An Extension of the Definition of Vector and Parallel Displacement.

By

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Introduction.

In an n-dimensional space whose coordinates are $x^1, \ldots\ldots, x^n$, we consider a *vector* which is expressed by N components $a^1, \ldots\ldots, a^N$. In § 1, we extend the definition of *summation* of vectors; and in § 2, we take an r-parameter continuous group as the transformations of coordinates in x-space, and thence we find the transformations of vectors on the assumptions that they form a group and transform the *sum* of any two vectors into that of the transformed vectors. In § 3, we define the parallel displacement of vectors on the assumption that the *sum* of any vectors is parallel to that of the vectors which are respectively parallel to the former, and in § 4 we obtain the transformation of coefficients of connection by the coordinate-transformation. In the remaining paragraphs we find the covariant derivative of a vector-field and curvature in our space.

§ 1. New definition of summation of vectors.

Let a^λ and b^λ $(\lambda = 1, \ldots\ldots, N)$ be any two vectors at a point $P(x)$ and c^λ $(\lambda = 1, \ldots\ldots, N)$ the sum of these two vectors. Then if we extend the idea of *summation*, c^λ may in general be expressed by a fuction of $a^1, \ldots\ldots, a^N, b^1, \ldots\ldots, b^N, x^1, \ldots\ldots, x^N$, viz.

$$c^\lambda = \varphi^\lambda(a^1, \ldots, a^N, b^1, \ldots, b^N, x^1, \ldots, x^n) \; (\lambda = 1, \ldots, N) \quad (1)$$

$$\equiv \varphi^\lambda(a, b, x),$$

which we express symbolically by

$$c = a + b .$$

Now we make the following assumptions.

(1, a) The functions $\varphi^\lambda(a, b, x)$ are analytic functions of the arguments, and equations (1) can be solved for as and bs.

(1, b) At every point the associative law holds with respect to the summation of more than three vectors: If a^λ, b^λ and c^λ $(\lambda = 1, \ldots\ldots, N)$ are any three vectors at any point $P(x)$, then

$$(a + b) + c = a + (b + c),$$

viz.
$$\varphi^\lambda(\varphi(a, b, x), c, x) = \varphi^\lambda(a, \varphi(b, c, x), x) \qquad (\lambda = 1, \ldots, N). \tag{2}$$

(1, c) When

$$c = a + b$$

there exists a vector d^λ $(\lambda = 1, \ldots\ldots, N)$ such that

$$c + d = a$$

Such a vector d^λ we denote by $-b$.

From the equations (2), we see that $\varphi^\lambda(a, b, x)$ are functions such that for any two elements T_a and T_b of a certain N-parameter continuous group G_N, say $y'^i = f^i(y^1, \ldots, y^s, a^1, \ldots, a^N, x^1, \ldots, x^n)$ and $y''^i = f^i(y', b, x)$ $(i = 1, \ldots, s)$, the law of connection is given by

$$T_b T_a = T_{\varphi(a, b, x)} \tag{3}$$

(where xs are regarded as constants). viz. that for each value of xs the two transformations as into a's:

$$a'^\lambda = \varphi^\lambda(a, b, x) \qquad (\lambda = 1, \ldots, N) \tag{4}$$

and

$$a'^\lambda = \varphi^\lambda(b, a, x) \qquad (\lambda = 1, \ldots, N), \tag{5}$$

(bs being regarded as the parameters), must be the first and second parameter groups of G_N.

If we take the general parameter group of G_N, say $a'^\lambda = \varphi^\lambda(a, b, x)$, then $c^\lambda = \varphi^\lambda(a, b, x)$ give the most general equations which define the summation of two vectors a^λ and b^λ satisfying the assumptions

(1, a), (1, b) and (1, c). However, to the structure-constants $d^{\lambda}_{\mu\nu}(x)$ of a group G_N there corresponds the unic canonical parameter group: $a'^{\lambda} = \Phi^{\lambda}(a, b, x)$ and the general parameter group which has the same structure as $d^{\lambda}_{\mu\nu}(x)$ is given by

$$g^{\lambda}(a', x) = \Phi^{\lambda}\left(g(a, x), g(b, x), x\right), \quad (\lambda = 1, \ldots, N)$$

where $g^{\lambda}(a', x)$ are N independent general analytic functions of as. Therefore in this paper we will confine ourselves to the case where (4) and (5) are the canonical parameter groups determined from the structure-constants $d^{\lambda}_{\mu\nu}(x)$ of a certain group G_N. In this case (1) can be written in the form

$$c^{\lambda} = e^{b^{\alpha} A^{\beta}_{\alpha}(a, x)\frac{\partial}{\partial a^{\beta}}} a^{\lambda} \tag{6}$$

and

$$c^{\lambda} = e^{a^{\alpha} B^{\beta}_{\alpha}(b, x)\frac{\partial}{\partial b^{\beta}}} b^{\lambda} \tag{7}$$

where $A^{\beta}_{\alpha}(a, x)$ and $B^{\beta}_{\alpha}(b, x)$ are defined by

$$A^{\beta}_{\alpha}(a, x) = \left[\frac{\partial \varphi^{\beta}(a, b, x)}{\partial b^{\alpha}}\right]_{b=0} ; \quad B^{\beta}_{\alpha}(b, x) = \left[\frac{\partial \varphi^{\beta}(a, b, x)}{\partial a^{\alpha}}\right]_{a=0} , \tag{8}$$

and satisfy the following relations:

$$\left(A^{\beta}_{\mu}\frac{\partial}{\partial a^{\beta}} , \quad A^{\beta}_{\nu}\frac{\partial}{\partial a^{\beta}}\right) = d^{\lambda}_{\mu\nu}(x) A^{\beta}_{\lambda}\frac{\partial}{\partial a^{\beta}} .$$

§ 2. Transformation of Vectors.

