On Relative Tensors.

By

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1. **Introduction.** In this paper, I shall treat the so-called relative scalars and relative tensors, and their covariant derivatives. Several mathematicians have already written on the properties of these quantities[(1)], but as their results appear to me to be unsatisfactory, I will make a more detailed study of the subject with a view to future applications.

In an n-dimensional space the equations

$$'x^{\nu} = 'x^{\nu}(x^1, \ldots\ldots, x^n), \quad \Delta \equiv \left| \frac{\partial' x^{\nu}}{\partial x^{\lambda}} \right| \neq 0,$$

$$(\alpha, \beta, \ldots\ldots, \lambda, \mu, \ldots\ldots = 1, \ldots\ldots, n) \quad (1.1)$$

define a transformation of coördinates. If $\overset{(m)}{v}$ and $'\overset{(m)}{v}$ are functions of the x's and $'x$'s such that

$$'\overset{(m)}{v} = \Delta^m \overset{(m)}{v} \quad (1.2)$$

in consequence of (1.1), $\overset{(m)}{v}$ and $'\overset{(m)}{v}$ are the components of a *relative scalar of weight* m in the respective coördinate system, where m is any real number. In the same way we can define a relative tensor of weight m by the transformation law

$$'\overset{(m)}{R}{}^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} = \Delta^m P^{\alpha_1}_{\gamma_1} \cdots P^{\alpha_r}_{\gamma_r} Q^{\delta_1}_{\beta_1} \cdots Q^{\delta_s}_{\beta_s} \overset{(m)}{R}{}^{\gamma_1 \cdots \gamma_r}_{\delta_1 \cdots \delta_s} \quad (1.3)$$

where $$P^{\alpha}_{\beta} = \frac{\partial' x^{\alpha}}{\partial x^{\beta}}, \qquad Q^{\alpha}_{\beta} = \frac{\partial x^{\alpha}}{\partial' x^{\beta}}.$$

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(1) For example, O. Veblen, *Invariants of Quadratic Differential Forms* (1929); Jour. London Math. Soc. **4**, (1929), 140–160.

If $m = 0$, the scalar and tensor are said to be *absolute*. In either case we indicate their weights by parenthesized indexes as mentioned above, but when $m = 0$ the indexes are omitted in the ordinary manner.

2. **Metric Property.** In the usual manner we define the *measure* of absolute vectors and the *angle* between two absolute vectors with a symmetric tensor $g_{\lambda\mu}$ whose rank is n as the fundamental tensor.

If we put $|g_{\lambda\mu}| = \overset{(-2)}{g}$, $\overset{(-2)}{g}$ is a relative scalar of weight -2. Therefore $\overset{(m)}{g} = (\overset{(-2)}{g})^{-\frac{m}{2}}$ is a relative scalar of weight m, and $\overset{(m)}{g}_{\lambda\mu} = \overset{(m)}{g}\, g_{\lambda\mu}$ is a relative tensor of weight m. $\overset{(m)}{g}$ and $\overset{(m)}{g}_{\lambda\mu}$ are called the *fundamental relative scalars and tensors* respectively. If $g^{\lambda\mu}$ is the reciprocal tensor of $g_{\lambda\mu}$ we can prove that $\overset{(-m)}{g}{}^{\lambda\mu}$, the reciprocal of $\overset{(m)}{g}_{\lambda\mu}$, is equal to $\overset{(-m)}{g}\, g^{\lambda\mu}$. In particular $\overset{(0)}{g} = 1$.

We define the *measure* of a relative scalar $\overset{(m)}{v}$ by the equation

$$|\overset{(m)}{v}| = \overset{(-m)}{g}\, \overset{(m)}{v}\,. \tag{2.1}$$

In particular, the measure of all the fundamental scalars is 1, because $|\overset{(m)}{g}| = \overset{(-m)}{g}\,\overset{(m)}{g} = 1$. In the same way the *measure* of a relative vector $\overset{(m)}{v}{}^\nu$ is defined by

$$|\overset{(m)}{v}{}^\nu|^2 = \overset{(-2m)}{g}_{\lambda\mu}\, \overset{(m)}{v}{}^\lambda\, \overset{(m)}{v}{}^\mu = \overset{(-m)}{v}_\mu\, \overset{(m)}{v}{}^\mu = \overset{(2m)}{g}{}^{\lambda\mu}\, \overset{(-m)}{v}_\lambda\, \overset{(-m)}{v}_\mu\,. \tag{2.2}$$

When $m = 0$ (2.2) coincides with the ordinary expression.

