Space of Differential Set Functions.

By

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In my previous papers,(1) I have investigated the space of set functions $\mathfrak{L}_2(\beta)$. It is defined as follows: Let $\mathfrak{K}$ be a closed family (σ-Körper) of all Borel subsets of a separable metric space Ω, and $\beta(E)$ be a completely additive, non-negative set function defined in $\mathfrak{K}$. If $\phi(E)$ be a completely additive set function defined in $\mathfrak{K}$, which is absolutely continuous with respect to $\beta(E)$ and $\int_\Omega |D_{\beta(E)}\phi(a)|^2 d\beta(E)$ is finite, then I said that $\phi(E)$ belongs to $\mathfrak{L}_2(\beta)$. $\mathfrak{L}_2(\beta)$ is a Hilbert space with the inner product

$$(\phi, \psi) = \int_\Omega D_{\beta(E)}\phi(a)\overline{D_{\beta(E)}\psi(a)}d\beta(E) .^{(2)}$$

In these previous papers, I have assumed that $\beta(\Omega)$ is finite. But in the applications, the case often occurs where $\beta(\Omega)$ is infinite. In this case, the usual definition of an integral is inconvenient. But A. Kolmogoroff(3) gave a new definition of an integral which is irrespective of the finiteness of $\beta(\Omega)$. In his definition of an integral, it is unnecessary that set functions are defined for all sets in a closed family; they need only be defined for decomposed sets of a multiplicative system. Such set functions, I call, in this paper, differential set functions. Using Kolmogoroff's integral, we can define the space of differential set functions in the same way as the space of ordinary set functions.

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(1) F. Maeda, "On the Space of Real Set Functions," this journal, **3** (1933), 1–42; "On Kernels and Spectra of Bounded Linear Transformations," ibid., 243–273; "Kernels of Transformations in the Space of Set Functions," this journal, **5** (1935), 107–116; "Transitivities of Conservative Mechanism," this volume, 1–18.

(2) If we do not demand the separability of $\mathfrak{L}_2(\beta)$, we can, more generally, take the closed family $\mathfrak{K}$ in an abstract space Ω as the domain of definition of set functions.

(3) A. Kolmogoroff, "Untersuchungen über den Integralbegriff," Math. Ann. **103** (1930), 654–682.

In this paper, I investigate this space of differential set functions, and I show that almost all properties of the space of ordinary set functions hold also in this space of differential set functions. And next, I apply these theories to certain problems of wave mechanics. In the space of point functions, the characteristic functions of the operators $\boldsymbol{Q} = q \ldots.$ and $\boldsymbol{P} = \frac{h}{2\pi i} \frac{d}{dq} \ldots.$ cannot be obtained in the strict mathematical sense. The characteristic function of $\boldsymbol{Q}$ is expressed, using the improper Dirac δ-function, by $\delta(q-\lambda)$, and the characteristic functions of $\boldsymbol{P}$ is $e^{\frac{2\pi i}{h}\lambda q}$ which is not quadratically integrable.(1) In the last part of this paper, I show that in the space of differential set functions, the characteristic functions of these operators can be obtained in the strict mathematical sense.

Differential Set Systems and Differential Set Functions.

1. Let $\mathfrak{M}$ be a system of sets in an abstract space Ω, then $\mathfrak{M}$ is called a multiplicative system when the product of any two sets E and E' belongs to $\mathfrak{M}$ with E and E'. Now, assume that $\mathfrak{M}$ contains Ω.

Let A be a set in $\mathfrak{M}$. The decomposition of A in a sum of finite or enumerably infinite distinct sets $\{E_n\}$ belonging to $\mathfrak{M}$:

$$A = \sum_n E_n\,,$$

is expressed by

$$\mathfrak{D}A \equiv \sum_n E_n\,,$$

and the sets E_n are called the elements of the decomposition $\mathfrak{D}$.

Let
$$\mathfrak{D}'A \equiv \sum_m E'_m$$

be another decomposition of A, such that E'_m is a subset of any one of the elements E_n of $\mathfrak{D}$, then we say that $\mathfrak{D}'$ is an extension of $\mathfrak{D}$, and write

$$\mathfrak{D}' \supset \mathfrak{D}\,.$$

(1) Cf. J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, (1932), 69.

Of course $\mathfrak{D}$ is an extension of $\mathfrak{D}$ itself.

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be any two decompositions of A

$$\mathfrak{D}A \equiv \sum_n E_n\,, \qquad \mathfrak{D}'A \equiv \sum_m E'_m\,,$$

then

$$[\mathfrak{D}\mathfrak{D}']A \equiv \sum_{n,m} E_n E'_m$$

is called the product of $\mathfrak{D}$ and $\mathfrak{D}'$, and is the extension of either $\mathfrak{D}$ and $\mathfrak{D}'$.

Denote by $\mathfrak{M}\mathfrak{D}A$ the aggregate of the elements of all extensions of a decomposition $\mathfrak{D}A$. We say $\mathfrak{M}\mathfrak{D}A$ is a *differential set system* in A. It is also a multiplicative system.

Let $\mathfrak{M}\mathfrak{D}'A$ be another differential set system in A. Then $[\mathfrak{D}\mathfrak{D}']A$ being a decomposition of A, we have a differential set system in A, i. e. $\mathfrak{M}[\mathfrak{D}\mathfrak{D}']A$. Since $[\mathfrak{D}\mathfrak{D}']A$ is an extension of $\mathfrak{D}A$ and $\mathfrak{D}'A$, we have

$$\mathfrak{M}\mathfrak{D}A \geqq \mathfrak{M}[\mathfrak{D}\mathfrak{D}']A\,, \qquad \mathfrak{M}\mathfrak{D}'A \geqq \mathfrak{M}[\mathfrak{D}\mathfrak{D}']A\,. \tag{1}$$

If E_0 be any set which belongs to both $\mathfrak{M}\mathfrak{D}A$ and $\mathfrak{M}\mathfrak{D}'A$, then there exist $\mathfrak{D}_1A$ and $\mathfrak{D}'_1A$ so that

$$\mathfrak{D}_1 \supset \mathfrak{D}\,, \qquad \mathfrak{D}'_1 \supset \mathfrak{D}'\,,$$

and E_0 is the element of both $\mathfrak{D}_1A$ and $\mathfrak{D}'_1A$. Then E_0 is the element of $[\mathfrak{D}_1\mathfrak{D}'_1]A$. Since $[\mathfrak{D}_1\mathfrak{D}'_1]A$ is an extension of $[\mathfrak{D}\mathfrak{D}']A$, E_0 belongs to $\mathfrak{M}[\mathfrak{D}\mathfrak{D}']A$. That is

$$(\mathfrak{M}\mathfrak{D}A)(\mathfrak{M}\mathfrak{D}'A) \leqq \mathfrak{M}[\mathfrak{D}\mathfrak{D}']A\,. \tag{2}$$

Combining (1) and (2), we have

$$(\mathfrak{M}\mathfrak{D}A)(\mathfrak{M}\mathfrak{D}'A) = \mathfrak{M}[\mathfrak{D}\mathfrak{D}']A\,.$$

That is, *the product of two differential set systems in A is also a differential set system in A.*

Denote by $\mathfrak{M}A$ the aggregate of the elements of all decompositions of A. Then $\mathfrak{M}A$ is also a differential set system in A, and

$$\mathfrak{M}A \geqq \mathfrak{M}\mathfrak{D}A$$

for all $\mathfrak{M}\mathfrak{D}A$. $\mathfrak{M}A$ contains A itself.

2. If, for any elements E of a differential set system $\mathfrak{M}\mathfrak{D}A$, one or many complex valued function $\xi(E)$ are defined, then we say that $\xi(E)$ is a *differential set function* in A.

Let $\xi'(E)$ be another differential set function in A. If

$$\xi(E) = \xi'(E)$$

for all sets E in a differential set system, then we consider that $\xi(E)$ and $\xi'(E)$ are the same differential set function in A.

When $\xi(E)$ is one-valued, and

$$\xi(E) = \sum_n \xi(E_n)$$

for any decomposition of E in $\mathfrak{M}\mathfrak{D}A$

$$E = \sum_n E_n$$

then we say that $\xi(E)$ is *completely additive*. Since the convergence of $\sum_n \xi(E_n)$ must be independent of the order of the terms of the series, $\sum_n \xi(E_n)$ must be absolutely convergent.

