Transitivities of Conservative Mechanism.

By

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Some statistical phenomena are connected with conservative mechanisms. These mechanisms may be mathematically described in the following way. Let $\beta(E)$ be a completely additive, non-negative set function defined in a closed family (σ-Körper) $\mathfrak{K}$ in an abstract space Ω. And let T_t be a one parameter group of one to one transformations of Ω into itself, with the properties

(1) $T_tT_sA = T_{t+s}A,\ \ T_0A = A,$ for any set A in $\mathfrak{K}$;

(2) T_tA belongs to $\mathfrak{K}$, and

$$\beta(T_tA) = \beta(A)$$

for any set A in $\mathfrak{K}$ and t;

(3) T_t is continuous in the following sense:

$$\lim_{t\to 0} \beta(T_tA\cdot B) = \beta(BA)$$

for any sets A and B in $\mathfrak{K}$.

A number of writers have already discussed the conditions in which the ergodic theory

$$\frac{1}{q-p}\int_p^q \beta(T_tA\cdot B)dt \to \frac{\beta(A)\beta(B)}{\beta(\Omega)} \qquad (q-p\to\infty),$$

and the mixing property

$$\beta(T_tA\cdot B) \to \frac{\beta(A)\beta(B)}{\beta(\Omega)} \qquad (t\to\infty)$$

hold in the conservative mechanism.(1)

(1) For detailed discussions, cf. J. v. Neumann, „Zur Operatorenmethode in der klassischen Mechanik", Annals of Math., (2) **33** (1932), 587–642; and E. Hopf, "On Causality, Statistics and Probability," Jour. of Math. and Physics, Massachusetts Inst. of Technology, **13** (1934), 51–102.

In this paper I intend to treat these problems generally from standpoint of the theory of set functions. First, I discuss the beh of set functions under the one parameter group of unitary transfo tions. And then, I apply these results to the conservative mechar

Set Functions under One Parameter Group of Unitary Transformations.

1. Let $\beta(E)$ be a completely additive, non-negative set func defined in a closed family (σ-Körper) $\mathfrak{K}$ of sets in an abstract s Ω, including Ω itself. When a complex-valued completely add set function $\phi(E)$ is absolutely continuous with respect to $\beta(E)$, $\int_\Omega | D_{\beta(E)}\phi(a) |^2 d\beta(E)$ is finite, then we say that $\phi(E)$ belongs to $\mathfrak{L}$ Then $\mathfrak{L}_2(\beta)$ is a linear space with the inner product

$$(\phi, \psi) = \int_\Omega D_{\beta(E)}\phi(a)\overline{D_{\beta(E)}\psi(a)}d\beta(E) ,$$

and is complete.(1)

Thus $\mathfrak{L}_2(\beta)$ satisfies the essential axioms of the abstract Hil space except the axiom of separability. Hence almost all the con tions of the abstract Hilbert space can be used also in $\mathfrak{L}_2(\beta)$.

If $U(t)$, $-\infty < t < +\infty$, is a family of unitary transformat with the group property

$$U_{s+t}\phi = U_s U_t \phi \qquad \text{for all } \phi \text{ in } \mathfrak{L}_2(\beta),$$

and the continuity property

$$\lim_{t \to s} \| (U_t - U_s)\phi \| = 0 \qquad \text{for all } \phi \text{ in } \mathfrak{L}_2(\beta),$$

then there exists a resolution of identity $E(U)$ defined in the spac real numbers R_1 such that

$$U_t\phi = \int_{R_1} e^{it\lambda} dE(U)\phi . \tag{1.1}$$

This last expression is an integral with respect to the vector va

(1) Cf. F. Maeda, "On the Space of Real Set Functions," this journal, **3** (3–5; and "Space of Differential set Functions," this volume, 29.

set function,[1] i. e. its functional value is the set function in $\mathfrak{L}_2(\beta)$. This theorem can be proved, with slight modifications of J. v. Neumann's method, without using the separability of the Hilbert space.[2]

Conversely, let $\boldsymbol{E}(U)$ be a resolution of identity defined in R_1, then

$$\int_{R_1} e^{it\lambda} d\boldsymbol{E}(U)\phi$$

is a unitary transformation[3] with parameter t, which has the group property[4] and the continuity property.[5]

Denote by V the sum of all open sets O such that $\boldsymbol{E}(O) = 0$, and put $W = R_1 - V$. If λ is a point in W, such that $\boldsymbol{E}(\lambda) \neq 0$,[6] we say that λ is a characteristic value of $\boldsymbol{E}(U)$. The set of all such characteristic values is the point spectrum, and the other part of W is the continuous spectrum.

Denote by $\mathfrak{M}_U$ the set of all set functions ϕ in $\mathfrak{L}_2(\beta)$, which satisfy

$$\boldsymbol{E}(U)\phi = \phi .$$

Then $\mathfrak{M}_U$ is a closed linear manifold, and $\boldsymbol{E}(U)$ is the projection on $\mathfrak{M}_U$. When $UU' = 0$, then $\mathfrak{M}_U$ and $\mathfrak{M}_{U'}$ and orthogonal.[7] When U is composed of only one element λ, then all set functions in $\mathfrak{M}_\lambda$ are the characteristic functions of $\boldsymbol{E}(U)$ with respect to λ.