$\overset{(-m)}{v}_\lambda = \overset{(-2m)}{g}_{\lambda\mu}\, \overset{(m)}{v}{}^\mu$ is called the *conjugate* vector of $\overset{(m)}{v}{}^\nu$.

If we put $\overset{(m)}{v}{}^\nu = \alpha\, v^\nu$ and $\overset{(l)}{w}{}^\nu = \beta\, w^\nu$, the vectors $\overset{(m)}{v}{}^\nu$ and v^ν are in the same direction. The *angle* between $\overset{(m)}{v}{}^\nu$ and $\overset{(l)}{w}{}^\nu$ can be defined by

$$\cos(\overset{(m)}{v}{}^\nu, \overset{(l)}{w}{}^\nu) = \cos(v^\nu, w^\nu) = g_{\lambda\mu} v^\lambda w^\mu / \sqrt{g_{\lambda\mu} v^\lambda v^\mu} \cdot \sqrt{g_{\lambda\mu} w^\lambda w^\mu}\,,$$

while we can easily show that this agrees with the ordinary expressions:

$$\cos(\overset{(m)}{v}{}^\nu, \overset{(l)}{w}{}^\nu) = \overset{(-m-l)}{g}_{\lambda\mu}\, \overset{(m)}{v}{}^\lambda\, \overset{(l)}{w}{}^\mu \Big/ \sqrt{\overset{(-2m)}{g}_{\lambda\mu}\, \overset{(m)}{v}{}^\nu\, \overset{(m)}{v}{}^\mu} \cdot \sqrt{\overset{(-2l)}{g}_{\lambda\mu}\, \overset{(l)}{w}{}^\lambda\, \overset{(l)}{w}{}^\mu}$$

Further the equations

$$|\overset{(m)}{v}\,\overset{(l)}{w}| = \overset{(-m-l)}{g}\, \overset{(m)}{v}\,\overset{(l)}{w} = |\overset{(m)}{v}|\,|\overset{(l)}{w}|$$

and

$$|\overset{(m)}{v}\,\overset{(l)}{w}{}^\nu|^2 = \overset{(-2(m+l))}{g}_{\lambda\mu}\, \overset{(m)}{v}\,\overset{(l)}{w}{}^\lambda\, \overset{(m)}{v}\,\overset{(l)}{w}{}^\mu = |\overset{(m)}{v}|^2\, |\overset{(l)}{w}{}^\nu|^2$$

lead to the result: *The measure of the product of relative scalars, and that of relative scalar and vector, are equal to the product of the measure of each factor.*

From this result, unit relative vectors $\overset{(m)}{e}{}^{\nu}$ $(|\overset{(m)}{e}{}^{\nu}|^2=1)$ are obtained by multiplying the ordinary unit vectors by $\overset{(m)}{g}$.

3. **Covariant Differentiation.** As in the case of absolute tensors and scalars, we may consider the covariant differentiation of relative tensors and scalars. Here we make use of the system of axioms used by Schouten.[1] Then the covariant differentials of absolute vectors based on these axioms, assume the usual form:

$$\delta v^{\nu} = dv^{\nu}+\Gamma^{\nu}_{\lambda\mu}v^{\lambda}dx^{\mu}, \quad \delta w_{\lambda} = dw_{\lambda}-\Gamma'^{\nu}_{\lambda\mu}w_{\nu}dx^{\mu}.$$

where $\Gamma^{\nu}_{\lambda\mu}$ and $\Gamma'^{\nu}_{\lambda\mu}$ are arbitrary functions of the x's. Corresponding transformation laws are as follows:

$$'\Gamma^{\kappa}_{\omega\pi} = Q^{\lambda}_{\omega}Q^{\mu}_{\pi}P^{\kappa}_{\nu}\Gamma^{\nu}_{\lambda\mu}+\frac{\partial Q^{\nu}_{\omega}}{\partial x^{\mu}}Q^{\mu}_{\pi}P^{\kappa}_{\nu}, \quad '\Gamma'^{\kappa}_{\omega\pi} = Q^{\lambda}_{\omega}Q^{\mu}_{\pi}P^{\kappa}_{\nu}\Gamma'^{\nu}_{\lambda\mu}+\frac{\partial Q^{\nu}_{\omega}}{\partial x^{\mu}}Q^{\mu}_{\pi}P^{\kappa}_{\nu}. \tag{3.1}$$