In our space, the transformations of coordinates xs are considered as always belonging to a given r-parameter continuous group, the equations being given by

$$\begin{aligned} x'^{i} &= e^{u^{h} X_{h}} x^{i} \qquad (i = 1, \ldots, n;\ h = 1, \ldots, r), \\ &\equiv f^{i}(x, u) \end{aligned} \tag{9}$$

where

$$X_h \equiv \xi^{i}_{h}(x^1, \ldots, x^n)\frac{\partial}{\partial x^{i}} \quad (h = 1, \ldots, r)$$

are generators and u^h the parameters of the group.

Now we make the two following assumptions.

(2, a) By the transformations (9) any vector a^λ $(\lambda = 1, \ldots, N)$ at a point $P(x)$ in our space, is transformed by the following equations:

$$a'^\lambda = \psi^\lambda(a^1, \ldots, a^N, x^1, \ldots, x^n, u^1, \ldots, u^r) \qquad (\lambda = 1, \ldots, N) \quad (10)$$
$$\equiv \psi^\lambda(a, x, u)$$

and the transformations $(x \to x', a \to a')$ obtained by putting (9) and (10) together, form a group. We denote this by Γ.

(2, b) The sum of any two vectors a^λ and b^λ $(\lambda = 1, \ldots, N)$ at any point $P(x)$, is transformed into the sum of the transformed vectors:

$$(a + b)' = a' + b'$$

viz.

$$\psi^\lambda(\varphi(a, b, x), x, u) = \varphi^\lambda(\psi(a, x, u), \psi(b, x, u), f(x, u)) \qquad (\lambda = 1, \ldots, N) . \quad (11)$$

The equations (11) show that when we consider a transformation as into a's:

$$a'^\lambda = \varphi^\lambda(a, b, x) \qquad (\lambda = 1, \ldots, N) \qquad (12)$$

(bs and xs being parameters), and apply a change of variables in the above:

$$\mathfrak{a}^\lambda = \psi^\lambda(a, x, u), \ \mathfrak{a}'^\lambda = \psi^\lambda(a', x, u) \quad (\lambda = 1, \ldots, N) \qquad (13)$$

(xs and us being as parameters), the equations of the transformation (12) become as follows

$$\mathfrak{a}'^\lambda = \varphi^\lambda(\mathfrak{a}, \psi(b, x, u), f(x, u)) \quad (\lambda = 1, \ldots, N) . \qquad (14)$$

However since, as can be seen from (6), (12) can be written in the form

$$a'^{\lambda} = e^{b^{\alpha} A^{\mu}_{\alpha}(a, x)\frac{\partial}{\partial a^{\mu}}} a^{\lambda}, \tag{15}$$

this is transformed, by the change of variables (13), into[1]

$$\mathfrak{a}'^{\lambda} = e^{b^{\alpha} C^{\mu}_{\alpha}(\mathfrak{a}, x, u)\frac{\partial}{\partial \mathfrak{a}^{\mu}}} \mathfrak{a}^{\lambda} \tag{16}$$

where

$$C^{\mu}_{\alpha}(\mathfrak{a}, x, u) = A^{\beta}_{\alpha}(a, x)\frac{\partial \psi^{\mu}(a, x, u)}{\partial a^{\beta}}$$

by the relation (13).

On the other hand, as in (15), the equations (14) can be written in the form

$$\mathfrak{a}'^{\lambda} = e^{\psi^{\alpha}(b, x, u) A^{\mu}_{\alpha}(\mathfrak{a}, f(x, u))\frac{\partial}{\partial \mathfrak{a}^{\mu}}} \mathfrak{a}^{\lambda} \quad (\lambda = 1, \ldots, N). \tag{17}$$

So from (16) and (17), we have

$$\psi^{\alpha}(b, x, u) A^{\mu}_{\alpha}\big(\mathfrak{a}, f(x, u)\big) = b^{\alpha} C^{\mu}_{\alpha}(\mathfrak{a}, x, u),$$

or substituting (13) for $\mathfrak{a}^{\lambda}$ in the above

$$\psi^{\alpha}(b, x, u) A^{\mu}_{\alpha}\big(\psi(a, x, u), f(x, u)\big) = b^{\alpha} A^{\beta}_{\alpha}(a, x)\frac{\partial \psi^{\mu}(a, x, u)}{\partial a^{\beta}}. \tag{18}$$