Denote by D the set of all characteristic values except 0, and by C the continuous spectrum, then we have

$$\mathfrak{L}_2(\beta) = \mathfrak{M}_0 \oplus \mathfrak{M}_D \oplus \mathfrak{M}_C .^{(8)}$$

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(1) I investigated the integral with respect to the vector valued set function in my previous paper (this journal, **4** (1934), 57–91). Almost all the theorems in this previous paper also hold without the separability.

(2) J. v. Neumann, Annals of Math., (2) **33** (1932), 569–573.

(3) M. H. Stone, *Linear Transformations in Hilbert Space*, (1932), 232.

(4) Cf. ibid., 222.

(5) Cf. F. Maeda, this journal, **4** (1934), 68. For

$$\underset{t\to s}{[\lim]} | e^{it\lambda} - e^{is\lambda} | = 0 \qquad \text{in } \mathfrak{L}_2(\| \boldsymbol{E}(U)\phi \|^2).$$

(6) $\boldsymbol{E}(\lambda)$ means the resolution of identity $\boldsymbol{E}(U)$ where U is the set which has only one element λ.

(7) Cf. F. Maeda, this journal, **4** (1934), 78.

(8) $\mathfrak{M}_1 \oplus \mathfrak{M}_2$ means the closed linear manifold determined by the element of $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Cf. M. H. Stone, loc. cit., 21.

In what follows, I investigate the properties of set functions which belong to these three closed linear manifolds.

2. Since $U_t\phi$ is a continuous function of t in the sense of strong convergence, like the Riemann integral, we can define the integral of $e^{-i\lambda_0 t}U_t\phi$ as follows:

$$\int_p^q e^{-i\lambda_0 t}U_t\phi dt = [\lim_{\Delta t\to 0}] \sum_\nu e^{-i\lambda_0 t_\nu} U_{t_\nu}\phi \Delta t_\nu \,.^{(1)}$$

Since the integral is defined by the strong convergence, the integral itself is a set function in $\mathfrak{L}_2(\beta)$. Now, by (1.1)

$$\int_p^q e^{-i\lambda_0 t}U_t\phi dt = [\lim_{\Delta t\to 0}] \int_{R_1} \sum_\nu e^{i(\lambda-\lambda_0)t_\nu}\Delta t_\nu dE(U)\phi \,. \tag{2.1}$$

Let I_ε be the closed interval $[\lambda_0-\varepsilon, \lambda_0+\varepsilon]$, and put

$$\phi_\varepsilon = \phi - E(I_\varepsilon)\phi\,, \qquad \phi_0 = \phi - E(\lambda_0)\phi\,,$$

then

$$[\lim_{\varepsilon\to 0}]\, \phi_\varepsilon = \phi_0 \,.^{(2)} \tag{2.2}$$

Since, in $\mathfrak{L}_2(\| E(U)\phi_\varepsilon \|^2)$,

$$[\lim_{\Delta t\to 0}] \sum_\nu e^{i(\lambda-\lambda_0)t_\nu}\Delta t_\nu = \int_p^q e^{i(\lambda-\lambda_0)t}dt$$

$$= \frac{1}{i(\lambda-\lambda_0)}\left(e^{i(\lambda-\lambda_0)q} - e^{i(\lambda-\lambda_0)p}\right)$$

when $\lambda \neq \lambda_0$, we have[3]

$$\int_p^q e^{-i\lambda_0 t}U_t\phi_\varepsilon dt = \int_{R_1} \frac{1}{i(\lambda-\lambda_0)}\left(e^{i(\lambda-\lambda_0)q} - e^{i(\lambda-\lambda_0)p}\right) dE(U)\phi_\varepsilon \,.$$

But, since

$$[\lim_{q-p\to\infty}] \frac{e^{i(\lambda-\lambda_0)q} - e^{i(\lambda-\lambda_0)p}}{i(\lambda-\lambda_0)(q-p)} = 0 \qquad \text{in } \mathfrak{L}_2(\| E(U)\phi_\varepsilon \|^2),$$

$$[\lim_{q-p\to\infty}] \frac{1}{q-p}\int_p^q e^{-i\lambda_0 t}U_t\phi_\varepsilon dt = 0\,. \tag{2.3}$$

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(1) Cf. S. Bochner, Acta Math., **61** (1933), 165–166; and Fund. Math., **20** (1933), 262–276. [lim] means the strong convergence in $\mathfrak{L}_2(\beta)$.

(2) Cf. F. Maeda, this journal, **4** (1934), 79.

(3) Cf. F. Maeda, ibid., 68.

