\mathfrak{S}$, and $\int_E |f(\lambda)| d\beta(E)$ $E \varepsilon \mathfrak{S}$, is bounded, then we shall say that $f(\lambda)$ is integrable on V with respect to α. Then

(1) $\mathfrak{q}(U)$ satisfies the conditions: 1° $\mathfrak{q}(U) = \sum_n (\mathfrak{q} U_n)$, when $U = \sum_n U_n$, and 2° $(\mathfrak{q}(U), \mathfrak{q}(U')) = 0$, when $UU' = 0$. Cf. E. Maeda, loc. cit., 60–61.

$$\varepsilon, E, \quad E' \supset E \quad (\varepsilon \mathfrak{S}) \quad \int_{E'-E} |f(\lambda)|\, d\beta(E) < \varepsilon .^{(1)}$$

hence $$\varepsilon, E, \quad E' \supset E \quad (\varepsilon \mathfrak{S}) \quad \left| \int_{E'-E} f(\lambda) d\beta(E) \right| < \varepsilon$$

Thus there exists an element $F \varepsilon \mathfrak{R}$ or $\mathfrak{N}$ such that

$$\varepsilon, E, \quad E' \supset E \quad (\varepsilon \mathfrak{S}) \quad \left| F - \int_{E'} f(\lambda) d\alpha(E) \right| < \varepsilon .$$

Then we call F the integral of $f(\lambda)$, and write it as $\int_V f(\lambda) d\alpha(E)$.

For this extension the theorem in section 2 will also hold.[2]

5. Properties of Integral.

1° If $F(V) = \int_V f(\lambda) d\alpha(E)$ exists, the total variation of $F(E)$ on V, is

$$\int_V |f(\lambda)|\, d\beta(E)$$

Proof. By section 4 (3) it suffices to prove 1° when $V \varepsilon \mathfrak{S}$. First let $f(\lambda)$ be bounded, say $|f(\lambda)| < M$. Then from the definition

$$\varepsilon, \mathfrak{D}, \quad \mathfrak{D}' \geqq \mathfrak{D} \quad \sum_{\mathfrak{D}'} |F(E) - f(\lambda)\alpha(E)| < \varepsilon \quad \text{and} \quad \sum_{\mathfrak{D}'} \big(\beta(E) - |\alpha(E)|\big) < \varepsilon .$$

Hence

$$\begin{aligned}
\varepsilon, \mathfrak{D}, \quad \mathfrak{D}' \geqq \mathfrak{D} \quad \sum_{\mathfrak{D}'} |f(\lambda)|\, \beta(E) &= \sum_{\mathfrak{D}'} |f(\lambda)| \big(\beta(E) - |\alpha(E)|\big) + \sum_{\mathfrak{D}'} |f(\lambda)|\, |\alpha(E)| \\
&< M\varepsilon + \sum_{\mathfrak{D}'} \left| \int_E f(\lambda) d\alpha(E) - f(\lambda)\alpha(E) \right| \\
&\quad + \sum_{\mathfrak{D}'} \left| \int_E f(\lambda) d\alpha(E) \right| \\
&< (M+1)\varepsilon + \sum_{\mathfrak{D}'} \left| \int_E f(\lambda) d\alpha(E) \right|
\end{aligned}$$

Let $T(V)$ be the total variation of $F(E)$ on V, then we have from above

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(1) That is, for a given positive number ε there exists a set $E \varepsilon \mathfrak{S}$ such that for any set $E' \supset E$ and $\varepsilon \mathfrak{S}$, $\int_{E'-E} |f(\lambda)|\, d\beta(E) < \varepsilon$.

(2) E is α-normal in this case, if EV_n is α-normal in V_n for every n.

$$\varepsilon, \mathfrak{D}, \quad \mathfrak{D}' \geqq \mathfrak{D} \quad \sum_{\mathfrak{D}} |f(\lambda)|\, \beta(E) < (M+1)\varepsilon + T(V)$$

therefore
$$\int_V |f(\lambda)|\, d\beta(E) \leqq T(V).$$

But $\left|\int_E f(\lambda) d\alpha(E)\right| \leqq \int_E |f(\lambda)|\, d\beta(E)$, hence $T(V) \leqq \int_V |f(\lambda)|\, d\beta(E)$.