Let $\xi(E)$ be a differential set function in A. If there exists a finite number I such that for any positive number ε, a decomposition $\mathfrak{D}_0 A$ exists so that for any decomposition $\mathfrak{D} \supset \mathfrak{D}_0$

$$\sup \left| \sum_n \xi(E_n) - I \right| < \varepsilon \qquad (\mathfrak{D}A \equiv \sum_n E_n),$$

then we say, after Kolmogoroff, that I is the *integral* of $\xi(E)$ in A, and write

$$I = \int_A \xi(dE).^{(1)}$$

If $\int_A \xi(dE)$ exists, then $\int_E \xi(dE)$ exists for all elements E in $\mathfrak{M}A$, and is a completely additive differential set function in A.

When $f(a)$ is a point function defined in A, then from $f(a)$, we can construct a many valued differential set function $f(E)$ so that it takes all values $f(a)$ when a is a point in E. After Kolmogoroff,[(2)] we can define the integral $\int_A f(a)\xi(dE)$ by $\int_A f(dE)\xi(dE)$.

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(1) We can define the integral in the case where I is infinite. But such an integral is not used in this paper.

(2) A. Kolmogoroff, loc. cit., 676.

These integrals, introduced by Kolmogoroff, have almost all the fundamental properties of the ordinary integrals.[1]

Space of Differential Set Functions.

3. Let $\beta(E)$ be a completely additive, non-negative differential set function in Ω, and $\xi(E)$ be another completely additive differential set function in Ω. Let $\mathfrak{M}\mathfrak{D}\Omega$ be a differential set system for all elements of which $\beta(E)$ and $\xi(E)$ are both defined. If $\xi(E)=0$ for all sets E in $\mathfrak{M}\mathfrak{D}\Omega$, where $\beta(E)=0$, then we say that $\xi(E)$ is *absolutely continuous* with respect to $\beta(E)$.

If $\xi_1(E)$ and $\xi_2(E)$ are absolutely continuous with respect to $\beta(E)$, then

$$\frac{\xi_1(E)\overline{\xi_2(E)}}{\beta(E)} \tag{1}$$

takes a definite value or $\frac{0}{0}$ at all sets E in a differential set system. Hence, if we define the value of (1) as 0 when (1) becomes $\frac{0}{0}$, then (1) is a differential set function in Ω.

If a completely additive differential set function $\xi(E)$ is absolutely continuous with respect to $\beta(E)$ and

$$\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)} \tag{2}$$

is finite, then we say that $\xi(E)$ *belongs to* $\mathfrak{L}_2(\beta)$. And we denote the positive square root of (2) by $\|\xi\|$.

In the inequality

$$|\sum_i a_ib_i|^2 \leqq \sum_i |a_i|^2 \sum_i |b_i|^2 ,$$

put $a_ib_i=\xi(E_i)$, $|b_i|^2=\beta(E_i)$, then we have

$$\frac{|\xi(E)|^2}{\beta(E)} \leqq \sum_i \frac{|\xi(E_i)|^2}{\beta(E_i)} \qquad (E=\sum_i E_i)\,, \tag{3}$$

since $\xi(E)$ and $\beta(E)$ are completely additive. Hence if $\int_E \frac{|\xi(dE)|^2}{\beta(dE)}$ is finite, then by the definition of the integral, it is the upper bound of

(1) For detailed discussion, cf. A. Kolmogoroff, loc. cit., 661–682.

$\sum_i \frac{|\xi(E_i)|^2}{\beta(E_i)}$ $(E = \sum_i E_i)$. Therefore, if $\xi(E)$ belongs to $\mathfrak{L}_2(\beta)$, then

$$\varkappa(E) = \int_E \frac{|\xi(dE)|^2}{\beta(dE)}$$

is a completely additive non-negative, differential set function defined for all sets E in $\mathfrak{M}\Omega$, and by (3)

$$|\xi(E)|^2 \leqq \beta(E)\varkappa(E) \tag{4}$$

for any set E in a differential set system. Since $\varkappa(E) \leqq \varkappa(\Omega) = \|\xi\|^2$, we have

$$|\xi(E)|^2 \leqq \beta(E)\|\xi\|^2. \tag{5}$$

Conversely, if there exists a completely additive, non-negative differential set function $\varkappa(E)$ in Ω which satisfies (4), and $\int_\Omega \varkappa(dE)$ is finite, then $\xi(E)$ is absolutely continuous with respect to $\beta(E)$, and

$$\sum_i \frac{|\xi(E_i)|^2}{\beta(E_i)} \leqq \sum_i \varkappa(E_i) = \int_\Omega \varkappa(dE) \qquad (\Omega = \sum_i E_i).$$

Hence $\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)}$ is finite. That is $\xi(E)$ belongs to $\mathfrak{L}_2(\beta)$.

Thus, we have the following theorem:

Let $\xi(E)$ be a completely additive, differential set function in Ω. Then $\xi(E)$ belongs to $\mathfrak{L}_2(\beta)$, when and only when there exists a non-negative completely additive differential set function $\varkappa(E)$ in Ω, which satisfies

$$|\xi(E)|^2 \leqq \beta(E)\varkappa(E)$$

for any sets E in a differential set system, and $\int_\Omega \varkappa(dE)$ is finite.

In this case

$$\|\xi\|^2 \leqq \int_\Omega \varkappa(dE).$$

4. $\mathfrak{L}_2(\beta)$ have the following properties:

(i) $\mathfrak{L}_2(\beta)$ is *linear*; i.e. $c_1\xi_1(E)+c_2\xi_2(E)$ belongs to $\mathfrak{L}_2(\beta)$, together with $\xi_1(E)$ and $\xi_2(E)$; c_1, c_2 being any complex numbers. For, it is evident that $c_1\xi_1(E)+c_2\xi_2(E)$ is completely additive, and absolutely continuous with respect to $\beta(E)$. In Minkowski's inequality

$$[\sum_i |a_i+b_i|^2]^{\frac{1}{2}} \leqq [\sum_i |a_i|^2]^{\frac{1}{2}} + [\sum_i |b_i|^2]^{\frac{1}{2}},$$

put $a_i = \dfrac{c_1\xi_1(E_i)}{\sqrt{\beta(E_i)}}$, $b_i = \dfrac{c_2\xi_2(E_i)}{\sqrt{\beta(E_i)}}$, $(\Omega = \sum_i E_i)$.

Then

$$\left[\sum_i \frac{|c_1\xi_1(E_i)+c_2\xi_2(E_i)|^2}{\beta(E_i)}\right]^{\frac{1}{2}} \leqq |c_1|\left[\sum_i \frac{|\xi_1(E_i)|^2}{\beta(E_i)}\right]^{\frac{1}{2}} + |c_2|\left[\sum_i \frac{|\xi_2(E_i)|^2}{\beta(E_i)}\right]^{\frac{1}{2}}.$$

Since the right hand side of the above inequality is not greater than $|c_1|\cdot||\xi_1|| + c_2|\cdot||\xi_2||$, $c_1\xi_1(E)+c_1\xi_2(E)$ belongs to $\mathfrak{L}_2(\beta)$, and

$$||c_1\xi_1+c_2\xi_2|| \leqq |c_1|\cdot||\xi_1|| + |c_2|\cdot||\xi_2||.$$

(ii) In $\mathfrak{L}_2(\beta)$, the *inner product* (ξ, η) is defined by the value

$$\int_\Omega \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)}. \tag{1}$$

Since, for any positive number ε, a decomposition $\mathfrak{D}_0\Omega$ exists so that for any decomposition $\mathfrak{D} \supset \mathfrak{D}_0$

$$\left|\sum_i \frac{\xi(E_i)\overline{\eta(E_i)}}{\beta(E_i)} - \sum_{i,n} \frac{\xi(E_{in})\overline{\eta(E_{in})}}{\beta(E_{in})}\right| < \varepsilon^{(1)} \qquad \left(\begin{matrix}\mathfrak{D}_0\Omega \equiv \sum_i E_i \\ \mathfrak{D}E_i \equiv \sum_n E_{in}\end{matrix}\right),$$

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(1) To prove this inequality, put $a = \gamma_n \dfrac{\sum_n a_n}{\sum_n \gamma_n}$, $b = a_n - \gamma_n \dfrac{\sum_n a_n}{\sum_n \gamma_n}$ in $(a+b)^2 = a^2+b^2+2ab$, where $\gamma_n \geqq 0$, and a_n are real. Next, divide by γ_n and sum with respect to n. Then we have

$$\sum_n \frac{|a_n|^2}{\gamma_n} = \frac{|\sum_n a_n|^2}{\sum_n \gamma_n} + \sum_n \frac{\left|a_n - \gamma_n \dfrac{\sum_n a_n}{\sum_n \gamma_n}\right|^2}{\gamma_n}. \tag{i}$$