Since $$\| e^{-i\lambda_0 t} U_t\phi_\varepsilon - e^{-i\lambda_0 t} U_t\phi_0 \| = \| \phi_\varepsilon - \phi_0 \| ,$$

we have

$$\| \frac{1}{q-p}\int_p^q e^{-i\lambda_0 t} U_t\phi_\varepsilon dt - \frac{1}{q-p}\int_p^q e^{-i\lambda_0 t} U_t\phi_0 dt \| \leqq \| \phi_\varepsilon - \phi_0 \| .$$

Hence, by (2.2) and (2.3), we have

$$\underset{q-p\to\infty}{[\lim]} \frac{1}{q-p}\int_p^q e^{-i\lambda_0 t} U_t\phi_0 dt = 0 . \tag{2.4}$$

From (2.1), we have directly

$$\frac{1}{q-p}\int_p^q e^{-i\lambda_0 t} U_t E(\lambda_0)\phi dt = E(\lambda_0)\phi . \tag{2.5}$$

By (2.4) and (2.5), we have the results

$$\underset{q-p\to\infty}{[\lim]} \frac{1}{q-p}\int_p^q e^{-i\lambda_0 t} U_t\phi dt = E(\lambda_0)\phi . \tag{2.6}$$

Especially, when $\lambda_0 = 0$,

$$\underset{q-p\to\infty}{[\lim]} \frac{1}{q-p}\int_p^q U_t\phi dt = E(0)\phi .\text{(1)} \tag{2.7}$$

3. Now put $\phi_t = U_t\phi$. And I proceed to investigate the property of ϕ_t when ϕ belongs to $\mathfrak{M}_0$, $\mathfrak{M}_D$ and $\mathfrak{M}_C$ respectively.(2)

From (2.7), it is evident that when

$$\phi_t = \phi \qquad \text{for all } t,$$

then $$\phi = E(0)\phi .$$

Conversely, if $\phi = E(0)\phi$, by (1.1)

$$\phi_t = \int_{R1} e^{it\lambda} dE(U)E(0)\phi = E(0)\phi ,$$

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(1) This has already been proved by J. v. Neumann (Annals of Math., (2) **33** (1932), 599–600). Here I give another proof using the theory of vector valued set functions.

(2) E. Hopf (Sitzber. Preus. Akad. Wiss., (1932) 183) decomposed U_t. Here I decompose set functions.

that is
$$\phi_t = \phi .$$

Hence we have the theorem:

ϕ belongs to $\mathfrak{M}_0$, that is $\phi = \boldsymbol{E}(0)\phi$, when and only when $\phi_t = \phi$ for all t.

4. Let ϕ be a set function in $\mathfrak{M}_D$, that is $\boldsymbol{E}(D)\phi = \phi$. Then by (1.1)

$$\phi_t = \int_{R_1} e^{it\lambda} d\boldsymbol{E}(U)\boldsymbol{E}(D)\phi = \int_D e^{it\lambda} d\boldsymbol{E}(U)\phi . \tag{4.1}$$

Since $\| \boldsymbol{E}(U)\phi \|^2$ is a completely additive finite set function, the set of discontinuous points is at most enumerable.[1] Denote these discontinuous points by $\{\lambda_n\}$. Then (4.1) becomes

$$\phi_t [=] \sum_n e^{it\lambda_n} \boldsymbol{E}(\lambda_n)\phi .^{(2)} \tag{4.2}$$

Since $\| \phi_t - \sum_{n=1}^{N} e^{it\lambda_n} \boldsymbol{E}(\lambda_n)\phi \|^2 = \| \sum_{n=N+1}^{\infty} e^{it\lambda_n} \boldsymbol{E}(\lambda_n)\phi \|^2 = \sum_{n=N+1}^{\infty} \| \boldsymbol{E}(\lambda_n)\phi \|^2$,

(4.2) converges uniformly for all t. Hence *ϕ_t is an almost periodic function of t* in the sense that for any positive number ε, the set of τ which satisfy

$$\| \phi_{t+\tau} - \phi_t \| \leqq \varepsilon \qquad \text{for all } t,$$

is relatively dense.[3]

Especially when λ is a characteristic value of $\boldsymbol{E}(U)$ and ϕ is in $\mathfrak{M}_\lambda$, that is $\boldsymbol{E}(\lambda)\phi = \phi$, then by (4.2)

$$\phi_t = e^{it\lambda}\boldsymbol{E}(\lambda)\phi = e^{it\lambda}\phi .$$

Hence *ϕ_t is a periodic function with period* $\dfrac{2\pi}{\lambda}$.

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(1) Cf. H. Hahn, *Theorie der reellen Funktionen*, I (1921), 410.

(2) [=] means the strong convergence of the series.

(3) Cf. S. Bochner, Acta Math., **61** (1933), 167–168. When $\phi \in \mathfrak{L}_2(\beta)$, (2.6) shows that $\boldsymbol{E}(\lambda)\phi$ is the Fourier coefficient of ϕ_t. And Bessel's inequality $\| \phi_t \|^2 \geqq \sum_n \| \boldsymbol{E}(\lambda_n)\phi \|^2$ holds. Especially when $\phi \in \mathfrak{M}_0 \oplus \mathfrak{M}_D$, from (4.2) Parseval's equality $\| \phi_t \|^2 = \sum_n \| \boldsymbol{E}(\lambda_n)\phi \|^2$ holds.