The relation (18) are the condition that the functions $\psi^{\lambda}(a, x, u)$ in the vector-transformation (10), must satisfy in order that the assumption (2, b) may be fulfilled. If we solve for $\psi^{\lambda}(b, x, u)$ from (18) we see that $\psi^{\lambda}(b, x, u)$ are linear and homogeneous with respect to b^{α} ($\alpha = 1, \ldots\ldots, N$), that is

$$\psi^{\lambda}(b, x, u) = \psi^{\lambda}_{\alpha}(x, u) b^{\alpha} \quad (\lambda = 1, \ldots, N). \tag{19}$$

Hence, from (19) we have the expressions for the vector-transformations:

$$\begin{aligned} a'^{\lambda} &= \psi^{\lambda}(a, x, u) \\ &= \psi^{\lambda}_{\alpha}(x, u) a^{\alpha} \quad (\lambda = 1, \ldots\ldots\ldots\ldots, N). \end{aligned} \tag{20}$$

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(1) S. Lie. *Theorie der Transformationsgruppen*. **1.** (1930) 58.

So we have the result: *Under the assumption* (2, *b*) *the vector-transformation of our space is linear and homogeneous with respect to the vector-components.*

Now from the assumption (2, a) the transformations ($x \rightarrow x'$, $a \rightarrow a'$) obtained by putting (9) and (20) together, must form a gruop Γ. Hence, if we expand the right hand side of (20) in powers of $u^1, \ldots\ldots, u^r$:

$$a'^{\lambda} = a^{\lambda} + \left(\frac{\partial \psi^{\lambda}_{\beta}(x, u)}{\partial u^h}\right)_{u=0} a^{\beta} u^h + \ldots\ldots,$$

and put

$$\eta^{\lambda}_{\beta h}(x) = \left(\frac{\partial \psi^{\lambda}_{\beta}(x, u)}{\partial u^h}\right)_{u=0}$$

$$Z_h = \xi^i_h(x) \frac{\partial}{\partial x^i} + \eta^{\lambda}_{\beta h}(x) a^{\beta} \frac{\partial}{\partial a^{\lambda}} (h=1, \ldots\ldots, r) , \qquad (21)$$

then Z_h must be the generators of the group Γ. Therefore it must be that

$$(Z_h , Z_k) = c^l_{hk} Z_l \quad (h, k, l = 1, \ldots\ldots r)$$

where c^l_{hk} are the structure-constants of the fundamental group (9). Then, comparing both sides of the above equations, we have[1]

(1) The condition of integrability of the equations (22) is easily seen to be satisfied. For if we put

$$\begin{cases} \eta^{\lambda}_{\beta h} = \dfrac{\partial \xi^{\lambda}_h}{\partial x^{\beta}} & (\lambda, \beta = 1, \ldots, n) \\ \eta^{\lambda}_{\beta h} = 0 & (\lambda \text{ or } \beta > n) \end{cases} \qquad (h = 1, \ldots, r)$$

or

$$\begin{cases} \eta^{\lambda}_{\beta h} = -\dfrac{\partial \xi^{\beta}_h}{\partial x^{\lambda}} & (\lambda, \beta = 1, \ldots, n) \\ \eta^{\lambda}_{\beta h} = 0 & (\lambda \text{ or } \beta > n) \end{cases} \qquad (h = 1, \ldots, r)$$

we see that (22) is satisfied by using the relations

$$\xi^i_h \frac{\partial \xi^{\lambda}_k}{\partial x^i} - \xi^i_k \frac{\partial \xi^{\lambda}_h}{\partial x^i} = c^l_{hk} \xi^{\lambda}_l \qquad (\lambda = 1, \ldots, n)$$

differentiated with respect to x^{β} ($\beta = 1, \ldots, n$), or

$$-\xi^i_h \frac{\partial \xi^{\beta}_k}{\partial x^i} + \xi^i_k \frac{\partial \xi^{\beta}_h}{\partial x^i} = -c^l_{kh} \xi^{\beta}_l \qquad (\beta = 1, \ldots, n)$$

differentiated with respect to x^{λ} ($\lambda = 1, \ldots, n$). This shows that (22) actually has particular solutions.

$$\xi^i_h \frac{\partial \eta^\lambda_{\beta k}}{\partial x^i} - \xi^i_{.k} \frac{\partial \eta^\lambda_{\beta h}}{\partial x^i} + \eta^\alpha_{\beta h}\eta^\lambda_{\alpha k} - \eta^\alpha_{\beta k}\eta^\lambda_{\alpha h} = c^l_{hk}\eta^\lambda_{\beta l} \,. \tag{22}$$

In order that (21) may form a group (assumption (2, a)) $\eta^\lambda_{\beta h}$ must satisfy the condition given by (22).

Hence if we substitute the general solution of (22) for $\eta^\lambda_{\beta h}$, (9) and (20) can be written in the forms

$$\left.\begin{aligned} x'^i &= e^{u^h Z_h} x^i \\ &\equiv f^i(x,\, u) \\ a'^\lambda &= e^{u^h Z_h} a^\lambda \\ &\equiv \psi^\lambda_\beta(x,\, u) a^\beta \end{aligned}\right\} \tag{23}$$

and (23) form a group Γ, viz. they satisfy the assumption (2, a).