When a_n are complex, the above equality holds for the real and imaginary parts. Hence (i) holds when a_n are complex. Next, put $a_n = \xi(E_{in})$, $\gamma_n = \beta(E_{in})$ in (i), and sum with respect to i. Then we have

$$\sum_{i,n} \frac{|\xi(E_{in})|^2}{\beta(E_{in})} = \sum_i \frac{|\xi(E_i)|^2}{\beta(E_i)} + \sum_{i,n} \frac{\left|\xi(E_{in}) - \beta(E_{in})\dfrac{\xi(E_i)}{\beta(E_i)}\right|^2}{\beta(E_{in})}. \tag{ii}$$

Now,

$$\left|\sum_i \frac{\xi(E_i)\overline{\eta(E_i)}}{\beta(E_i)} - \sum_{i,n} \frac{\xi(E_{in})\overline{\eta(E_{in})}}{\beta(E_{in})}\right|^2 = \left|\sum_{i,n} \frac{\overline{\eta(E_{in})}\left\{\beta(E_{in})\dfrac{\xi(E_i)}{\beta(E_i)} - \xi(E_{in})\right\}}{\beta(E_{in})}\right|^2$$

$$\leqq \sum_{i,n} \frac{|\eta(E_{in})|^2}{\beta(E_{in})} \sum_{i,n} \frac{\left|\beta(E_{in})\dfrac{\xi(E_i)}{\beta(E_i)} - \xi(E_{in})\right|^2}{\beta(E_{in})}$$

$$\leqq \sum_{i,n} \frac{|\eta(E_{in})|^2}{\beta(E_{in})} \left(\sum_{i,n} \frac{|\xi(E_{in})|^2}{\beta(E_{in})} - \sum_i \frac{|\xi(E_i)|^2}{\beta(E_i)}\right) \qquad \text{by (ii).}$$

the integral (1) exists. And by the following inequality

$$\left| \sum_i \frac{\xi(E_i)\overline{\eta(E_i)}}{\beta(E_i)} \right|^2 \leqq \sum_i \frac{|\xi(E_i)|^2}{\beta(E_i)} \sum_i \frac{|\eta(E_i)|^2}{\beta(E_i)},$$

we have
$$|(\xi, \eta)| \leqq \|\xi\| \cdot \|\eta\|.$$

Thus $\mathfrak{L}_2(\beta)$ satisfies the two essential axioms of the abstract Hilbert space, where the null element is the differential set function which vanishes identically for all sets in a differential set system. For when $\|\xi\| = 0$, then by (5) of the preceding section $\xi(E) = 0$ for any set in a differential set system. Hence almost all conceptions in the abstract Hilbert space can be used also in $\mathfrak{L}_2(\beta)$. We use these conceptions in this paper without explanations.(1)

Put
$$\xi(E) = \beta(EE'),$$
then
$$\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)} = \int_\Omega \frac{|\beta(dE\cdot E')|^2}{\beta(dE)} = \int_{E'} \beta(dE) = \beta(E')$$
is finite, when E' belongs to a differential set system, say $\mathfrak{M}\mathfrak{D}\Omega$. In this case $\beta(EE')$ belongs to $\mathfrak{L}_2(\beta)$.

Denote by $\mathfrak{L}_2'(\beta)$ the linear manifold determined by the system $\{\beta(EE')\}$, E' being any set in $\mathfrak{M}\mathfrak{D}\Omega$. Let $\xi(E)$ be any function in $\mathfrak{L}_2(\beta)$. Then for any ε, we can take a decomposition $\mathfrak{D}\Omega$ and k so that

$$\|\xi - \phi\| < \varepsilon$$

where
$$\phi(E) = \sum_{i=1}^{k} \frac{\xi(E_i)}{\beta(E_i)} \beta(EE_i) \qquad (\mathfrak{D}\Omega \equiv \sum_i E_i).$$

(Cf. p. 25 footnote (ii)). Hence $\mathfrak{L}_2'(\beta)$ *is dense in* $\mathfrak{L}_2(\beta)$.

Weak and Strong Convergences.(2)

5. Let $\{\xi_n(E)\}$ be a sequence of differential set functions in $\mathfrak{L}_2(\beta)$, and $\xi(E)$ be a differential set function in $\mathfrak{L}_2(\beta)$.

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(1) For the axioms of the abstract Hilbert space, and conceptions in this space, cf. M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 2–23. When $\mathfrak{M}$ is a σ-Körper containing Ω, and $\beta(E)$, $\xi(E)$ are defined in all sets E in $\mathfrak{M}$, then $\mathfrak{L}_2(\beta)$ is nothing else than the space of ordinary set functions, which I considered in previous papers.

(2) The following theorems about the weak and strong convergences hold also in the space of ordinary set functions.

If
$$\lim_{n\to\infty} \| \xi_n - \xi \| = 0,$$
then we say that $\{\xi_n(E)\}$ *converges strongly* to $\xi(E)$, and write
$$[\lim_{n\to\infty}]\, \xi_n(E) = \xi(E).$$
In this case, of course,
$$\lim_{n\to\infty} \| \xi_n \| = \| \xi \|.$$

And if
$$\lim_{n\to\infty} (\xi_n, \eta) = (\xi, \eta) \tag{1}$$
for all $\eta(E)$ in $\mathfrak{L}_2(\beta)$, then we say that $\{\xi_n(E)\}$ *converges weakly* to $\xi(E)$, and write
$$(\lim_{n\to\infty})\, \xi_n(E) = \xi(E).$$
In this case, let n becomes infinite in the following inequality:
$$|(\xi_n, \xi)| \leqq \| \xi_n \| \cdot \| \xi \|,$$
we have
$$\| \xi \|^2 \leqq \varliminf_{n\to\infty} \| \xi_n \| \cdot \| \xi \|.$$
That is
$$\varliminf_{n\to\infty} \| \xi_n \| \geqq \| \xi \|. \tag{2}$$

Since
$$|(\xi_n - \xi, \eta)| \leqq \| \xi_n - \xi \| \cdot \| \eta \|,$$
the strong convergence implies the weak convergence.

And since
$$\| \xi_n - \xi \|^2 = \| \xi_n \|^2 - (\xi_n, \xi) - (\xi, \xi_n) + \| \xi \|^2,$$
if $(\lim_{n\to\infty})\, \xi_n(E) = \xi(E)$ *and* $\lim_{n\to\infty} \| \xi_n \| = \| \xi \|$,

then
$$[\lim_{n\to\infty}]\, \xi_n(E) = \xi(E). \tag{3}$$

Let all differential set functions $\xi_n(E)$ be defined for all sets in a common differential set system. In (1), put $\eta(E) = \beta(EE')$, then
$$\lim_{n\to\infty} \xi_n(E) = \xi(E)^{(1)}$$

(1) For $\left(\xi(E), \beta(EE')\right) = \int_\Omega \frac{\xi(dE)\beta(dE \cdot E')}{\beta(dE)} = \int_{E'} \xi(dE) = \xi(E').$

for any set E in a differential set system. Hence, *the weak convergence, and therefore the strong convergence, implies the ordinary convergence in a differential set system.*[1]

If $\lim_{n\to\infty} \xi_n(E) = \xi(E)$ *in a differential set system and* $\| \xi_n \| < M$ *for any* n, M *being a constant, then*

$$(\lim_{n\to\infty}) \xi_n(E) = \xi(E)\,.$$

For, by the footnote of the preceding page, $\lim_{n\to\infty} \xi_n(E) = \xi(E)$ can be written in the form

$$\lim_{n\to\infty} (\xi_n, \eta) = (\xi, \eta) \tag{4}$$

where $\eta(E) = \beta(EE')$, E' are sets in a differential set system. Hence (4) holds for all $\eta(E)$ in $\mathfrak{L}_2'(\beta)$. But

$$|(\xi_n, \eta)| \leqq \| \xi_n \| \cdot \| \eta \| < M \| \eta \|\,,$$

and $\mathfrak{L}_2'(\beta)$ is dense in $\mathfrak{L}_2(\beta)$. Hence (4) holds for all $\eta(E)$ in $\mathfrak{L}_2(\beta)$. That is, $\{\xi_n(E)\}$ converges weakly to $\xi(E)$.[2]

If $(\lim_{n\to\infty}) \xi_n(E) = \xi(E)$, then it is already proved that $\{\| \xi_n \|\}$ is bounded when $\mathfrak{L}_2(\beta)$ is complete.[2] Hence we have the following theorem:

When all $\xi_n(E)$ *are defined in a differential set syetem and* $\mathfrak{L}_2(\beta)$ *is complete,* $\{\xi_n(E)\}$ *converges weakly to* $\xi(E)$, *when and only when*

$$\lim_{n\to\infty} \xi_n(E) = \xi(E)$$

in a differential set system, and $\{\| \xi_n \|\}$ *is bounded.*

If $\lim_{n\to\infty} \xi_n(E) = \xi(E)$ *in a differential set system and*

$$\overline{\lim_{n\to\infty}} \| \xi_n \| \leqq \| \xi \|\,, \tag{5}$$

(1) In the case of the strong convergence, the implication of the ordinary convergence can be proved directly by

$$|\xi_n(E) - \xi(E)| \leqq \{\beta(E)\}^{\frac{1}{2}} \| \xi_n - \xi \|\,.$$

(Cf. sec. 3, (5)).