5. Let ϕ be any set function in $\mathfrak{L}_2(\beta)$, then by (1.1)

$$U_t E(C)\phi = \int_{R_1} e^{it\lambda} dE(U)E(C)\phi .$$

Put the closed interval $(-\infty, \lambda]$ instead of U, and let

$$g(\lambda) = \left(E((-\infty, \lambda])E(C)\phi, \phi\right) = \| E((-\infty, \lambda])E(C)\phi \|^2 ,$$

then
$$\left(U_t E(C)\phi, \phi\right) = \int_{-\infty}^{+\infty} e^{it\lambda} dg(\lambda) .$$ [(1)]

Since $g(\lambda)$ is continuous, non-decreasing and bounded in $(-\infty, +\infty)$, by the theorem proved by E. Hopf,[(2)]

$$\lim_{q-p\to\infty} \frac{1}{q-p} \int_p^q \left| \left(U_t E(C)\phi, \phi\right) \right| dt = 0 .$$

Hence as B. O. Koopman and J. v. Neumann did,[(3)] we find a set I in R_1 such that

$$\lim_{q-p\to\infty} \frac{m(I[p \leqq t \leqq q])}{q-p} = 0 , \tag{5.1}$$ [(4)]

and
$$\lim_{\substack{t\to\pm\infty \\ t \text{ not on } I}} \left(U_t E(C)\phi, \phi\right) = 0 .$$

Hence
$$\lim_{\substack{t\to\pm\infty \\ t \text{ not on } I}} \left(U_t E(C)\phi, \psi\right) = 0 .$$ [(5)]

Therefore, putting $\psi(E) = \beta(EE')$, we have

$$\lim_{\substack{t\to\pm\infty \\ t \text{ not on } I}} U_t E(C)\phi(E) = 0 .$$

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(1) F. Maeda, this journal, **4** (1934), 80.

(2) E. Hopf, Proc. Nat. Acad. Sci., **18** (1932), 208.

(3) B. O. Koopman and J. v. Neumann, ibid., 256–259.

(4) m is the Lebesgue measure in R_1. $I[p \leqq t \leqq q]$ means the set of points t in I where $p \leqq t \leqq q$. I depends on ϕ unless $\mathfrak{L}_2(\beta)$ is separable.

(5) For $\left(U_t E(C)\frac{\phi+\psi}{2}, \frac{\phi+\psi}{2}\right) - \left(U_t E(C)\frac{\phi-\psi}{2}, \frac{\phi-\psi}{2}\right) = \frac{1}{2}\left(U_t E(C)\phi, \psi\right) + \frac{1}{2}\left(U_t E(C)\psi, \phi\right)$, and put $i\phi$ instead of ϕ. I depends on ϕ and ψ, and it satisfies (5.1).

When ϕ be a set function in $\mathfrak{M}_C$, that is $E(C)\phi = \phi$, we have

$$\lim_{\substack{t\to\pm\infty \\ t \text{ not on } I}} U_t\phi(E) = 0 \,.$$ [1]

Conservative Mechanism.

6. Let T_t be the linear one parameter group of one to one transformations of the abstract space Ω into itself, with the properties

(i) $T_tT_sA = T_{t+s}A$, $T_0A = A$ for any set A in $\mathfrak{K}$;

(ii) T_tA belongs to $\mathfrak{K}$, and

$$\beta(T_tA) = \beta(A) \qquad \text{for all } t \text{ and } A \text{ in } \mathfrak{K};$$

(iii) $\lim_{t\to 0} \beta(T_tA\cdot B) = \beta(AB)$ for any sets A, B in $\mathfrak{K}$.

Let $\phi(E)$ be any set function in $\mathfrak{L}_2(\beta)$. And put

$$\phi_t(E) = \phi(T_tE)\,.$$

Then, as B. O. Koopman showed,[2] there is a family of unitary transformations U_t

$$\phi_t = U_t\phi$$

which has the group property and continuity property. For, by (i)

$$U_tU_s\phi = U_{t+s}\phi\,, \qquad U_0\phi = \phi\,. \tag{6.1}$$

And
$$(U_t\phi, \psi) = \int_\Omega D_{\beta(E)}\phi(T_ta)\overline{D_{\beta(E)}\psi(a)}d\beta(E)$$

$$= \int_\Omega D_{\beta(E)}\phi(a)\overline{D_{\beta(E)}\psi(T_{-t}a)}d\beta(E) = (\phi, U_{-t}\psi)\,,$$

for any ϕ, ψ in $\mathfrak{L}_2(\beta)$. Hence

$$U_t^*\phi = U_{-t}\phi\,.$$

Therefore, by (6.1)

(1) I depends on ϕ and E.

(2) B. O. Koopman, Proc. Nat. Acad. Sci., **17** (1931), 315-518.

$$U_t U_t^* \phi = \phi \qquad \text{and} \qquad U_t^* U_t \phi = \phi .$$

Consequently, U_t is a unitary transformation with the group property (6.1).