We shall next find the condition that $\eta^\lambda_{\beta h}$ must satisfy in order that the transformations (23) may fulfill the assumption (2, b). For this purpose we take the equation (18). Substituting $\psi^\mu_\beta(x, u)a^\beta$ for $\psi^\mu(a, x, u)$, (18) can be written in the form :

$$\psi^\alpha(b,\, x,\, u) A^\mu_\alpha(\psi(a,\, x,\, u),\, f(x,\, u)) = b^\alpha A^\beta_\alpha(a,\, x)\psi^\mu_\beta(x,\, u) \,. \tag{24}$$

Now we consider the following infinitesimal transformations in $3N+n$ variables $a^1, \ldots, a^N, b^1, \ldots, b^N, c^1, \ldots, c^N$, and $x^1, \ldots, x^n$:

$$W_h = \xi^i_h \frac{\partial}{\partial x^i} + \eta^\lambda_{\beta h}\left\{a^\beta \frac{\partial}{\partial a^\alpha} + b^\beta \frac{\partial}{\partial b^\alpha} + c^\beta \frac{\partial}{\partial c^\alpha}\right\} \quad (h = 1,\, \ldots,\, r) \,. \tag{25}$$

Then from the relations (22), we easily see that

$$(W_h W_k) = c^l_{hk} W_l \quad (h,\, k,\, l = 1,\, \ldots,\, r) \tag{26}$$

viz. W_h form a group, and from (21), and (23), we have

$$f^i(x,\, u) = e^{u^h W_h} x^i \,, \quad \psi^\lambda(a,\, x,\, u) = e^{u^h W_h} a^\lambda ,$$

$$\psi^\lambda(b,\, x,\, u) = e^{u^h W_h} b^\lambda \,, \quad \psi^\lambda(c,\, x,\, u) = e^{u^h W_h} c^\lambda \,.$$

Hence the equations (24) can be written in the form :

$$e^{u^h W_h}\big(b^\alpha A^\mu_\alpha(a,\, x)\big) = \Big[e^{u^h W_h} c^\lambda\Big]_{c^\beta = b^\alpha A^\beta_\alpha(a,\, x)} \,.$$

This relation shows that the system of equations

$$c^{\lambda}-b^{\alpha}A_{\alpha}^{\lambda}(a, x) = 0 \qquad (\lambda = 1, \ldots\ldots, N)\,, \tag{27}$$

in $3N+n$ variables as, bs, cs, and xs, admits the group generated by W_h. ($h = 1, \ldots\ldots\ldots, r$), therefore the system of equations (27) admits the infinitesimal transformations W_h[1]. So we have

$$W_h b^{\mu} A_{\mu}^{\lambda}(a, x) = [W_h c^{\lambda}]_{c^{\lambda} = b^{\mu}A_{\mu}^{\lambda}(a, x)} \quad (h = 1, \ldots\ldots, r)$$

i.e.

$$\xi_h^i \frac{\partial A_{\mu}^{\lambda}}{\partial x^i} + \eta_{\beta h}^{\alpha} a^{\beta} \frac{\partial A_{\mu}^{\lambda}}{\partial a^{\alpha}} + \eta_{\mu h}^{\alpha} A_{\alpha}^{\lambda} = \eta_{\alpha h}^{\lambda} A_{\mu}^{\alpha} \tag{28}$$

However A_{μ}^{λ} have the form :[2]

$$A_{\mu}^{\lambda} = \delta_{\mu}^{\lambda} + \varkappa_1 U_{\mu}^{\lambda} + \varkappa_2 U_{\beta_1}^{\lambda} U_{\mu}^{\beta_1} + \ldots + \varkappa_m U_{\beta_1}^{\lambda} U_{\beta_2}^{\beta_1} U_{\beta_3}^{\beta_2} \ldots U_{\alpha}^{\beta_{m-1}} + \ldots, \tag{29}$$

where $\varkappa_m$ are the coefficients of x^m in the power series of

$$\frac{x}{e^x - 1}$$

and U_{μ}^{λ} are defined by

$$U_{\mu}^{\lambda} = d_{\mu\beta}^{\lambda}(x) a^{\beta}\,,$$

$d_{\mu\beta}^{\lambda}(x)$ being the structure-constants of the parameter group (4). In order that the equations (28) should hold for every value of as, it is necessary and sufficient that the equations hold when A_{μ}^{λ} are replaced by their coefficients of the expansion (29). In particular, it is necessary that

$$\xi_h^i \frac{\partial U_{\mu}^{\lambda}}{\partial x^i} + \eta_{\beta h}^{\alpha} a^{\beta} \frac{\partial U_{\mu}^{\lambda}}{\partial a^{\alpha}} + \eta_{\mu h}^{\alpha} U_{\alpha}^{\lambda} = \eta_{\alpha h}^{\lambda} U_{\mu}^{\alpha}\,. \tag{30}$$

But if (30) are satisfied we can easily show that they are also satisfied when U_{μ}^{λ} are replaced by $U_{\beta_1}^{\lambda} U_{\mu}^{\beta_1}$, and so on. Hence (30) are also sufficient for the consistency of (28). In fact, (30) can be written in the form :