If we attempt to obtain the covariant differential of a relative scalar $\overset{(m)}{v}$, we shall find

$$\delta\overset{(m)}{v} = d\overset{(m)}{v}+m\overset{(m)}{v}\Delta_{\alpha}dx^{\alpha}, \tag{3.2}$$

where Δ_{α} is any function of the x's[2], the transformation law of which is

$$'\Delta_{\tau} = Q^{\alpha}_{\tau}\left(\Delta_{\alpha}-\frac{\partial\log\Delta}{\partial x^{\alpha}}\right). \tag{3.3}$$

So Δ_{μ} is not a vector. The general rule for covariant differentiation of relative tensor is as follows:

$$\begin{aligned}\delta\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} &= d\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}+m\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\Delta_{\rho}dx^{\rho}\\ &+\sum_{u}^{1,\ldots,r}\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_{u-1}\tau\alpha_{u+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s}\Gamma^{\alpha_u}_{\tau\rho}dx^{\rho}\\ &-\sum_{u}^{1,\ldots,s}\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{u-1}\tau\beta_{u+1}\ldots\beta_s}\Gamma'^{\tau}_{\beta_u\rho}dx^{\rho}.\end{aligned} \tag{3.4}$$

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(1) J. A. Schouten, *Der Ricci-Kalkül.* (1924), 63.

(2) J. A. Schouten and V. Hlavatý obtained these formulas from a general point of view. Where we use Δ as multiplier, they take any arbitrary function of the x's. (Math. Zeitschr. **30** (1929), 414-432).

If $\delta\Phi=0$ holds for a point $P(x)$ and line-element (dx), Φ at $Q(x+dx)$ is said to be obtained by parallel displacement from P to Q.

4. Various Kinds of Tensors Characteristic of Space. Let us enumerate those important tensors described in "Der Ricci-Kalkül".

$$C_{\mu\lambda}^{\cdot\cdot\nu} = \Gamma_{\lambda\mu}^{\nu} - \Gamma_{\lambda\mu}^{\prime\nu} = \nabla_{\mu} A_{\mu}^{\nu}, \quad S_{\lambda\mu}^{\cdot\cdot\nu} = \Gamma_{[\lambda\mu]}^{\nu}, \quad S'^{\cdot\cdot\nu}_{\lambda\mu} = \Gamma_{[\lambda\mu]}^{\prime\nu},$$

$$\nabla_{\mu} g^{\lambda\nu} = Q_{\mu}^{\cdot\lambda\nu}, \quad \nabla_{\mu} g_{\lambda\nu} = Q'_{\mu\lambda\nu}, \quad T_{\lambda\mu}^{\cdot\cdot\nu} = \Gamma_{\lambda\mu}^{\nu} - \left\{ {\nu \atop \lambda\mu} \right\}. \tag{4.1}$$

In addition to these we may consider certain new tensors characterizing the space by the introduction of Λ_{μ}. Contracting (3.1) for κ and ω we have

$$'\Gamma_{\alpha\pi}^{\alpha} = Q_{\pi}^{\mu}\left(\Gamma_{\alpha\mu}^{\alpha} - \frac{\partial \log \Delta}{\partial x^{\mu}}\right), \quad '\Gamma_{\alpha\pi}^{\prime\alpha} = Q_{\pi}^{\mu}\left(\Gamma_{\sigma\mu}^{\prime\alpha} - \frac{\partial \log \Delta}{\partial x^{\mu}}\right), \tag{4.2}$$

that is, the transformation laws of $\Gamma_{\alpha\mu}^{\alpha}$ and $\Gamma_{\alpha\mu}^{\prime\alpha}$ coincide with that of Λ_{μ}. Hence if we put

$$\Gamma_{\alpha\pi}^{\alpha} - \Lambda_{\pi} = D_{\pi} \quad \text{and} \quad \Gamma_{\alpha\pi}^{\prime\alpha} - \Lambda_{\pi} = D'_{\pi}, \tag{4.3}$$

D_{π} and D'_{π} are vectors peculiar to the space.

Next, expressing the condition of integrability of the equation (3.3), the transformation law of Λ_{μ}, we have

$$'M_{\rho\sigma} P_{\lambda}^{\rho} P_{\mu}^{\sigma} = M_{\lambda\mu},$$

where

$$M_{\lambda\mu} = \partial_{[\mu} \Lambda_{\lambda]} = \frac{1}{2}\left(\frac{\partial \Lambda_{\lambda}}{\partial x^{\mu}} - \frac{\partial \Lambda_{\mu}}{\partial x^{\lambda}}\right). \tag{4.4}$$

Hence the rotation of Λ_{μ} is a tensor.