(2) Cf. J. v. Neumann, Math. Ann., **102** (1929), 380. The completeness of $\mathfrak{L}_2(\beta)$ is proved in the next section.

then
$$\underset{n\to\infty}{[\lim]}\, \xi_n(E) = \xi(E)\,.$$

For, since $\| \xi_n \|$ is bounded, by the preceding theorem,

$$\underset{n\to\infty}{(\lim)}\, \xi_n(E) = \xi(E)\,.$$

Hence, by (2)
$$\varliminf_{n\to\infty} \| \xi_n \| \geqq \| \xi \|\,.$$

Combining with (5), we have

$$\lim_{n\to\infty} \| \xi_n \| = \| \xi \|\,.$$

Therefore, by (3)
$$\underset{n\to\infty}{[\lim]}\, \xi_n(E) = \xi(E)\,.$$

Combining the above theorems, we have the following theorem:
When all $\xi_n(E)$ are defined in a differential set system, $\{\xi_n(E)\}$ converges strongly to $\xi(E)$, when and only when

$$\lim_{n\to\infty} \xi_n(E) = \xi(E)$$

in a differential set system, and

$$\lim_{n\to\infty} \| \xi_n \| = \| \xi \|\,.$$

Completeness of $\mathfrak{L}_2(\beta)$.[1]

6. *When all differential set functions in $\mathfrak{L}_2(\beta)$ are defined in a common differential set system, then $\mathfrak{L}_2(\beta)$ is complete.* That is, if $\{\xi_n(E)\}$ be a sequence of differential set functions in $\mathfrak{L}_2(\beta)$, such that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \| \xi_m - \xi_n \| = 0\,, \tag{1}$$

then there exists a differential set function $\xi(E)$ in $\mathfrak{L}_2(\beta)$, so that

$$\underset{n\to\infty}{[\lim]}\, \xi_n(E) = \xi(E)\,.$$

(1) This method of proof can be applied to the case of the space of ordinary set functions. In my previous paper (F. Maeda, this journal, **3** (1933, 4), I deduced the completeness from that of the space of point functions.

Since, by sec. 3, (5)

$$|\xi_m(E)-\xi_n(E)|^2 \leqq \beta(E)\,||\,\xi_m-\xi_n\,||^2,$$

$\xi_n(E)$ converges to a differential set function $\xi(E)$. And

$$|\,\xi_n(E)-\xi(E)\,| \leqq \varepsilon\beta(E) \tag{2}$$

for any E when $n > N$. Hence, when $E = \sum_{i=1}^{i=\infty} E_i$

$$\left|\sum_{i=1}^{k}\xi_n(E_i)-\sum_{i=1}^{k}\xi(E_i)\right| \leqq \varepsilon\sum_{i=1}^{k}\beta(E_i) \leqq \varepsilon\beta(E) \tag{3}$$

for any k when $n > N$. Now take n so that (3) and

$$|\,\xi(E)-\xi_n(E)\,| \leqq \varepsilon\beta(E) \tag{4}$$

hold. For this definite n, there exists an integer K so that

$$\left|\,\xi_n(E)-\sum_{i=1}^{k}\xi_n(E_i)\,\right| \leqq \varepsilon\beta(E) \tag{5}$$

when $k > K$. Hence, by (3), (4) and (5)

$$\left|\,\xi(E)-\sum_{i=1}^{k}\xi(E_i)\,\right| \leqq 3\varepsilon\beta(E)$$

when $k > K$. That is, $\xi(E)$ is completely additive.

Now, by (1) $\{||\,\xi_n\,||\}$ is bounded, and by sec. 3 $||\,\xi_n\,||^2$ is the upper bound of $\sum_{i=1}^{\infty}\frac{|\,\xi_n(E_i)\,|^2}{\beta(E_i)}$ $(\Omega = \sum_i E_i)$. Hence

$$\sum_{i=1}^{k}\frac{|\,\xi_n(E_i)\,|^2}{\beta(E_i)} \leqq ||\,\xi_n\,||^2 < M \qquad (\Omega = \sum_i E_i),$$

where M is a constant number independent of n. Since

$$\lim_{n\to\infty}\xi_n(E_i) = \xi(E_i)$$

for any E_i, we have

$$\sum_{i=1}^{k}\frac{|\,\xi(E_i)\,|^2}{\beta(E_i)} \leqq M.$$

Therefore

$$\sum_{i=1}^{\infty}\frac{|\,\xi(E_i)\,|^2}{\beta(E_i)} \leqq M$$

for any decomposition $\Omega = \sum_i E_i$. Hence, by sec. 3, $\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)}$ is finite. That is $\xi(E)$ belongs to $\mathfrak{L}_2(\beta)$.

$\{\|\xi_n\|\}$ being bounded, by sec. 5, we have

$$(\lim_{n\to\infty})\, \xi_n(E) = \xi(E).$$

Hence
$$(\lim_{n\to\infty})\, \{\xi_m(E)-\xi_n(E)\} = \xi_m(E)-\xi(E).$$

And by sec. 5 (2)

$$\varliminf_{n\to\infty} \|\xi_m-\xi_n\| \geqq \|\xi_m-\xi\|. \tag{6}$$

But, since
$$\|\xi_m-\xi_n\| < \varepsilon \qquad \text{when } m, n > N,$$

N being an integer which depends to ε, from (6), we have

$$\|\xi_m-\xi\| \leqq \varepsilon \qquad \text{when } m > N.$$

That is
$$[\lim_{m\to\infty}]\, \xi_m(E) = \xi(E).$$

Thus, the theorem is proved.

In what follows, we consider the case where $\mathfrak{L}_2(\beta)$ is complete.

Kernels of Transformations.

7. Let $\mathfrak{K}(E, E')$ be a completely additive differential set function defined in a differential set system, when $\mathfrak{K}(E, E')$ is considered as a function of set E, E' being a parameter, ~~and when $\mathfrak{K}(E, E')$ is considered as a function of set E', E being a parameter~~. When $\mathfrak{K}(E, E')$ is considered as a function of set E, E' being a parameter, we denote it by $\mathfrak{K}(E, _{(E')})$. Similarly for $\mathfrak{K}(_{(E)}, E')$.

When $\mathfrak{K}(E, _{(E')})$ and $\mathfrak{K}(_{(E)}, E')$ belong to $\mathfrak{L}_2(\beta)$, we say that $\mathfrak{K}(E, E')$ *belongs to* $\mathfrak{L}_2(\beta, \beta)$.

When $\mathfrak{K}(E, E')$ belongs to $\mathfrak{L}_2(\beta, \beta)$, if for some differential set function $\xi(E)$ in $\mathfrak{L}_2(\beta)$,

$$\int_\Omega \frac{\mathfrak{K}(E, dE')\xi(dE')}{\beta(dE')} \tag{1}$$

is a differential set function in $\mathfrak{L}_2(\beta)$, say $\eta(E)$, then (1) represents a transformation in $\mathfrak{L}_2(\beta)$, which we denote by

$$\boldsymbol{T}_{\mathfrak{K}}\xi(E) = \eta(E)\,.$$

And we say that $\mathfrak{K}(E, E')$ is the kernel of $\boldsymbol{T}_{\mathfrak{K}}$. The domain of $\boldsymbol{T}_{\mathfrak{K}}$, which we denote by $\mathfrak{D}_{\mathfrak{K}}$, is the aggregate of all functions $\xi(E)$ in $\mathfrak{L}_2(\beta)$ so that (1) is a function in $\mathfrak{L}_2(\beta)$. It is evident that $\mathfrak{D}_{\mathfrak{K}}$ is a linear manifold, and $\boldsymbol{T}_{\mathfrak{K}}$ is a linear transformation.