(1) S. Lie, Ibid.,

(2) F. Schur, Zur Theorie der endlichen Transformationsgruppen. Math. Annalen. **38** (1891), 271.

$$\xi_h^i \frac{\partial d_{\mu\nu}^\lambda}{\partial x^i} + \eta_{\mu h}^\beta d_{\beta\nu}^\lambda + \eta_{\nu h}^\beta d_{\mu\beta}^\lambda - \eta_{\beta h}^\lambda d_{\mu\nu}^\beta = 0 \ . \tag{31}$$

This is the required condition that $\eta_{\beta h}^\lambda$ must satisfy in order that the transformations (23) may fulfill the assumption (2, b).

Hence, taking account of the equations (22), we have the result: *If we take for $\eta_{\beta h}^\lambda(x)$ the functions of xs which satisfy both equations (22) and (31), then (23) gives the most general transformations which satisfy the assumptions (2, a) and (2, b).*

If $\eta_{\beta h}^\lambda(x)$ are determined uniquely from the equations (31) we see that these $\eta_{\beta h}^\lambda(x)$ necessarily satisfy the equations (22). For, if we take these $\eta_{\beta h}^\lambda(x)$, the system of equations (27) admits the following infinitesimal transformations:

$$V_h = \xi_h^i \frac{\partial}{\partial x^i} + \eta_{\beta h}^\lambda \left\{ a^\beta \frac{\partial}{\partial a^\alpha} + b^\beta \frac{\partial}{\partial b^\alpha} + c^\beta \frac{\partial}{\partial c^\alpha} \right\} \ (h = 1, \ldots\ldots, r) \ ,$$

and therefore also admits (V_l, V_m) $(l, m = 1, \ldots\ldots, r)$. In fact, if we put

$$\eta'^\lambda_{\beta lm} = \xi_l^i \frac{\partial \eta_{\beta m}^\lambda}{\partial x^i} - \xi_m^i \frac{\partial \eta_{\beta l}^\lambda}{\partial x^i} + \eta_{\beta l}^\alpha \eta_{\alpha m}^\lambda - \eta_{\beta m}^\alpha \eta_{\alpha l}^\lambda \ ,$$

(V_l, V_m) take the forms:

$$(V_l, V_m) = c_{lm}^h \xi_h^i \frac{\partial}{\partial x^i} + \eta'^\lambda_{\beta lm} \left\{ a^\beta \frac{\partial}{\partial a^\alpha} + b^\beta \frac{\partial}{\partial b^\beta} + c^\beta \frac{\partial}{\partial c^\beta} \right\} \ ,$$

and from the fact that the system of equations (27) admits (V_l, V_m), we have, as in (31),

$$c_{lm}^h \xi_h^i \frac{\partial d_{\mu\nu}^\lambda}{\partial x^i} + \eta'^\beta_{\mu lm} d_{\beta\nu}^\lambda + \eta'^\beta_{\nu lm} d_{\mu\beta}^\lambda - \eta'^\lambda_{\beta lm} d_{\mu\nu}^\beta = 0 \tag{32}$$

On the other hand, multiplying each side of (31) by c_{lm}^h and adding them for all values of h from *1* to r, we have

$$c_{lm}^h \xi_h^i \frac{\partial d_{\mu\nu}^\lambda}{\partial x^i} + c_{lm}^h \eta_{\mu h}^\beta d_{\beta\nu}^\lambda + c_{lm}^h \eta_{\nu h}^\beta d_{\mu\nu}^\lambda - c_{lm}^h \eta_{\beta h}^\lambda d_{\mu\nu}^\beta = 0 \tag{33}$$

Hence by the hypothesis that $\eta_{\beta h}^\alpha$ are determined uniquely from (31), it must be that

$$\eta'^{\alpha}_{\beta lm} - c^{h}_{lm}\eta^{\alpha}_{\beta h} = 0$$

which shows that (22) is fulfilled. So we have the result: *If $N.^2r$ functions $\eta^{\alpha}_{\beta h}(x)$ are determined uniquely from the equations (31), then for such $\eta^{\alpha}_{\beta h}(x)$, (23) gives the most general transformation which satisfy the assumptions (2, a) and (2, b).*

§ 3. Parallel Displacement of Vectors

We define parallel displacement of vectors in our space, making the two following assumptions.

(3, a) Parallelism between two vectors at any point $P(x)$ and any neighbouring point $Q(x+dx)$ is a reversively one-to-one correspondence.

(3, b) The sum of any two vectors at $P(x)$ is parallel to the sum of the vectors at $Q(x+dx)$ which are parallel to the former respectively.

Let $\bar{a}^{\lambda}$ at $P(x)$ be parallel to a^{λ} at $Q(x+dx)$, then from the assumption (3, a), the relation will be expressed by the following equations[1]:

$$\bar{a}^{\lambda} = a^{\lambda} + \Gamma^{\lambda}_{j}(a, x)\,dx^{j} \qquad \begin{pmatrix} \lambda = 1, \ldots, N, \\ j = 1, \ldots, n \end{pmatrix} \tag{34}$$

or, in simpler form

$$\equiv P^{\lambda}(a,\ x,\ dx)\ ,$$

where $\Gamma^{\lambda}_{j}(a, x)$ are certain functions of $a^1, \ldots, a^N, x^1, \ldots, x^n$, which will be hereafter determined.