Thus we have
$$T(V) = \int_V |f(\lambda)|\, d\beta(E).$$

When $f(\lambda)$ is not bounded, let $E_n = \underset{\lambda}{E}[|f(\lambda)| \leqq n]$, then we have

$$T(E_n) = \int_{E_n} |f(\lambda)|\, d\beta(E);$$

hence
$$T(V) = \int_V |f(\lambda)|\, d\beta(E).$$

2° Is $F(E) = \int_E f(\lambda) d\alpha(E)$ exists on $E \varepsilon \mathfrak{S}$, then

$$\int_V g(\lambda) dF(E) \doteq \int_V f(\lambda) g(\lambda) d\alpha(E), \tag{1}$$

when either side of this equation exists.

Proof. As in 1° there is no loss of generality in supposing that $V \varepsilon \mathfrak{S}$. Since $\mathfrak{K}_\alpha$ is F-equivalent to $\mathfrak{K}$, we use only division into sets of $\mathfrak{K}_\alpha$.

Then $\quad \varepsilon, \mathfrak{D}, \quad \sum_{\mathfrak{D}} |f(\lambda)|\, \beta(E)$ conv. and $\sum_{\mathfrak{D}} Osc(f \cdot E) \beta(E) < \varepsilon$.

First assume that $g(\lambda)$ is bounded, say $|g(\lambda)| < M$. If $\int_V g(\lambda) dF(E)$ exists, then

$$\varepsilon, \mathfrak{D}, \quad \sum_{\mathfrak{D}} Osc(g \cdot E) T(E) < \varepsilon$$

where $T(E)$ has the same meaning as in the proof of 1°.

Hence $\quad \varepsilon, \mathfrak{D}, \quad \sum_{\mathfrak{D}} |f(\lambda) g(\lambda)|\, \beta(E) < M |f(\lambda)|\, \beta(E)$,

$$\sum_{\mathfrak{D}} Osc(fg, E) \beta(E) < \sum_{\mathfrak{D}} Osc(g \cdot E) |f|\, \beta(E) + M \sum_{\mathfrak{D}} Osc(f \cdot E) \beta(E)$$

$$\leqq \sum_{\mathfrak{D}} Osc(g\cdot E)\Big||f|\beta(E)-T(E)\Big|+\sum_{\mathfrak{D}} Osc(g\cdot E)T(E)+M\varepsilon$$

$$< 2M\varepsilon+\sum_{\mathfrak{D}} Osc(g\cdot E)T(E)+M\varepsilon$$

$$< (3M+1)\varepsilon\,,$$

and

$$\Big|\int_V gdF-\sum_{\mathfrak{D}} f\cdot g\alpha(E)\,\Big| \leqq \Big|\int_V gdF-\sum_{\mathfrak{D}} gF(E)\,\Big| + \Big|\sum_{\mathfrak{D}} gF(E)-\sum_{\mathfrak{D}} f\cdot g\alpha(E)\Big|$$

$$< \varepsilon+M\sum_{\mathfrak{D}} |\,F(E)-f\alpha(E)\,|$$

$$< (M+1)\varepsilon\,.$$

Thus $\int_V f(\lambda)g(\lambda)d\alpha(E)$ exists, and we have (1).

If $\int_V f(\lambda)g(\lambda)d\alpha(E)$ exists, then we shall have (1) by the same reasoning.

When $g(\lambda)$ is not bounded, let $V_0=E[|f(\lambda)|\neq 0]$, and $E_n=\underset{\lambda}{E}[|g(\lambda)|\leqq n,\ \lambda\varepsilon V_0]$. Then E_n is in $\mathfrak{K}_a$ if either $\int_V g(\lambda)dF(E)$ or $\int_V f(\lambda)g(\lambda)d\alpha(E)$ exists. Hence we have

$$\int_{E_n} g(\lambda)dF(E)=\int_{E_n} f(\lambda)g(\lambda)d\alpha(E)\,,$$

From 1° $$\int_{E_n} |\,g(\lambda)\,|\,dT(E)=\int_{E_n} |f(\lambda)|\,|\,g(\lambda)d\beta(E)\,;$$

therefore $\int_{V_0} g(\lambda)dF(E)=\int_{V_0} f(\lambda)g(\lambda)d\alpha(E)$. Hence we have (1).

This theorem holds also for Maeda's integral.

2′ If $\mathfrak{P}(V)=\int_V f(\lambda)d\mathfrak{q}(U)$ exists, then

$$\int_V g(\lambda)d\mathfrak{P}(U)=\int_V f(\lambda)g(\lambda)d\mathfrak{q}(U)\,.$$

when either side of the equation exists.