For example, *$\beta(EE')$ is the kernel of the identical transformation in $\mathfrak{L}_2(\beta)$.* For

$$\int_{\Omega}\frac{\beta(E\cdot dE')\xi(dE')}{\beta(dE')} = \int_{E}\xi(dE') = \xi(E)\,,$$

for any function $\xi(E)$ in $\mathfrak{L}_2(\beta)$.

All the theorems in the previous paper[(1)] concerning the kernels of transformations in the space of ordinary set functions hold also in the space of differential set functions. Here I cite one theorem which is used in what follows.

If $\mathfrak{U}(E, E')$ be a differential set function in $\mathfrak{L}_2(\beta, \beta)$, which satisfies the following relations:

$$\big(\mathfrak{U}(E, {}_{(E')}), \mathfrak{U}(E, {}_{(E'')})\big) = \beta(E'E'')\,, \tag{2}$$

$$\big(\mathfrak{U}^*(E, {}_{(E')}), \mathfrak{U}^*(E, {}_{(E'')})\big) = \beta(E'E'')\,, \tag{3}$$

$$\mathfrak{U}^*(E, E') = \overline{\mathfrak{U}(E', E)}\,,$$

then $\mathfrak{U}(E, E')$ *is the kernel of a unitary transformation.*[(2)]

In this paper, I use the following abbreviation:

$$\int_{\Omega}\frac{\mathfrak{K}(E, dE')\xi(dE')}{\beta(dE')} = \mathfrak{K}\xi(E)\,,$$

$$\int_{\Omega}\frac{\mathfrak{K}_1(E, dE'')\mathfrak{K}_2(dE'', E')}{\beta(dE'')} = \mathfrak{K}_1\mathfrak{K}_2(E, E')\,.$$

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(1) F. Maeda, this journal, **5** (1935), 107–116.

(2) Ibid., 115.

Then (2), (3) can be written as follows:

$$\mathfrak{U}^*\mathfrak{U}(E'', E') = \beta(E'E''),$$

$$\mathfrak{U}\mathfrak{U}^*(E'', E') = \beta(E'E'').$$

Vector Valued Differential Set Functions and Resolution of Identity.

8. The theory of vector valued set functions, which I considered in my previous papers,[1] can be applied to the case of vector valued differential set functions.

Let $\mathfrak{H}$ be a space of vectors, which satisfies the following axioms[2]:

(i) $\mathfrak{H}$ is a linear space,

(ii) in $\mathfrak{H}$ an inner product is defined,

(iii) $\mathfrak{H}$ is complete.

And let $\mathfrak{R}$ be a multiplicative system of sets in an abstract space V. If for all sets U of a differential set system $\mathfrak{R}\mathfrak{D}V$, a vector $\mathfrak{q}(U)$ in $\mathfrak{H}$ be determined, then $\mathfrak{q}(U)$ may be called a *vector valued differential set function*. And $\mathfrak{q}(U)$ is said to be *completely additive*, if

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = 0$$

when $UU' = 0$, and

$$\mathfrak{q}(U)\,[=]\,\mathfrak{q}(U_1) + \mathfrak{q}(U_2) + \dots + \mathfrak{q}(U_n) + \dots \text{[3]}$$

when $U = \sum_n U_n$. Then

$$\sigma(U) = \|\,\mathfrak{q}(U)\,\|^2$$

is a completely additive differential set function in V.[4]

Let $\xi(U)$ be a completely additive differential set function in V, which is absolutely continuous with respect to $\sigma(U)$. If there exists a vector $\mathfrak{f}$ in $\mathfrak{H}$ such that for any positive number ε a decomposition $\mathfrak{D}_0V$ exists so that for any decomposition $\mathfrak{D} \supset \mathfrak{D}_0$

(1) F. Maeda, this journal, **4** (1934), 57–91, 141–160.

(2) The separability is unnecessary, since all essential properties in the abstract Hilbert space hold also without it.

(3) [=] means the strong convergence of the series.

(4) Cf. F. Maeda, this journal, **4** (1934), 60.

$$\left\| \sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)} - \mathfrak{f} \right\| < \varepsilon \qquad (\mathfrak{D}V \equiv \sum_n U_n),$$

then we say that $\mathfrak{f}$ is the *integral* of $\xi(U)$ by $\mathfrak{q}(U)$, and write

$$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}. \tag{1}$$

When (1) exist, since

$$\left\| \sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)} \right\|^2 = \sum_n \frac{|\xi(U_n)|^2}{\sigma(U_n)},$$

$\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$ and

$$\| \mathfrak{f} \| = \| \xi \|. \tag{2}$$

Conversely, when $\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$, for any positive number ε, a decomposition $\mathfrak{D}_\varepsilon V$ exists so that for any decomposition $\mathfrak{D} \supset \mathfrak{D}_\varepsilon$

$$\left\| \sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)} - \sum_{n,i} \frac{\xi(U_{ni})\mathfrak{q}(U_{ni})}{\sigma(U_{ni})} \right\| < \varepsilon.^{(1)} \qquad \begin{pmatrix} \mathfrak{D}_\varepsilon V \equiv \sum_n U_n \\ \mathfrak{D}U_n \equiv \sum_i U_{ni} \end{pmatrix}. \tag{3}$$

Let $\{\varepsilon_\nu\}$ be a sequence of positive numbers, such that $\lim_{\nu\to\infty} \varepsilon_\nu = 0$.

Put $$\sum_{\mathfrak{D}} = \sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)} \qquad \text{when } \mathfrak{D}V \equiv \sum_n U_n,$$

and $$\sum_{\mathfrak{D}_\nu} = \sum_{\mathfrak{D}_{\varepsilon_\nu}}.$$

Then, by (3) there exists a sequence $\left\{\sum_{\mathfrak{D}_\nu}\right\}$ such that

$$\left\| \sum_{\mathfrak{D}_\nu} - \sum_{\mathfrak{D}} \right\| < \varepsilon_\nu \qquad \text{for any } \mathfrak{D} \supset \mathfrak{D}_\nu. \tag{4}$$

Put $$\mathfrak{D}^{(\nu)} = [\mathfrak{D}_1\mathfrak{D}_2 \dots \mathfrak{D}_\nu].$$

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(1) For $$\left\| \sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)} - \sum_{n,i} \frac{\xi(U_{ni})\mathfrak{q}(U_{ni})}{\sigma(U_{ni})} \right\|^2$$

$$= \sum_n \frac{|\xi(U_n)|^2}{\sigma(U_n)} + \sum_{n,i} \frac{|\xi(U_{ni})|^2}{\sigma(U_{ni})} - \sum_{n,i} \frac{\xi(U_n)\overline{\xi(U_{ni})}}{\sigma(U_n)} - \sum_{n,i} \frac{\overline{\xi(U_n)}\xi(U_{ni})}{\sigma(U_n)}$$

$$= \sum_{n,i} \frac{|\xi(U_{ni})|^2}{\sigma(U_{ni})} - \sum_n \frac{|\xi(U_n)|^2}{\sigma(U_n)}.$$

By (4) $$\| \sum_{\mathfrak{D}^{(\nu)}} - \sum_{\mathfrak{D}^{(\mu)}} \| < 2\varepsilon_\nu \qquad \text{when } \mu > \nu.$$

Since $\mathfrak{H}$ is complete, a vector $\mathfrak{f}$ exists so that

$$\| \mathfrak{f} - \sum_{\mathfrak{D}^{(\nu)}} \| < 2\varepsilon_\nu .$$

But, by (3) $$\| \sum_{\mathfrak{D}^{(\nu)}} - \sum_{\mathfrak{D}} \| < 2\varepsilon_\nu \qquad \text{for any } \mathfrak{D} \supset \mathfrak{D}_\nu,$$

Hence $$\| \mathfrak{f} - \sum_{\mathfrak{D}} \| < 4\varepsilon_\nu \qquad \text{for any } \mathfrak{D} \supset \mathfrak{D}_\nu.$$

That is, $$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} .$$

Consequently, *if* $\xi(U)$ *be a completely additive differential set function, which is absolutely continuous with respect to* $\sigma(U)$, *then* $\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}$ *exists when and only when* $\xi(U)$ *belongs to* $\mathfrak{L}_2(\sigma)$.