Next if $\bar{a}^{\lambda}$ and $\bar{b}^{\lambda}$ are two vectors at $P(x)$, parallel to vectors a^{λ} and b^{λ} at $Q(x+dx)$ respectively, the assumption (3, b), namely

$$(\bar{a}+\bar{b}) \text{ is parallel to } (a+b),$$

is expressed analytically

$$\varphi^{\lambda}\big(P(a, x, dx), P(b, x, dx), x\big) = P^{\lambda}\big(\varphi(a, b, x+dx), x, dx\big) \qquad (\lambda = 1, \ldots, N)\ . \tag{35}$$

(1) Here we assume that the equations of parallel displacement are linear with respect to dx s in the usual manner and when $dx=0$ the two vectors $\bar{a}^{\lambda}$ and a^{λ} coincide.

From another point of view this relation is interpreted as follows. The equations of the transformation $(a \rightarrow a')$:

$$a'^{\lambda} = \varphi^{\lambda}(a,\, b,\, x+dx) \qquad (\lambda = 1,\, \ldots,\, N)$$

(regarding bs and $(x+dx)$s as parameters), are changed by the change of variables $(a \rightarrow \mathfrak{a}\,;\, a' \rightarrow \mathfrak{a}')$:

$$\mathfrak{a}^{\lambda} = P^{\lambda}(a,\, x,\, dx)\,, \quad \mathfrak{a}'^{\lambda} = P^{\lambda}(a^1,\, x,\, dx) \quad (\lambda = 1,\, \ldots,\, N)$$

(regarding xs and dxs as parameters), into

$$\mathfrak{a}'^{\lambda} = \varphi^{\lambda}\big(\mathfrak{a},\, P(b,\, x,\, dx),\, x\big) \quad (\lambda = 1,\, \ldots,\, N)\,.$$

From this relation, by the same method by which the equations (18) was obtained from (11), we have equations for $P^{\lambda}(a,\, x,\, dx)$:

$$P^{\alpha}(b,\, x,\, dx)A^{\mu}_{\alpha}\big(P(a,\, x,\, dx),\, x\big) = b^{\alpha}A^{\beta}_{\alpha}(a,\, x+dx)\frac{\partial P^{\mu}(a,\, x,\, dx)}{\partial a^{\beta}} \qquad (36)$$

By solving the above for $P^{\lambda}(b,\, x,\, dx)$, we see that $P^{\lambda}(b,\, x,\, dx)$ must be linear and homogeneous with respect to $b^{\alpha}\,(d=1,\, \ldots,\, N)$; therefore it must have the form:

$$P^{\lambda}(b,\, x,\, dx) = Q^{\lambda}_{\alpha}(x,\, dx)b^{\alpha} \quad (\lambda = 1,\, \ldots,\, N) \qquad (37)$$

However since

$$P^{\lambda}(b,\, x,\, dx) \equiv b^{\lambda} + \Gamma^{\lambda}_{j}(b,\, x)dx^{j}\,,$$

we have expressions for $Q^{\lambda}_{\alpha}(x,\, dx)$:

$$Q^{\lambda}_{\alpha}(x,\, dx) = \delta^{\lambda}_{\alpha} + \Gamma'^{\lambda}_{\alpha j}(x)dx^{j}\,,$$

where $\Gamma'^{\lambda}_{\alpha\beta}$ are functions of xs, therefore (34) becomes

$$P^{\lambda}(b,\, x,\, dx) = b^{\lambda} + \Gamma'^{\lambda}_{\alpha j}(x)b^{\alpha}dx^{j} \qquad (38)$$

Hence (34) must have the form:

$$\begin{aligned} \bar{a}^{\lambda} &= P^{\lambda}(a,\, x,\, dx) \\ &= a^{\lambda} + \Gamma'^{\lambda}_{\alpha j}(x)a^{\alpha}dx^{j} \end{aligned} \qquad (39)$$

So we have the result: *Under the assumption (3, b) the equations of the parallel displacement of our space are linear and homogeneous with respect to the vector-components.*

Next we shall determine $\Gamma^{\lambda}_{aj}(x)$ in (39), such that the equations of the parallel displacement (39) satisfy the assumption (3, b). Substituting (38) and (39) into (36), expanding them in powers of $dx^1, \ldots\ldots, dx^n$, and comparing the coefficients of dx^j on both sides of (36), we have

$$\frac{\partial A^{\lambda}_{\mu}}{\partial x^h} - \Gamma^{\alpha}_{\beta h} a^{\beta} \frac{\partial A^{\lambda}_{\mu}}{\partial a^{\alpha}} - \Gamma^{\alpha}_{\mu h} A^{\lambda}_{\alpha} + \Gamma^{\lambda}_{\alpha h} A^{\alpha}_{\mu} = 0$$

From the above, by the same method by which the equations (31) was obtained from (28), we have

$$\frac{\partial d^{\lambda}_{\mu\nu}}{\partial x^i} - \Gamma^{\beta}_{\mu h} d^{\lambda}_{\beta\nu} - \Gamma^{\beta}_{\nu h} d^{\lambda}_{\mu\beta} + \Gamma^{\lambda}_{\beta h} d^{\beta}_{\mu\nu} = 0 \tag{40}$$

This is the condition that $\Gamma^{\lambda}_{aj}(x)$ must satisfy in order that the assumption (3, b) may be fulfilled.