When $\psi(E) = \beta(EA)$, then by (i) and (iii)

$$\lim_{t \to s} \psi_t(E) = \psi_s(E) . \tag{6.2}$$

Hence (6.2) holds for any set function in $\mathfrak{L}_2'(\beta)$.(1) But

$$\| U_t \psi \| = \| U_s \psi \| , \qquad \text{that is} \qquad \| \psi_t \| = \| \psi_s \| ,$$

Hence,

$$[\lim_{t \to s}] \psi_t(E) = \psi_s(E) .^{(2)} \tag{6.3}$$

Next, let $\phi(E)$ be any set function in $\mathfrak{L}_2(\beta)$. Then there exists a sequence $\{\psi^{(\nu)}\}$ of set functions in $\mathfrak{L}_2'(\beta)$, such that

$$[\lim_{\nu \to \infty}] \psi^{(\nu)}(E) = \phi(E) . \tag{6.4}$$

Since

$$\begin{aligned} \| \phi_t - \phi_s \| &\leqq \| \phi_t - \psi_t^{(\nu)} \| + \| \phi_s - \psi_s^{(\nu)} \| + \| \psi_t^{(\nu)} - \psi_s^{(\nu)} \| \\ &= 2 \| \phi - \psi^{(\nu)} \| + \| \psi_t^{(\nu)} - \psi_s^{(\nu)} \| , \end{aligned}$$

we have, by (6.3) and (6.4),

$$[\lim_{t \to s}] \phi_t(E) = \phi_s(E) .$$

That is, U_t has the continuity property.

Thus $U_t \phi(E) = \phi(T_t E)$ being a unitary transformation with the group property and the continuity property, we can apply the theorems of the preceding sections to the conservative mechanism.

For instance, from (2.7) we have

$$[\lim_{q-p \to \infty}] \frac{1}{q-p} \int_p^q \phi(T_t E) dt = E(0) \phi(E) . \tag{6.5}$$

If $\mathfrak{M}_0$ *is of one dimension, then*

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(1) $\mathfrak{L}_2'(\beta)$ is the linear manifold determined by the system $\{\beta(EE')\}$, E' being the parameter. $\mathfrak{L}_2'(\beta)$ is dense in $\mathfrak{L}_2(\beta)$. Cf. F. Maeda, this journal, **5** (1935), 109.

(2) In $\mathfrak{L}_2(\beta)$, if $\lim \phi_n(E) = \phi(E)$ and $\lim \| \phi_n \| = \| \phi \|$, then $[\lim] \phi_n(E) = \phi(E)$. For the proof, cf. F. Maeda, this volume, 29.

$$E(0)\phi(E) = \frac{\phi(\Omega)}{\beta(\Omega)}\beta(E) \tag{6.6}$$

for any set function $\phi(E)$ *in* $\mathfrak{L}_2(\beta)$.

For, since $\beta(E)$ belongs to $\mathfrak{M}_0$,

$$E(0)\phi(E) = c\beta(E)\,.$$

Put $E=\Omega$ in (6.5), then since $\phi(T_t\Omega)=\phi(\Omega)$, we have

$$\phi(\Omega) = c\beta(\Omega)\,.$$

Thus, (6.6) holds.

Especially when $\phi(E) = \beta(EA)$,

$$E(0)\beta(EA) = \frac{\beta(A)}{\beta(\Omega)}\beta(E) \tag{6.7}$$

for any set A.

Conversely, *when* (6.7) *holds for any set* A, $\mathfrak{M}_0$ *is of one dimension.*

For, let $\phi(E)$ be any set function in $\mathfrak{L}_2(\beta)$, then

$$\big(E(0)\beta(EA),\, E(0)\phi(E)\big) = \big(\beta(EA), E(0)\phi(E)\big) = E(0)\phi(A)\,.$$

On the other hand

$$\big(E(0)\beta(EA), E(0)\phi(E)\big) = \big(E(0)\beta(EA), \phi(E)\big)$$

$$= \left(\frac{\beta(A)}{\beta(\Omega)}\beta(E), \phi(E)\right) = \frac{\beta(A)}{\beta(\Omega)}\phi(\Omega)\,.$$

Therefore
$$E(0)\phi(A) = \frac{\phi(\Omega)}{\beta(\Omega)}\beta(A)$$

for any set A. Hence, $\mathfrak{M}_0$ is composed of set functions of the form $c\beta(E)$, that is $\mathfrak{M}_0$ is of one dimension.