Let $\xi(U)$ and $\eta(U)$ be two differential set functions in $\mathfrak{L}_2(\sigma)$, and put

$$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}, \qquad \mathfrak{g} = \int_V \frac{\eta(dU)\mathfrak{q}(dU)}{\sigma(dU)} .$$

Then by the definition of the integral, there exists a sequence of decompositions

$$\mathfrak{D}_1 \subset \mathfrak{D}_2 \subset \ldots . \subset \mathfrak{D}_\nu \subset \ldots .$$

so that

$$\mathfrak{f} = [\lim_{\nu \to \infty}] \sum_n \frac{\xi(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)}, \qquad \mathfrak{g} = [\lim_{\nu \to \infty}] \sum_n \frac{\eta(U_n)\mathfrak{q}(U_n)}{\sigma(U_n)},$$

$$(\xi, \eta) = \lim_{\nu \to \infty} \sum_n \frac{\xi(U_n)\overline{\eta(U_n)}}{\sigma(U_n)} \qquad (\mathfrak{D}_\nu V \equiv \sum_n U_n) .$$

Hence $$(\mathfrak{f}, \mathfrak{g}) = \lim_{\nu \to \infty} \sum_n \frac{\xi(U_n)\overline{\eta(U_n)}}{\sigma(U_n)} .$$

That is $$(\mathfrak{f}, \mathfrak{g}) = (\xi, \eta) .$$

It is evident that

$$\int_V \frac{\{\xi(dU)+\eta(dU)\}\mathfrak{q}(dU)}{\sigma(dU)} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} + \int_V \frac{\eta(dU)\mathfrak{q}(dU)}{\sigma(dU)}. \tag{5}$$

Let $\xi_\nu(U)$ and $\xi(U)$ be differential set functions in $\mathfrak{L}_2(\sigma)$. If

$$[\lim_{\nu\to\infty}]\, \xi_\nu(U) = \xi(U) \qquad [\text{in } \mathfrak{L}_2(\sigma)],$$

then $$[\lim_{\nu\to\infty}] \int_V \frac{\xi_\nu(dU)\mathfrak{q}(dU)}{\sigma(dU)} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad [\text{in } \mathfrak{H}]. \tag{6}$$

For, by (2) and (5)

$$\left\| \int_V \frac{\xi_\nu(dU)\mathfrak{q}(dU)}{\sigma(dU)} - \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} \right\| = \| \xi_\nu - \xi \| .$$

Let U' be a set in $\mathfrak{R}V$. If $\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$, then, since $\xi(UU')$ belongs to $\mathfrak{L}_2(\sigma)$,

$$\mathfrak{p}(U') = \int_{U'} \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} = \int_V \frac{\xi(dU\cdot U')\mathfrak{q}(dU)}{\sigma(dU)}$$

exists. And $\mathfrak{p}(U')$ is a completely additive vector valued differential set function. For, when $U'U'' = 0$,

we have $$\big(\mathfrak{p}(U'), \mathfrak{p}(U'')\big) = \big(\xi(UU'), \xi(UU'')\big) = 0 .$$

And when $U' = \sum_i U'_i$, since

$$\xi(UU') = [\lim_{\nu\to\infty}] \sum_{i=1}^{\nu} \xi(UU'_i) \qquad [\text{in } \mathfrak{L}_2(\sigma)],$$

we have, by (6) $$\mathfrak{p}(U') = [\lim_{\nu\to\infty}] \sum_{i=1}^{\nu} \mathfrak{p}(U'_i) \qquad [\text{in } \mathfrak{H}].$$

Lastly, we define $\int_V f(\lambda)\mathfrak{q}(dU)$ by

$$\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \qquad \xi(U) = \int_U f(\lambda)\sigma(dU) .$$

As in my previous paper, we may consider $\{\mathfrak{q}(U)\}$ as a normalized orthogonal system in $\mathfrak{H}$, and when $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, then any vector $\mathfrak{f}$ in $\mathfrak{H}$ is expressed in the form

$$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \qquad \xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big)\,.$$

And almost all the theorems concerning the normalized orthogonal system $\{\mathfrak{q}(U)\}$ in my previous paper hold also in this case.(1)

For example, let $\mathfrak{U}(E, U)$ be the kernel of a unitary transformation in $\mathfrak{L}_2(\beta)$.(2) Since

$$\big(\mathfrak{U}(E, {}_{(U)}), \mathfrak{U}(E, {}_{(U')})\big) = \beta(UU')\,,$$

we have

$$\big(\mathfrak{U}(E, {}_{(U)}), \mathfrak{U}(E, {}_{(U')})\big) = 0 \tag{7}$$

when $UU' = 0$, and

$$\mathfrak{U}(E, {}_{(U)})\, [=]\, \mathfrak{U}(E, {}_{(U_1)}) + \mathfrak{U}(E, {}_{(U_2)}) + \ldots\ldots + \mathfrak{U}(E, {}_{(U_n)}) + \ldots\ldots$$

where $U = \sum_n U_n$.(3) Hence, $\mathfrak{U}(E, U)$ is a completely additive vector valued differential set function defined for all sets U of a differential set system $\mathfrak{M}\mathfrak{D}\Omega$, and its functional values are the elements of $\mathfrak{L}_2(\beta)$. Therefore, $\{\mathfrak{U}(E, {}_{(U)})\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta)$, and since

$$\big(\mathfrak{U}^*(U, {}_{(E)}), \mathfrak{U}^*(U, {}_{(E')})\big) = \beta(EE')\,,$$

it is complete in $\mathfrak{L}_2(\beta)$.(4) For, then $\mathfrak{U}(E, U)$ and $\mathfrak{U}^*(U, E)$ are the kernels of unitary transformations, we have

$$\xi(U) = \big(\psi(E), \mathfrak{U}(E, {}_{(U)})\big) = \mathfrak{U}^*\psi(U)\,,$$

(1) F. Maeda, this journal **4** (1934), 70–76. In the proof of sec. 13 (ibid., 76), the complete additivity of $\mathfrak{q}(U)$ must now be proved as follows: Let $U = \sum_i U_i$, then there is an integer k independent of j such that

$$\Big\| \sum_{i=1}^{j} \sum_{n=1}^{k} \mathfrak{q}_n(U_i) - \sum_{i=1}^{j} \mathfrak{q}(U_i) \Big\|^2 = \sum_{i=1}^{j} \Big\| \sum_{n=1}^{k} \mathfrak{q}_n(U_i) - \mathfrak{q}(U_i) \Big\|^2 = \sum_{i=1}^{j} \sum_{n=k+1}^{\infty} \sigma_n(U_i) \leqq \sum_{n=k+1}^{\infty} \sigma_n(U) < \varepsilon^2\,.$$

We then apply the method used for the proof of the complete additivity of $\xi(E)$ in sec. 6.

(2) Cf. sec. 7.

(3) By the complete additivity of $\mathfrak{U}({}_{(E)}, U)$, and the orthogonality (7).

(4) These notions have already been mentioned in my previous paper (this journal, **3** (1933), 253).

and $$\int_{\Omega} \frac{\xi(dU)\mathfrak{U}(E, dU)}{\beta(dU)} = \mathfrak{U}\xi(E) = \psi(E),$$

for any $\psi(E)$ in $\mathfrak{L}_2(\beta)$.(1)

9. Let $\boldsymbol{E}(U)$ be a self-adjoint transformation in $\mathfrak{H}$, which is defined for all sets U in $\mathfrak{R}V$. If $\boldsymbol{E}(U)$ satisfies the following conditions, then I say that $\boldsymbol{E}(U)$ is a *resolution of identity*.

(α) $\boldsymbol{E}(U)\boldsymbol{E}(U')\mathfrak{f} = \boldsymbol{E}(UU')\mathfrak{f}$;

(β) $\boldsymbol{E}(U)\mathfrak{f}\,[=]\,\boldsymbol{E}(U_1)\mathfrak{f}+\boldsymbol{E}(U_2)\mathfrak{f}+\ldots.+\boldsymbol{E}(U_i)\mathfrak{f}+\ldots.$

where $U = \sum_i U_i$,

(γ) $\boldsymbol{E}(V)\mathfrak{f} = \mathfrak{f}$;

all for any vector $\mathfrak{f}$.

As in my previous paper,(2) $\boldsymbol{E}(U)$ is a projoction on some closed linear manifold which depends on U. And

$$\mathfrak{p}(U) = \boldsymbol{E}(U)\mathfrak{f}$$

is a completely additive vector valued differential set function in V.