So we have the result: *If we take for $\Gamma^{\lambda}_{ah}(x)$ the functions of xs which satisfy the equations (40), (39) gives the most general equations of the parallel displacement of vectors under the assumptions (3, a) and (3, b).*

4. Transformation of Γ^{λ}_{aj}.

Let a vector $\bar{a}^{\lambda}$ at $P(x)$ be parallel to a vector a^{λ} at $Q(x+dx)$, viz.

$$\bar{a}^{\lambda} = a^{\lambda} + \Gamma^{\lambda}_{\alpha j} a^{\alpha} dx^j \quad (\lambda = 1, \ldots\ldots, N), \tag{41}$$

and suppose that $\bar{a}^{\lambda}$, a^{λ}, $\Gamma^{\lambda}_{\alpha j}$ and dx^j are transformed, by the coordinate-transformation (23), into $\bar{a}'^{\lambda}$, a'^{λ}, Γ'^{λ}_{aj} and dx'^j respectively. Then it must be that

$$\bar{a}'^{\lambda} = a'^{\lambda} + \Gamma'^{\lambda}_{\alpha j} a'^{\alpha} dx'^j \tag{42}$$

Actually, if we substitute the equations of these transformations:

$$x'^i = f^i(x, u)\,; \quad dx'^i = \frac{\partial f^i}{\partial x^j} dx^j$$
$$a'^{\lambda} = \psi^{\lambda}_{\alpha}(x+dx, u) a^{\alpha}$$
$$\bar{a}'^{\lambda} = \psi^{\lambda}_{\alpha}(x, u) \bar{a}^{\alpha}$$

into (42), then by using (41), we have

$$\psi^\lambda_\alpha(x, u)\{a^\alpha+\Gamma^\alpha_{\beta j}a^\beta dx^j\} = \psi^\lambda_\alpha(x+dx, u)a^\alpha+\Gamma'^\lambda_{\alpha i}\psi^\alpha_\beta(x+dx, u)a^\beta\frac{\partial f^i(x, u)}{\partial x^j}dx^j$$

Expanding both sides of the above in the powers of $dx^1, \ldots\ldots, dx^n$, and comparing the coefficients of dx^j, we have

$$\psi^\lambda_\alpha(x, u)\,\Gamma^\alpha_{\beta j} = \frac{\partial\psi^\lambda_\beta(x, u)}{\partial x^j}+\Gamma'^\lambda_{\alpha i}\psi^\alpha_\beta(x, u)\frac{\partial f^i(x, u)}{\partial x^j}$$

$$\left(\begin{array}{l}\lambda, \alpha, \beta = 1, \ldots\ldots, N\\ j = 1, \ldots\ldots, n\end{array}\right). \qquad (43)$$

These are the required equations for the transformation of $\Gamma^\lambda_{\beta j}$.

§ 5. Covariant derivative of $v^\lambda(x)$.

We proceed to find the *covariant derivative* of any vector-field $v^\lambda(x)$ in our space.

For this purpose we must define the *difference* of two vectors.

Since we know that the equations(1) :

$$c^\lambda = \varphi^\lambda(a, b, x) \qquad (\lambda = 1, \ldots\ldots, N)$$

give the sum of two vectors a^λ and b^λ, $(a+b)$, we give the following definition for the *difference* of two vectors: In the equations

$$c^\lambda = \varphi^\lambda(a, b, x) \qquad (\lambda = 1, \ldots\ldots, N)$$

we call b^λ the *difference of the first kind* between c^λ and a^λ and a^λ the *difference of the second kind* between c^λ and b^λ.

Now let $v^\lambda(x)$ give an arbitrary vector-field, and let a vector $\bar{v}^\lambda$ at $P(x)$ be parallel to a vector $v^\lambda(x+dx)$ at $Q(x+dx)$. Here we will obtain the differences of the first kind and the second kind between $\bar{v}^\lambda$ and $v^\lambda(x)$, which we denote by $\delta_1 v^\lambda$ and $\delta_2 v^\lambda$ respectively.