7. *Let* $\phi(E)$ *be a real set function in* $\mathfrak{L}_2(\beta)$, *which has the period* t_0, *that is,*

$$\phi_{t_0}(E) = \phi(E)\,. \tag{7.1}$$

And put $A = \Omega[D_{\beta(E)}\phi(a) > p]$,[1] *$p$ being any real number, then, $T_{t_0}A$ coincides with A except the set whose β-value is zero.* In this case, we write as follows

$$T_{t_0}A \equiv A \qquad (\beta) . \tag{7.2}$$

Since
$$\phi(E) = \int_E D_{\beta(E)}\phi(a)d\beta(E) ,$$

it is evident that

$$\phi(E) \geqq p\beta(E) \qquad \text{for all} \qquad E \leqq A , \tag{7.3}$$

and
$$\phi(E) \leqq p\beta(E) \qquad \text{for all} \qquad E \leqq \Omega - A . \tag{7.4}$$

When $\beta(A) \neq 0$, (7.3) can be replaced by

$$\phi(E) > p\beta(E) \qquad \text{for all} \qquad E \leqq A, \beta(E) \neq 0 . \tag{7.5}$$

For, let A_1 be a subset of A, such that

$$\phi(A_1) = p\beta(A_1) , \qquad \beta(A_1) \neq 0 .$$

Then, by (7.3)

$$\phi(E) = p\beta(E) \qquad \text{for all} \qquad E \leqq A_1 ,$$

therefore,
$$D_{\beta(E)}\phi(a) = p$$

in A_1, which is absurd.

When $\beta(A) = 0$, (7.2) is evident. If $\beta(A) \neq 0$, and (7.2) does not hold, then there exists a subset E of A, such that

$$T_{t_0}E \leqq \Omega - A , \qquad \beta(E) \neq 0 .$$

By (7.4), we have

$$\phi(T_{t_0}E) \leqq p\beta(T_{t_0}E) .$$

That is, by (7.1),

(1) $\Omega[f(a) > p]$ means the set of all points in Ω for which $f(a) > p$. For the definition of the derivative $D_{\beta(E)}\phi(a)$, cf. F. Maeda, this journal, **4** (1934), 143.

$$\phi(E) \leqq p\beta(E),$$

which contradicts (7.5). Hence, the theorem is proved.

Put $$B = \Omega[D_{\beta(E)}\phi(a) \leqq q],$$

then since $T_{t_0}\Omega = \Omega$, by (7.2) we have

$$T_{t_0}B \equiv B \qquad (\beta).$$

Next, put $$C = \Omega[p < D_{\beta(E)}\phi(a) \leqq q], \tag{7.6}$$

then since $C = AB$, by (7.2) and (7.6) we have

$$T_{t_0}C \equiv C \qquad (\beta). \tag{7.7}$$

8. *When $\phi(E)$ has the period t_0, that is $\phi_{t_0}(E) = \phi(E)$, then $\phi(E)$ is a strongly convergent limit of the sequence of set functions which are written in the form*

$$\sum_i p_i\beta(EA_i),$$

where $$T_{t_0}A_i \equiv A_i \qquad (\beta) \qquad\qquad \textit{for all } i.$$

Since, the real and imaginary parts of $\phi(E)$ have also the period t_0, we can assume that $\phi(E)$ is real. Consider a sequence $\{p_i\}$ of real numbers such that

$$\ldots\ldots < p_{-n} < \ldots\ldots < p_{-1} < p_0 < p_1 < \ldots\ldots < p_n < \ldots\ldots$$

$$\lim_{n\to\infty} p_n = +\infty, \qquad \lim_{n\to\infty} p_{-n} = -\infty,$$

$$p_{i+1} - p_i < \varepsilon \qquad (i = 0, \pm 1, \pm 2, \ldots\ldots),$$

ε being any positive number. And put

$$A_i = \Omega[p_i < D_{\beta(E)}\phi(a) \leqq p_{i+1}] \qquad (i = 0, \pm 1, \pm 2, \ldots\ldots).$$

Then, by (7.7)

$$T_{t_0}A_i \equiv A_i \qquad (\beta) \qquad\qquad \text{for all } i.$$

Since $\phi(EA_i) - p_i\beta(EA_i)$ and $\phi(EA_j) - p_j\beta(EA_j)$ are orthogonal when $i \neq j$, we have

$$|| \phi(E) - \sum_i p_i \beta(EA_i) ||^2 = \sum_i || \phi(EA_i) - p_i \beta(EA_i) ||^2 \leqq \varepsilon^2 \beta(\Omega) .$$

Thus the theorem is proved.

9. By sec. 3, $\mathfrak{M}_0$ is composed of the set functions $\phi(E)$, such that

$$\phi_t(E) = \phi(E) \qquad \text{for all } t.$$

Now, we have the following theorem:

$\mathfrak{M}_0$ is a closed linear manifold determined by the set functions of the form $\beta(EA)$ where

$$T_t A \equiv A \quad (\beta) \qquad \textit{for all } t. \quad (9.1)$$

Let A be the set which satisfies (9.1). Then

$$\beta(T_t E \cdot A) = \beta(E \cdot T_{-t} A) = \beta(EA) .$$

Hence $\beta(EA)$ belongs to $\mathfrak{M}_0$.

Next let $\phi(E)$ be any set function in $\mathfrak{M}_0$. Then, since

$$\phi_t(E) = \phi(E) \qquad \text{for all } t,$$

by the preceding section,[1] $\phi(E)$ is a strongly convergent limit of the sequence of the set functions of the form

$$\sum_i p_i \beta(EA_i) ,$$

where $\quad T_t A_i \equiv A_i \quad (\beta) \quad \text{for all } t, \quad (i = 0, \pm 1, \pm 2, \ldots .).$

Hence, the theorem is proved.