$\Gamma_{\lambda\mu}^{\nu}$ and $\Gamma_{\lambda\mu}^{\prime\nu}$ are determined by the tensors $C_{\mu\lambda}^{\cdot\cdot\nu}$, $S_{\lambda\mu}^{\cdot\cdot\nu}$, $g_{\lambda\mu}$ and $Q_{\mu}^{\cdot\lambda\nu}$ as shown by J. A. Schouten. Then from (4.3) we can deduce that Λ_{μ} is also determined if vector D_{π} is given. Consequently *the properties of the space are completely determined by the tensors* $C_{\mu\lambda}^{\cdot\cdot\nu}$, $S_{\lambda\mu}^{\cdot\cdot\nu}$, $g_{\lambda\mu}$, $Q_{\mu}^{\cdot\lambda\nu}$ *and* D_{π}.

5. Field of Parallel Relative Scalar. From (3.2) it follows that when a function $\overset{(m)}{g}$ satisfies the equations

$$\nabla_{\mu} \overset{(m)}{g} = \frac{\partial \overset{(m)}{g}}{\partial x^{\mu}} + m \overset{(m)}{g} \Lambda_{\mu} = 0, \tag{5.1}$$

any two scalars of the scalar field are parallel. The condition of integrability of these equations is

$$M_{\mu\lambda} = 0 \, . \tag{5.2}$$

Hence: *A necessary and sufficient condition for the existence of the field of parallel relative scalar is (5.2).*

We can express the same condition in another form. Let $\overset{(m)}{g}$ be any field of parallel scalar, that is $\delta \overset{(m)}{g} = 0$ $(m \neq 0)$, then $\overset{(-2)}{g} = (\overset{(m)}{g})^{-\frac{2}{m}}$ is a field of parallel relative scalar of weight -2. Let $f_{\lambda\mu}$ be an arbitrary symmetric tensor such that $|f_{\lambda\mu}| = \overset{(-2)}{f} \neq 0$, and $p = (\overset{(-2)}{g}/\overset{(-2)}{f})^{\frac{1}{n}}$ is an absolute scalar. If $g_{\lambda\mu}$ denotes the tensor $p\, f_{\lambda\mu}$ we shall find that the determinant $|g_{\lambda\mu}| = p^n |f_{\lambda\mu}| = \overset{(-2)}{g}$. In other words, if there exists any field of parallel relative scalar, we can find a tensor $g_{\lambda\mu}$ such that its determinant is a parallel scalar field. From the equation (5.1) in which $m = -2$ it follows that

$$\Lambda_{\mu} = \left\{ {}^{\alpha}_{\alpha\mu} \right\}_{(g_{\lambda\mu})} \, . \tag{5.3}$$

Obviously the condition thus obtained is sufficient. Hence: *A necessary and sufficient condition for the existence of the field of parallel relative scalar, whose weight is not 0, is that there exists a symmetric tensor $g_{\lambda\mu}$ satisfying the condition (5.3).*

Where this condition is satisfied, if we take $g_{\lambda\mu}$ above-mentioned as the fundamental tensor, from the equation

$$\delta(\overset{(-m)}{g}\overset{(m)}{v}) = \overset{(-m)}{g}\,\delta \overset{(m)}{v} \, ,$$

it follows that the measure of relative scalar remains unchanged by our parallel displacement. Conversely, for the given fundamental tensor $g_{\lambda\mu}$, if parallel relative scalars are equal in measure, we have $\delta \overset{(-m)}{g} = 0$, from which (5.3) for given $g_{\lambda\mu}$ is obtained. Hence we have the result: *The system of equations (5.3) is the necessary and sufficient condition that the measures of relative scalars remain unchanged by their parallel displacement.*

Obviously the contracted Christoffel symbol $\{ {}^{\alpha}_{\alpha\mu} \}$ is transformed in the same way as Λ_{μ}, so the equation (5.3) has the significance independent of the coördinate system, and in fact (5.3) can be replaced by tensor equations

$$D_{\mu} = T_{c\mu}^{\cdot\cdot\,\alpha} \, .$$

6. Geometry of Riemannian Type. The characteristic equations of Riemannian geometry are

$$\delta g_{\lambda\mu} = 0\ ,\ \ C_{\mu\lambda}^{\cdot\cdot\nu} = 0\ ,\ \ S_{\lambda\mu}^{\cdot\cdot\nu} = 0\ ,\ \text{i.e.}\ \ \Gamma^{\nu}_{\lambda\mu} = \{{}^{\nu}_{\lambda\mu}\}\ .$$