If $\mathfrak{q}(U)$ is a completely additive vector valued differential set function in V, which satisfies

$$\boldsymbol{E}(U')\mathfrak{q}(U) = \mathfrak{q}(U'U),$$

then I say that $\mathfrak{q}(U)$ is *generated* by $\boldsymbol{E}(U)$. When $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, we can find the resolution of identity $\boldsymbol{E}(U)$, which generates $\mathfrak{q}(U)$, as follows. Let $\mathfrak{f}$ be any vector in $\mathfrak{H}$, and put

$$\mathfrak{p}(U) = \int_U \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \qquad \xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big).$$

Then, by sec. 8, $\mathfrak{p}(U)$ is a completely additive vector valued differential set function defined in $\mathfrak{R}V$. The transformation $\boldsymbol{E}(U)$ defined in $\mathfrak{R}V$, which transforms $\mathfrak{f}$ to $\mathfrak{p}(U)$ is the required resolution of identity. This can be proved as in my preceding paper.(3)

When $\mathfrak{q}(U)$ *is generated by* $\boldsymbol{E}(U)$, *if*

$$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}, \tag{1}$$

(1) Cf. F. Maeda, this journal, **4** (1934), 74.
(2) Ibid., 78-79.
(3) F. Maeda, this journal, **4** (1934), 81-83.

then
$$E(U)\mathfrak{f} = \int_U \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.$$

For, from the definition of the integral (1),

$$\left\| \mathfrak{f} - \sum_i \frac{\xi(U_i)\mathfrak{q}(U_i)}{\sigma(U_i)} \right\| < \varepsilon \quad \text{for any} \quad \mathfrak{D} \supset \mathfrak{D}_0 \quad (\mathfrak{D}V \equiv \sum_i U_i).$$

Hence

$$\left\| E(U)\mathfrak{f} - \sum_i \frac{\xi(U_i)E(U)\mathfrak{q}(U_i)}{\sigma(U_i)} \right\| < \varepsilon \quad \text{for any} \quad \mathfrak{D} \supset \mathfrak{D}_0.$$

10. When H is a self-adjoint transformation, we can find a resolution of identity $E(U)$ defined for all sets U in a multiplicative system $\mathfrak{R}V$, where V is the space of real numbers, and

$$H\mathfrak{f} = \int_V \lambda E(dU)\mathfrak{f} \tag{1}$$

for all $\mathfrak{f}$, such that $\int_V \lambda^2 \| E(dU)\mathfrak{f} \|^2$ are finite.[1] When $\mathfrak{q}(U)$ is generated by $E(U)$, since $E(U)\mathfrak{q}(U') = \mathfrak{q}(UU')$, we have

$$H\mathfrak{q}(U) = \int_U \lambda \mathfrak{q}(dU). \tag{2}$$

In general, if a completely additive vector valued differential set function $\mathfrak{q}(U)$ satisfies (2), then we say that it is the *characteristic element* of H,[2] when $\mathfrak{q}(U)$ does not vanish identically. When $\mathfrak{q}(U)$ is defined at U which is a point λ_0, (2) becomes

$$H\mathfrak{q}(\lambda_0) = \lambda_0 \mathfrak{q}(\lambda_0). \tag{3}$$

When $\mathfrak{q}(\lambda_0)$ is not a null element, λ_0 is the socalled characteristic value of H and $\mathfrak{q}(\lambda_0)$ is the characteristic element of H with respect to λ_0.

Thus, when H has a continuous spectrum, equation (3) is not sufficient for defining the characteristic elements, and we must use

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(1) When $\mathfrak{R}V$ is the multiplicative system of intervals, cf. K. Friedrichs, Math. Ann., **110** (1934), 54–62. When $\mathfrak{R}V$ is a closed family (σ-Körper) of Borel sets, cf. F. Maeda, this journal, **4** (1934), 91; and T. Ogasawara, this journal, **5** (1935), 117–130.

(2) Cf. F. Maeda, this journal, **3** (1933), 261; **4** (1934), 91.

equation (2).[1] That is, characteristic elements are vector valued differential set functions.

When $\mathfrak{q}(U)$ is a characteristic element of $\boldsymbol{H}$, and $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, we can find by the method of the preceding section, a resolution of identity $\boldsymbol{E}(U)$ which generates $\mathfrak{q}(U)$. I will show that this resolution of identity $\boldsymbol{E}(U)$ satisfies (1).

Let $\mathfrak{f}$ be any vector in the domain of $\boldsymbol{H}$, then

$$\mathfrak{f}=\int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \qquad \xi(U)=\big(\mathfrak{f},\mathfrak{q}(U)\big),$$

$$\boldsymbol{H}\mathfrak{f}=\int_V \frac{\eta(dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \qquad \eta(U)=\big(\boldsymbol{H}\mathfrak{f},\mathfrak{q}(U)\big), \tag{4}$$

and

$$\|\boldsymbol{H}\mathfrak{f}\|=\|\eta\|.$$

Since

$$\boldsymbol{H}\mathfrak{q}(U')=\int_{U'}\lambda\mathfrak{q}(dU)=\int_V \frac{\zeta(U'\cdot dU)\mathfrak{q}(dU)}{\sigma(dU)} \qquad \text{where} \qquad \zeta(U)=\int_U \lambda\sigma(dU),$$

we have

$$\eta(U')=\big(\mathfrak{f},\boldsymbol{H}\mathfrak{q}(U')\big)=\int_V \frac{\xi(dU)\zeta(U'\cdot dU)}{\sigma(dU)}=\int_{U'}\lambda\xi(dU). \tag{5}$$

On the other hand, by the preceding section,

$$\boldsymbol{E}(U)\mathfrak{f}=\int_U \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.$$

Therefore, from (5)

$$\int_V \lambda\boldsymbol{E}(dU)\mathfrak{f}=\int_V \frac{\lambda\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}=\int_V \frac{\eta(dU)\mathfrak{q}(dU)}{\sigma(dU)},$$

and

$$\int_V |\lambda|^2\|\boldsymbol{E}(dU)\mathfrak{f}\|^2=\|\eta\|^2=\|\boldsymbol{H}\mathfrak{f}\|^2.$$

Hence, from (4) we have (1).

---

(1) Since the equality (2) holds for any U, we may say that $\boldsymbol{H}\mathfrak{q}(U)$ and $\lambda\mathfrak{q}(U)$ are differential-equivalent, and we may write $\boldsymbol{H}\mathfrak{q}(dU)=\lambda\mathfrak{q}(dU)$. (Cf. A. Kolmogoroff, loc. cit., 669). This expression is similar to (3).

Characteristic Functions of Q and P.

11. In Schrödinger's wave mechanics, for simplicity's sake, let the state space be of one dimension: $-\infty < q < \infty$, the wave function is denoted by $f(q)$ where $\int_{-\infty}^{+\infty} |f(q)|^2 dq$ is finite. The space of such point functions is a Hilbert space. In this space of point functions, two self-adjoint transformations $\boldsymbol{Q}$ and $\boldsymbol{P}$ are defined as follows:

$$\boldsymbol{Q}f(q) = qf(q), \qquad \boldsymbol{P}f(q) = \frac{h}{2\pi i}\frac{d}{dq}f(q).^{(1)}$$

With respect to the characteristic functions of $\boldsymbol{Q}$,

$$qf(q) = \lambda f(q) \tag{1}$$

has no solution except $f(q) \equiv 0$. But, if we use Dirac's improper δ function, then

$$q\delta(q-\lambda) = \lambda\delta(q-\lambda)$$

for any λ. That is, (1) is solved for any λ, and the characteristic functions are $\delta(q-\lambda)$.[(2)]

With respect to $\boldsymbol{P}$,

$$\frac{h}{2\pi i}\frac{d}{dq}f(q) = \lambda f(q)$$

has solutions $$f_\lambda(q) = ce^{\frac{2\pi i}{h}\lambda q}$$

for any λ. But since $\int_{-\infty}^{+\infty} |f_\lambda(q)|^2 dq = +\infty$, $f_\lambda(q)$ does not belong to the space of point functions.[(2)]

Thus, when the transformation has a continuous spectrum, its characteristic elements cannot be obtained as a function of λ, it must be a differential set function whose functional values are the elements

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(1) Cf. J. v. Neumann, loc. cit., 47–49. Accurately speaking, the operator $\frac{h}{2\pi i}\frac{d}{dq}$ is not self-adjoint, and $\boldsymbol{P}$ is the extension of $\frac{h}{2\pi i}\frac{d}{dq}$ (Cf. ibid.. 245, note 88.)

(2) J. v. Neumann, loc. cit., 69.

in the space of point functions.[1] Then, as we shall see in the next sections, it is convenient to treat the problems in the space of differential set functions rather than in the space of point functions.