The necessary and sufficient condition that $\mathfrak{M}_0$ be a linear manifold of one dimension, is that

$$\beta(A) = \beta(\Omega) \quad \textit{or} \quad \beta(A) = 0 ,$$

for any set A, which satisfies

$$T_t A \equiv A \quad (\beta) \qquad \textit{for all } t. \quad (9.2)$$

The sufficiency is evident. For, by the preceding theorem $\mathfrak{M}_0$ is determined by only one set function $\beta(EA) = \beta(E)$.

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(1) Since A_i depends only on $\phi(E)$, and not on t_0.

Next, when $\mathfrak{M}_0$ is of one dimension, all set functions $\phi(E)$ in $\mathfrak{M}_0$ being of the form $c\beta(E)$. Let A be a set which satisfies (9.2), then

$$\beta(EA) = c\beta(E) \qquad \text{for all } E.$$

When $\beta(A) \neq 0$, put $E = A$, then $c = 1$. Hence

$$\beta(EA) = \beta(E).$$

Put $E = \Omega$, then we have

$$\beta(A) = \beta(\Omega).$$

10. *The necessary and sufficient condition that $\lambda = 0$ be the only characteristic value of $\boldsymbol{E}(U)$ is that any set A which has the following relation*

$$T_t A \equiv A \quad (\beta), \qquad \beta(A) \neq 0, \tag{10.1}$$

for a definite value $t = t_0$, satisfies the same relation for all t.

To prove the necessity, assume that $\lambda = 0$ is the only characteristic value of $\boldsymbol{E}(U)$, and let A be the set which satisfies (10.1) for a definite value $t = t_0$. And put

$$\phi(E) = \beta(EA).$$

Then

$$\phi(T_{t+t_0}E) = \phi(T_t E) \qquad \text{for all } t. \tag{10.2}$$

Since, $\mathfrak{M}_D$ is absent, by sec. 1 $\phi(E)$ is decomposed as follows

$$\phi(E) = \phi_0(E) + \phi_C(E),$$

where

$$\phi_0(E) \in \mathfrak{M}_0, \qquad \phi_C(E) \in \mathfrak{M}_C.$$

Then, by (10.2)

$$\phi_0(T_{t+t_0}E) + \phi_C(T_{t+t_0}E) = \phi_0(T_t E) + \phi_C(T_t E) \qquad \text{for all } t,$$

But, by sec. 3

$$\phi_0(T_t E) = \phi_0(E) \qquad \text{for all } t.$$

Hence

$$\phi_C(T_{t+t_0}E) = \phi_C(T_t E) \qquad \text{for all } t.$$

That is, $\phi_C(T_t E)$ is periodic. This contradicts the result of sec. 5:

$$\lim_{\substack{t \to \pm\infty \\ t \text{ not on } I}} \phi_C(T_t E) = 0,$$

unless $$\phi_C(E) = 0 \, .$$ [1]

Hence $\phi(E)$ belongs to $\mathfrak{M}_0$. Consequently,

$$\phi(T_t E) = \phi(E) \qquad \text{for all } t,$$

that is, $$T_t A \equiv A \quad (\beta) \qquad \text{for all } t.$$

Next, to prove the sufficiency, assume that any set A which satisfies (10.1) for $t = t_0$, satisfies (10.1) for all t. And let λ_0 be a characteristic value which is not zero, and $\phi(E)$ be a characteristic function of $E(U)$ with respect to λ_0. Then, by sec. 4

$$\phi_{t_0}(E) = \phi(E) \, ,$$

where $\lambda_0 t_0 = 2\pi$. Then by sec. 8, $\phi(E)$ is a strongly convergent limit of the sequence of set functions

$$\sum_i p_i \beta(EA_i)$$

where $$T_{t_0} A_i \equiv A_i \quad (\beta) \, .$$

Then by the assumption

$$T_t A_i \equiv A_i \quad (\beta) \qquad \text{for all } t.$$

Therefore, by sec. 9, $\phi(E)$ is a set function in $\mathfrak{M}_0$, which is absurd.

Combining this theorem with that of the preceding section, we have the following theorem.

The necessary and sufficient condition that $\lambda = 0$ *be the only, and a simple, characteristic value of* $E(U)$ *is that*

$$\beta(A) = \beta(\Omega) \quad \textit{or} \quad \beta(A) = 0$$

should hold for any set A, *which satisfies*

$$T_t A \equiv A \quad (\beta)$$

for a definite value $t = t_0$. [2]

11. In the conservative mechanism, the *ergodic theory* may be stated as follows:

(1) Since $\phi_C(T_t E)$ is a continuous function of t, when $\phi_C(E) \neq 0$, $\phi_C(T_t E) \neq 0$ for all t such that $|t - nt_0| < \varepsilon$.