When $\delta g_{\lambda\mu} = 0$, let us consider the change of measure of relative vector $\overset{(m)}{v^{\nu}}$ produced by our parallel displacement. Since

$$\delta\,|\overset{(m)}{v^{\nu}}|^2 = m\,|\overset{(m)}{v^{\nu}}|^2\,\overset{(2)}{g}\ \delta\,\overset{(-2)}{g}\ , \tag{6.1}$$

$|\overset{(m)}{v^{\nu}}|$ remains unchanged if $\delta\overset{(-2)}{g} = 0$; otherwise $|\overset{(m)}{v^{\nu}}|$ remains unchanged only when $m = 0$, that is, when $\overset{(m)}{v^{\nu}}$ is an absolute vector. Hence, if the measure of any relative vector does not change, we have (5.3) from the condition $\delta\overset{(-2)}{g} = 0$.

Now, let us take the case in which the measure of vector changes, but the ratio of its components does not change as is the case with the space of Weyl. That is,

$$\delta\,|\overset{(m)}{v^{\nu}}|^2 = \alpha\,|\overset{(m)}{v^{\nu}}|^2 \quad \text{when} \quad \delta\,\overset{(m)}{v^{\nu}} = 0\ , \tag{6.2}$$

where α is an infinitesimal scalar of the same order as dx^{ν}. Hence, if we put $\alpha = -2m\,\varphi_{\alpha}\,dx^{\alpha}$, where φ_{α} is a covariant vector, we have from (6.1) and (6.2)

$$\Lambda_{\iota} = \{{}^{\beta}_{\beta\alpha}\} + \varphi_{\alpha} \qquad \text{or} \qquad D_{\alpha} = -\varphi_{\alpha}\ . \tag{6.3}$$

Conversely if (6.3) holds, it is evident that (6.2) holds. Hence: *In the space of Riemannian type where (6.3) holds, the change of measure of vector of weight m by parallel displacement is equal to the result obtained by replacing Q_{α} by $2m\,\varphi_{\alpha}$ in the change of the absolute vector in the space of Weyl, where $\delta g_{\lambda\mu} = -Q_{\alpha}\,g_{\lambda\mu}\,dx^{\alpha}$.*

7. Geometry of Weyl Type. The characteristic equations of Weyl geometry are

$$\delta g_{\lambda\mu} = -Q_{\sigma}\,g_{\lambda\mu}\,dx^{\alpha}\ ,\quad C_{\mu\lambda}^{\cdot\cdot\nu} = 0\ ,\quad S_{\mu\lambda}^{\cdot\cdot\nu} = 0\ , \tag{7.1}$$

where Q_{α} is a covariant vector. In this case

$$\Gamma^{\nu}_{\lambda\mu} = \{{}^{\nu}_{\lambda\mu}\} + \frac{1}{2}(Q_{\mu}\,A^{\nu}_{\lambda} + Q_{\lambda}\,A^{\nu}_{\mu} - Q_{\alpha}\,g^{\alpha\nu}\,g_{\lambda\mu})\ . \tag{7.2}$$

Contracting (7.2) for λ and ν, we have

$$\Gamma^{\alpha}_{\alpha\mu} = \{ {}^{\alpha}_{\alpha\mu} \} + \frac{n}{2} Q_{\mu} . \tag{7.3}$$

Now let us consider the case of the parallel displacement characterized by (7.1). Then the measure of $\overset{(m)}{v}{}^{\nu}$ changes as follows:

$$\delta |\overset{(m)}{v}{}^{\nu}|^2 = |\overset{(m)}{v}{}^{\nu}|^2 \Big[2m\Big(\{ {}^{\beta}_{\beta\alpha} \} - A \Big) - Q_{\alpha} \Big] dx^{\alpha} . \tag{7.4}$$

In the special case in which $A_{\alpha} = \{ {}^{\beta}_{\beta\alpha} \}$ or $D_{\alpha} = T_{\beta\alpha}^{\cdot\cdot\beta}$, as is seen from (7.4), corresponding to (7.1), we have

$$\delta \overset{(-2m)}{g}_{\lambda\mu} = -Q_{\alpha} \overset{(-2m)}{g}_{\lambda\mu} dx^{\alpha} .$$

In this case the measure of $\overset{(m)}{v}{}^{\nu}$ undergoes the same change as that of absolute vector. Let us consider another special case in which

$$\Gamma^{\alpha}_{\alpha\mu} = A_{\mu} = \{ {}^{\alpha}_{\alpha\mu} \} + \frac{n}{2} Q_{\mu} .$$