12. Let Ω be the Euclidean space of one dimension, and let the multiplicative system $\mathfrak{M}$ be composed of all open intervals and all points in Ω, including Ω itself. Then $\mathfrak{M}\Omega = \mathfrak{M}$. Put $\beta(E) = b-a$ when E is the finite open interval (a, b), and $\beta(E) = 0$ when E is a point. Then $\beta(E)$ is a completely additive, non-negative, differential set function defined in a differential set system $\mathfrak{M}\mathfrak{D}\Omega$ composed of finite open intervals and points. Then to the wave function $f(q)$, there corresponds a completely additive differential set function $\psi(E)$ belonging to $\mathfrak{L}_2(\beta)$, such that

$$\psi(E) = \int_E f(q)dq\,.$$

Hence the transformation $\boldsymbol{Q}$ in $\mathfrak{L}_2(\beta)$ must be defined as follows:

$$\boldsymbol{Q}\psi(E) = \int_E qf(q)dq = \int_E q\psi(dE)\,.$$

Therefore, *the kernel of* $\boldsymbol{Q}$ *is*

$$\mathfrak{Q}(E, E') = \int_{E'} q'\beta(E\cdot dE')\,.$$

For

$$\mathfrak{Q}\psi(E) = \int_\Omega \frac{\mathfrak{Q}(E, dE')\psi(dE')}{\beta(dE')} = \int_\Omega \frac{q'\beta(E\cdot dE')\psi(dE')}{\beta(dE')} = \int_E q\psi(dE)\,.^{(2)}$$

Since
$$\boldsymbol{Q}\beta(EU) = \int_U \lambda\beta(E\cdot dU)\,,$$

by sec. 10, $\beta(EU)$ *is the characteristic function of* $\boldsymbol{Q}$. When U is a point, then $\beta(EU) = 0$. And $\{\beta(EU)\}$ is complete in $\mathfrak{L}_2(\beta)$. Hence, $\boldsymbol{Q}$ has no discrete spectrum.

In order to find the resolution of identity which generates $\beta(EU)$, we use the method of sec. 9. Then since

$$\big(\psi(E), \beta(EU)\big) = \psi(U)\,,$$

(1) Cf. sec. 10.

(2) Therefore, $\boldsymbol{Q}$ and $\boldsymbol{T}_{\mathfrak{Q}}$ have the same domain.

$E(U)$ is the transformation defined in $\mathfrak{M}\Omega$ which transforms $\psi(E)$ to

$$\int_U \frac{\psi(dU)\beta(E\cdot dU)}{\beta(dU)} = \psi(EU).$$

That is

$$E(U)\psi(E) = \psi(EU)$$

for any differential set function $\psi(E)$ in $\mathfrak{L}_2(\beta)$.(1)

Now $$\int_\Omega \lambda E(dU)\psi(E) = \int_\Omega \lambda\psi(E\cdot dU) = \int_E \lambda\psi(dU).$$

Hence the above defined Q is a self-adjoint transformation and

$$Q\psi(E) = \int_\Omega \lambda E(dU)\psi(E).$$

13. With respect to the transformation P in the space of differential set function, it must be that

$$P\psi(E) = \int_E \frac{h}{2\pi i}\frac{d}{dq}f(q)dq \qquad \left(\psi(E) = \int_E f(q)dq\right) \tag{1}$$

when the right hand operators have significance.

Put $$\mathfrak{U}(E, U) = \frac{1}{\sqrt{h}}\int_U d\lambda \int_E e^{\frac{2\pi i}{h}\lambda q}dq.$$

Since, it is already known that

$$\frac{1}{h}\int_{-\infty}^{+\infty} d\lambda \int_a^b e^{\frac{2\pi i}{h}\lambda q}dq \int_c^d e^{-\frac{2\pi i}{h}\lambda q'}dq'$$

$$= \text{length common to } (a, b) \text{ and } (c, d),\text{(2)}$$

we have

$$\int_\Omega \frac{\mathfrak{U}(E, dU)\overline{\mathfrak{U}(E', dU)}}{\beta(dU)} = \frac{1}{h}\int_\Omega d\lambda \int_E e^{\frac{2\pi i}{h}\lambda q}dq \int_{E'} e^{-\frac{2\pi i}{h}\lambda q'}dq' = \beta(EE').$$

Hence $\mathfrak{U}(_{(E)}, U)$ belongs to $\mathfrak{L}_2(\beta)$, and we have

$$\mathfrak{U}\mathfrak{U}^*(E, E') = \beta(EE').$$

(1) J. v. Neumann (loc. cit., 67) obtained the resolution of identity $E(\lambda)$ by an inexact method in the space of point functions.

(2) N. Wiener, Acta Math., **55** (1930), 196.

Similarly $\mathfrak{U}^*({}_{(U)}, E)$ belongs to $\mathfrak{L}_2(\beta)$ and

$$\mathfrak{U}^*\mathfrak{U}(U, U') = \beta(UU').$$

Hence by sec. 7, $\mathfrak{U}(E, U)$ is the kernel of a unitary transformation, say $\boldsymbol{U}$, and by sec. 8, $\{\mathfrak{U}(E, {}_{(U)})\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$.

Put $\mathfrak{U}(E, {}_{(U)})$ instead of $\psi(E)$ in (1). Then since

$$f(q) = \frac{1}{\sqrt{h}} \int_U e^{\frac{2\pi i}{h}\lambda q} d\lambda,$$

$$\frac{h}{2\pi i} \frac{d}{dq} f(q) = \frac{1}{\sqrt{h}} \int_U \lambda e^{\frac{2\pi i}{h}\lambda q} d\lambda,$$

we have $$\boldsymbol{P}\mathfrak{U}(E, U) = \frac{1}{\sqrt{h}} \int_E dq \int_U \lambda e^{\frac{2\pi i}{h}\lambda q} d\lambda,$$

that is $$\boldsymbol{P}\mathfrak{U}(E, U) = \int_U \lambda \mathfrak{U}(E, dU).$$

Since $\{\mathfrak{U}(E, {}_{(U)})\}$ is complete in $\mathfrak{L}_2(\beta)$, $\boldsymbol{P}$ *must be the transformation which has* $\mathfrak{U}(E, U)$ *as the complete system of characteristic functions.* Hence $\boldsymbol{P}$ has no discrete spectrum.

To find the resolution of identity $\boldsymbol{F}(U)$ which generates $\mathfrak{U}(E, U)$, as preceding section, since

$$\left(\psi(E), \mathfrak{U}(E, {}_{(U)})\right) = \mathfrak{U}^*\psi(U),$$

$\boldsymbol{F}(U)$ is the transformation defined in $\mathfrak{M}\Omega$ which transforms $\psi(E)$ to

$$\int_U \frac{\mathfrak{U}^*\psi(dU')\mathfrak{U}(E, dU')}{\beta(dU')} = \int_V \frac{\mathfrak{U}^*\psi(U \cdot dU')\mathfrak{U}(E, dU')}{\beta(dU')}$$

$$= \int_V \frac{\boldsymbol{E}(U)\mathfrak{U}^*\psi(dU')\mathfrak{U}(E, dU')}{\beta(dU')} = \mathfrak{U}\boldsymbol{E}(U)\mathfrak{U}^*\psi(E),$$

where $\boldsymbol{E}(U)$ is the resolution of identity which corresponds to $\boldsymbol{Q}$.

That is $$\boldsymbol{F}(U)\psi(E) = \mathfrak{U}\boldsymbol{E}(U)\mathfrak{U}^*\psi(E).$$

Or we may write $$\boldsymbol{F}(U) = \boldsymbol{U}\boldsymbol{E}(U)\boldsymbol{U}^*.$$ [1]

(1) Cf. J. v. Neumann, loc. cit., 70.

Consequently, $$P\psi(E) = \int_{\Omega} \lambda F(dU)\psi(E)\,,$$

and $$P = UQU^* .^{(1)} \tag{2}$$

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(1) For, $$QU^*\psi(E) = \int_{\Omega} \lambda E(dU)U^*\psi(E)$$

for any $\psi(E)$, such that $U^*\psi(E)$ is contained in the domain of Q. Then

$$UQU^*\psi(E) = \int_{\Omega} \lambda UE(dU)U^*\psi(E) = \int_{\Omega} \lambda F(dU)\psi(E)$$

for any $\psi(E)$, such that

$$\int_{\Omega} \lambda^2 \| E(dU)U^*\psi \|^2 = \int_{\Omega} \lambda^2 \| UE(dU)U^*\psi \|^2 = \int_{\Omega} \lambda^2 \| F(dU)\psi \|^2$$

is finite. (Cf. F. Maeda, this journal, **4** (1934), 66.)