(2) Cf. E. Hopf, Proc. Nat. Acad. Sci., **18** (1932), 207.

$$\underset{q-p\to\infty}{[\lim]} \frac{1}{q-p}\int_p^q \beta(T_tE\cdot A)dt = \frac{\beta(A)}{\beta(\Omega)}\beta(E) \tag{11.1}$$

for any set A.

Since, by sec. 6,

$$\boldsymbol{E}(0)\beta(EA) = \frac{\beta(A)}{\beta(\Omega)}\beta(E)$$

when and only when $\mathfrak{M}_0$ is of one dimension, by (2.7), the ergodic theory holds when and only when $\mathfrak{M}_0$ is of one dimension.

Hence, from sec. 9, we have the following theorem:

The ergodic theory (11.1) *holds when and only when*

$$\beta(A) = \beta(\Omega) \qquad \textit{or} \qquad \beta(A) = 0$$

for any set A, *which satisfies*

$$T_tA \equiv A \quad (\beta) \qquad\qquad \textit{for all } t.$$[1]

In this case the conservative mechanism T_t is called *metrically transitive.*

When (11.1) holds, $\mathfrak{M}_0$ is of one dimension. Hence by (6.6),

$$\boldsymbol{E}(0)\phi(E) = \frac{\phi(\Omega)}{\beta(\Omega)}\beta(E).$$

Therefore, by (2.7),

$$\underset{q-p\to\infty}{[\lim]} \frac{1}{q-p}\int_p^q \phi(T_tE)dt = \frac{\phi(\Omega)}{\beta(\Omega)}\beta(E). \tag{11.2}$$

for any set function $\phi(E)$ in $\mathfrak{L}_2(\beta)$. Consequently, *in order that* (11.2) *may hold for any set function* $\phi(E)$ *in* $\mathfrak{L}_2(\beta)$, *it is sufficient that* (11.2) *holds only for any set function of the form* $\beta(EA)$.

12. When $$\lim_{\substack{t\to\pm\infty \\ t \text{ not on } I}} \beta(T_tE\cdot A) = \frac{\beta(A)}{\beta(\Omega)}\beta(E)^{(2)} \tag{12.1}$$

for any sets E and A, the conservative mechanism is said to have the *mixing property.*

(1) Cf. J. v. Neumann, Proc. Nat. Acad. Sci., **18** (1932), 79.

(2) I depends on E and A, and it satisfies (5.1).

The necessary and sufficient condition for the mixing property is that

$$\beta(A) = \beta(\Omega) \qquad or \qquad \beta(A) = 0$$

should hold for any set A, which satisfies

$$T_t A \equiv A \qquad (\beta)$$

for a definite value $t = t_0$.(1)

In this case, the conservative mechanism is said to be *completely transitive.*

To prove the necessity, let A be a set such that

$$T_{t_0} A \equiv A \qquad (\beta), \qquad \beta(A) \neq 0\,.$$

Then
$$\beta(T_{t+t_0} E \cdot A) = \beta(T_t E \cdot A)$$

for all t. Hence, when $\beta(A) \neq \beta(\Omega)$, $\beta(T_t E \cdot A)$ is periodic, which contradicts (12.1). Therefore it must be that $\beta(A) = \beta(\Omega)$.

Next, assume that the condition of the theorem is satisfied, then by sec. 10, $\lambda = 0$ is the only, and a simple, characteristic value. Then by sec. 1,

$$\beta(EA) = \phi_0(E) + \phi_C(E) \qquad \text{where} \qquad \phi_0(E) \in \mathfrak{M}_0,\ \phi_C(E) \in \mathfrak{M}_C\,.$$

But by sec. 3 and 5,

$$\phi_0(T_t E) = \phi_0(E) \qquad \text{for all } t, \qquad \lim_{\substack{t \to +\infty \\ t \text{ not on } I}} \phi_C(T_t E) = 0\,.$$

Therefore,
$$\lim_{\substack{t \to +\infty \\ t \text{ not on } I}} \beta(T_t E . A) = \phi_0(E) = E(0)\beta(EA)\,.$$

Hence by (6.7)

$$\lim_{\substack{t \to \pm\infty \\ t \text{ not on } I}} \beta(T_t E \cdot A) = \frac{\beta(A)}{\beta(\Omega)} \beta(E)\,.$$

That is, (12.1) holds. Hence the theorem is proved.

When (12.1) holds, then $\lambda = 0$ is the only, and a simple, characteristic value. Hence, as above, instead of (12.1) we have

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(1) Cf. B. O. Koopman and J. v. Neumann, Proc. Nat. Acad. Sci., **18** (1932), 256 and 259.

$$\lim_{\substack{t\to\pm\infty \\ t \text{ not on } I}} \phi(T_t E) = \frac{\phi(\Omega)}{\beta(\Omega)}\beta(E)\,. \tag{12.2}$$

That is, (12.2) *holds for any set function* $\phi(E)$ *in* $\mathfrak{L}_2(\beta)$, *only when* (12.2) *holds for any set function of the form* $\beta(EA)$.