From (7.4) we have

$$\delta |\overset{(m)}{v}{}^{\nu}|^2 = -(mn+1) Q_{\alpha} dx^{\alpha} |\overset{(m)}{v}{}^{\nu}|^2 .$$

Hence: *In the space of Weyl type, if we take* $\{ {}^{\alpha}_{\alpha\mu} \}$ as A_{μ}, *the change of the measure of vector of weight m is the same as that of the absolute vector, and, if we take* $\Gamma^{\alpha}_{\alpha\mu}$ *as* A_{μ}, *the change is equal to the result obtained by replacing* Q_{α} *by* $(mn+1)$ Q_{α} *in the change of the absolute vector.*

8. **Ricci Identity.** From the equation (3.4), we have

$$\overset{(m)}{\Gamma}_{[\mu} \overset{(m)}{w}_{\lambda]} = \partial_{[\mu} \overset{(m)}{w}_{\lambda]} + m \overset{(m)}{w}_{[\lambda} A_{\mu]} - S'{}_{\lambda\mu}^{\cdot\cdot\nu} \overset{(m)}{w}_{\nu} . \tag{8.1}$$

Hence the rotation of a covariant vector, the weight of which is not 0, is not a tensor different from the ordinary case. Even in the case of a symmetric connection the rotation of w_{λ} $(m \neq 0)$ formed with respect to Γ_{μ} is not equal to that formed with respect to ∂_{μ}. As a special case when $\overset{(m)}{w}_{\lambda} = \overset{(m)}{\Gamma}_{\lambda} \varphi = \overset{(m)}{\partial}_{\lambda} \varphi + m \overset{(m)}{\varphi} A_{\lambda}$, (8.1) becomes

$$\Gamma_{[\mu} \Gamma_{\lambda]} \overset{(m)}{\varphi} = m \overset{(m)}{\varphi} M_{\lambda\mu} - S'{}_{\lambda\mu}^{\cdot\cdot\nu} \Gamma_{\nu} \overset{(m)}{\varphi} . \tag{8.2}$$

Hence in a symmetric connection, when and only when $M_{\lambda\mu}=0$, we can change the order of differentiation in the second covariant derivatives of a relative scalar.

From axioms in §3 we have

$$\nabla_{[\omega}\nabla_{\mu]}(\Phi+\Psi)=\nabla_{[\omega}\nabla_{\mu]}\Phi+\nabla_{[\omega}\nabla_{\mu]}\Psi\,,$$

and

$$\nabla_{[\omega}\nabla_{\mu]}\Phi\Psi=(\nabla_{[\omega}\nabla_{\mu]}\Phi)\Psi+\Phi\nabla_{[\omega}\nabla_{\mu]}\Psi\,,$$

that is, the operator $\nabla_{[\omega}\nabla_{\mu]}$ is under the same laws as ∇_{ω}. By using this characteristic we can easily obtain the following equation, the generalization of Ricci identity,

$$\begin{aligned}\nabla_{[\omega}\nabla_{\mu]}\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}=&\frac{1}{2}\sum_{u}^{1,\ldots,s}R'{}_{\omega\mu\beta_u}^{\cdot\cdot\cdot\;\;\;\delta}\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{u-1}\delta\beta_{u+1}\ldots\beta_s}\\&-\frac{1}{2}\sum_{u}^{1,\ldots,r}R_{\omega\mu\delta}^{\cdot\cdot\cdot\;\;\alpha_u}\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_{u-1}\delta\alpha_{u+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s}\\&+m\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}M_{\mu\omega}+S'{}_{\omega\mu}^{\cdot\cdot\;\delta}\nabla_\delta\overset{(m)}{R}{}^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}\,.\end{aligned}\tag{8.3}$$

From this equation we have: *In a space of covariant symmetric connection, if Λ_μ is changed by*

$$\bar{\Lambda}_\alpha=\Lambda_\alpha+\varphi_\alpha\,,\tag{8.4}$$

where φ_α is a vector, the necessary and sufficient condition is that φ_α is a gradient of an absolute scalar, in order that the change, which is brought about when any relative tensor undergoes a parallel displacement along any infinitesimal circuit, should remain unaltered.