From the hypothesis, we have

$$\bar{v}^\lambda = v^\lambda(x+dx)+\Gamma^\lambda_{\alpha j}v^\alpha(x)\,dx^j$$

$$= v^\lambda(x) +\left[\frac{\partial v^\lambda(x)}{\partial x^i}+\Gamma^\lambda_{\alpha j}v^\alpha(x)\right]dx^j \quad (\lambda = 1, \ldots\ldots, N) \qquad (44)$$

and

$$\bar{v}^{\lambda} = \varphi^{\lambda}(v(x), \delta_1 v, x) \qquad (\lambda = 1, \ldots\ldots, N).$$

Expanding the right hand side of the last equation in the powers of $\delta_1 v^1, \ldots\ldots, \delta_1 v^N$, and substituting (40) in it, we have

$$\left[\frac{\partial v^{\lambda}(x)}{\partial x^j} + \Gamma^{\lambda}_{\alpha j} v^{\alpha}(x)\right] dx^j = A^{\alpha}_{\lambda}\big(v(x), x\big)\delta_1 v^{\alpha} + \ldots\ldots, \tag{45}$$

where $A^{\lambda}_{\alpha}(v(x), x)$ are the functions which we have defined in the equations (8). Then solving for $\delta_1 v^{\lambda}$ from the equations (45) and neglecting terms higher than the second order, we have

$$\delta_1 v^{\lambda} = \bar{A}^{\lambda}_{\alpha}\big(v(x), x\big)\left[\frac{\partial v^{\alpha}(x)}{\partial x^j} + \Gamma^{\alpha}_{\beta j} v^{\beta}(x)\right] dx^j \tag{46}$$

where $\bar{A}^{\lambda}_{\alpha}(v(x), x)$ are functions defined by

$$\bar{A}^{\lambda}_{\alpha}(v, x) A^{\alpha}_{\mu} = \delta^{\lambda}_{\mu} \quad \begin{cases} = 1, & \lambda = \mu\,. \\ = 0, & \lambda \neq \mu\,. \end{cases}$$

Similarly, from the equations:

$$\bar{v}^{\lambda} = \varphi^{\lambda}\big(\delta_2 v, v(x), x\big), \qquad (\lambda = 1, \ldots\ldots, N)$$

we have

$$\delta_2 v^{\lambda} = \bar{B}^{\lambda}_{\alpha}\big(v(x), x\big)\left[\frac{\partial v^{\alpha}(x)}{\partial x^i} + \Gamma^{\alpha}_{\beta j} v^{\beta}(x)\right] dx^j \ (\lambda = 1, \ldots\ldots, N), \tag{47}$$

where $\bar{B}^{\lambda}_{\alpha}(v(x), x)$ are defined by

$$\bar{B}^{\lambda}_{\alpha}(v, x) B^{\alpha}_{\mu}(v, x) = \delta^{\lambda}_{\mu}$$

in which $B^{\alpha}_{\mu}(v, x)$ are the functions which we have defined in the equations (8).

By virtue of the significance of (46) and (47), we call

$$\bar{A}^{\lambda}_{\alpha}\big(v(x), x\big)\left[\frac{\partial v^{\alpha}(x)}{\partial x^j} + \Gamma^{\alpha}_{\beta j} v^{\beta}(x)\right]$$

and

$$\overline{B}^{\lambda}_{\alpha}\Big(v(x),\, x\Big)\left[\frac{\partial v^{\alpha}(x)}{\partial x^{j}}+\Gamma^{\alpha}_{\beta j}v^{\beta}(x)\right]$$

the *covariant derivatives of the first and second kind* of a vector $v^{\lambda}(x)$ respectively

§ 6. Curvature of the space.

We proceed to find the *curvature* of our space. For this purpose we consider an infinitesimal circuit comprising the points $P(x)$, $Q(x+d_1x)$, $R(x+d_1x+d_2x)$, $S(x+d_2x)$, and P, and let a vector a^{λ} at a point $P(x)$, be parallel to vectors v_1^{λ} and v_2^{λ} at the point $R(x+d_1x+d_2x)$ along the curves PQR and PSR respectively. We can prove by the usual method the following relations:

$$v_1^{\lambda}-v_2^{\lambda} = R^{\lambda}_{\alpha ij}v^{\alpha}\,d_1x^{i}\,d_2x^{j}$$

where $R^{\lambda}_{\alpha ij}$ are defined by

$$R^{\lambda}_{\alpha ij} = \frac{\partial\Gamma^{\lambda}_{\alpha j}}{\partial x^{i}}-\frac{\partial\Gamma^{\lambda}_{\alpha i}}{\partial x^{j}}+\Gamma^{\beta}_{\alpha j}\Gamma^{\lambda}_{\beta i}-\Gamma^{\beta}_{\alpha i}\Gamma^{\lambda}_{\beta j}\,.$$

Hence if we denote by $\Delta_1 v^{\lambda}$ and $\Delta_2 v^{\lambda}$, the differences of the first and second kind between v_1^{λ} and v_2^{λ}, we have, by the same method as was used in obtaining (46) and (47),

$$\Delta_1 v^{\lambda} = \overline{A}^{\lambda}_{\alpha}(v,\, x)R^{\alpha}_{\beta ij}v^{\beta}\,d_1x^{i}\,d_2x^{j} \tag{48}$$

and

$$\Delta_2 v^{\lambda} = \overline{B}^{\lambda}_{\alpha}(v,\, x)R^{\alpha}_{\beta ij}v^{\beta}\,d_1x^{i}\,d_2x^{j}\,\cdot \tag{49}$$

By virtue of (48) and (49), we call the right hand sides of (48) and (49) the *curvatures of the first and second kind* respectively.

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