9. Transformation which does not alter the Parallelism of v^ν. As is easily seen from the transformation laws (3.1) and (3.3), the necessary and sufficient condition is that φ_λ and $\varphi^\nu_{\lambda\mu}$ should be the tensors of the nature shown by their indexes, in order that

$$\bar{\Lambda}_\mu=\Lambda_\mu-\varphi_\mu\tag{9.1}$$

and

$$\bar{\Gamma}^\nu_{\lambda\mu}=\Gamma^\nu_{\lambda\mu}+\varphi^\nu_{\lambda\mu}\tag{9.2}$$

may be used again as coefficients of connection. In what follows, let us consider the simultaneous transformation of Λ_μ and $\Gamma^\nu_{\lambda\mu}$ which does not bring any change to the parallelism of any $\overset{(m)}{v}{}^\nu$ for the fixed m. The condition is

$$\bar{\delta}\overset{(m)}{v}{}^\nu=0\qquad\text{when}\qquad\delta\overset{(m)}{v}{}^\nu=0\,.\tag{9.3}$$

From these equations we obtain

$$m\Lambda_\alpha A_\lambda^\nu + \Gamma_{\lambda\alpha}^\nu = m\bar{\Lambda}_\alpha A_\lambda^\nu + \bar{\Gamma}_{\lambda\alpha}^\nu ,$$

therefore
$$\bar{\Gamma}_{\lambda\mu}^\nu = \Gamma_{\lambda\mu}^\nu + m\varphi_\mu A_\lambda^\nu . \tag{9.4}$$

Hence: *The general transformation of the coefficients of connection which does not give any change to the parallelism in the strict sense, of relative vectors of weight m is given by*

$$\bar{\Lambda}_\mu = \Lambda_\mu - \varphi_\mu , \quad \bar{\Gamma}_{\lambda\mu}^\nu = \Gamma_{\lambda\mu}^\nu + m\varphi_\mu A_\lambda^\nu , \tag{9.5}$$

where φ_λ is any vector.

By parallelism in the strict sense of the word I mean the parallelism defined by $\delta \overset{(m)}{v^\nu} = 0$, while on the other hand the parallelism in a wide sense is the case in which the ratio of components, or direction, alone is considered; analyticaly $\delta \overset{(m)}{v^\nu} = \alpha \overset{(m)}{v^\nu}$. In the case of symmetric connection, since $\varphi_\mu A_\lambda^\nu$ is not symmetric, such transformation (9.5) as preserves the symmetricity does not exist.

In other words the above result may be stated as follows: *In the transformation of the coefficients of connection above-mentioned, if $\bar{\Lambda}_\mu$ is given by (9.1), the symmetric part of $\Gamma_{\lambda\mu}^\nu$ undergoes a projective change in which $\frac{m}{2}\varphi_\mu$ is the vector of transformation.*

Next, when we consider the transformation which does not change the parallelism in a wide sense, we must deal with the equations

$$\bar{\delta} \overset{(m)}{v^\nu} = \alpha \overset{(m)}{v^\nu} = p_\alpha dx^\alpha \overset{(m)}{v^\nu} \quad \text{when} \quad \delta \overset{(m)}{v^\nu} = 0 , \tag{9.6}$$

in which p_α stands for any vector, in place of (9.3). In the same way as before we have: *The general transformatian of the coefficients of connection which does not give any change to the parallelism of $\overset{(m)}{v^\nu}$ in a wide sense, is given by*

$$\left.\begin{aligned} \bar{\Lambda}_\lambda &= \Lambda_\lambda - \varphi_\lambda , \\ \bar{\Gamma}_{\lambda\mu}^\nu &= \Gamma_{\lambda\mu}^\nu + (p_\mu + m\varphi_\mu) A_\lambda^\nu , \end{aligned}\right\} \tag{9.7}$$

where φ_α and p_α are any vectors.

In this case the symmetric part of $\Gamma^{\nu}_{\lambda\mu}$ undergoes a projective change in which $\frac{1}{2}(m\varphi_{\mu}+p_{\mu})$ is the vector of transformation.

In what follows let us consider the tensors which are invariant under the transformations (9.5) and (9.7). Since both (9.5) and (9.7) are of the same form as that of the projective transformation with respect to $\Gamma^{\nu}_{(\lambda\mu)}$, $\Pi^{\nu}_{\lambda\mu}$, defined by

$$\Pi^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{(\lambda\mu)} - \frac{1}{n+1}(A^{\nu}_{\lambda}\Gamma^{\alpha}_{(\alpha\mu)} + A^{\nu}_{\mu}\Gamma^{\alpha}_{(\alpha\lambda)}) ,$$

is invariant under (9.5) and (9.7). Therefore it goes without saying that the Weyl tensor formed with respect to $\Pi^{\nu}_{\lambda\mu}$, thus defined, in place of the ordinary coefficients of projective connection, is invariant under both (9.5) and (9.7).

10. **Tensors** $\underset{m}{W}{}_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu}$. In this section let us consider the case of parallelism in the strict sense only. If we put

$$\underset{m}{\Phi}{}^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\lambda\mu} + mA_{\mu}A^{\nu}_{\lambda} \tag{10.1}$$

as is readily seen, $\underset{m}{\Phi}{}^{\nu}_{\lambda\mu}$ is invariant under (9.5); hence its symmetric part and anti-symmetric part are also invariant. Let us put

$$\underset{m}{\Sigma}{}^{\nu}_{\lambda\mu} = \underset{m}{\Phi}{}^{\nu}_{(\lambda\mu)} = \Gamma^{\nu}_{(\lambda\mu)} + \frac{m}{2}(A^{\nu}_{\lambda}A_{\mu} + A^{\nu}_{\mu}A_{\lambda}) . \tag{10.2}$$

The equations of transformations of $\underset{m}{\Sigma}{}^{\nu}_{\lambda\mu}$ are obtained from (3.1) and (3.3), as follows

$$\frac{\partial^{2\prime}x^{\alpha}}{\partial x^{\lambda}\partial x^{\mu}} = \underset{m}{\Sigma}{}^{\tau}_{\lambda\mu}\frac{\partial' x^{\alpha}}{\partial x^{\tau}} - {}'\underset{m}{\Sigma}{}^{\alpha}_{\beta\tau}\frac{\partial' x^{\beta}}{\partial x^{\lambda}}\frac{\partial' x^{\tau}}{\partial x^{\mu}} + \frac{\partial' x^{\alpha}}{\partial x^{\mu}}\frac{\partial\theta'}{\partial x^{\lambda}} + \frac{\partial' x^{\alpha}}{\partial x^{\lambda}}\frac{\partial\theta'}{\partial x^{\mu}} \tag{10.3}$$

where
$$\theta' = -\frac{m}{2}\log\Delta = -\frac{m(n+1)}{2}\theta . \tag{10.4}$$

The equation (10.3) is obtained by replacing θ by θ' in the transformation of the coefficients of projective connection. Hence we can go on discussing the problem in just the same way as L. P. Eisenhart did in his work.[1]

(1) L. P. Eisenhart, *Non-Riemannian Geometry* (1927), 98-100.

That is, $\underset{m}{W}{}_{\dot\omega\dot\mu\dot\lambda}{}^{\nu}$, *which is defined by*

$$\underset{m}{W}{}_{\dot\omega\dot\mu\dot\lambda}{}^{\nu} = \underset{m}{\Sigma}{}_{\dot\omega\dot\mu\dot\lambda}{}^{\nu} + \frac{1}{n-1}\left(A_{\mu}^{\nu}\underset{m}{\Sigma}{}_{\lambda\omega} - A_{\omega}^{\nu}\underset{m}{\Sigma}{}_{\lambda\mu}\right) \tag{10.5}$$

where by definition

$$\underset{m}{\Sigma}{}_{\dot\omega\dot\mu\dot\lambda}{}^{\nu} = \frac{\partial}{\partial x^{\mu}}\underset{m}{\Sigma}{}_{\lambda\omega}^{\nu} - \frac{\partial}{\partial x^{\omega}}\underset{m}{\Sigma}{}_{\lambda\mu}^{\nu} + \underset{m}{\Sigma}{}_{\lambda\omega}^{\rho}\underset{m}{\Sigma}{}_{\rho\mu}^{\nu} - \underset{m}{\Sigma}{}_{\lambda\mu}^{\rho}\underset{m}{\Sigma}{}_{\rho\omega}^{\nu}$$

and $$\underset{m}{\Sigma}{}_{\lambda\mu} = \underset{m}{\Sigma}{}_{\dot\alpha\dot\mu\dot\lambda}{}^{\alpha},$$

is a tensor invariant under the transformation (9.5).

If we seek the relation between $\underset{m}{W}{}_{\dot\omega\dot\mu\dot\lambda}{}^{\nu}$ and $R_{\dot\omega\dot\mu\dot\lambda}{}^{\nu}$ after the fashion of Eisenhart, we shall obtain the following expression.

$$\underset{m}{W}{}_{\dot\omega\dot\mu\dot\lambda}{}^{\nu} = R_{\dot\omega\dot\mu\dot\lambda}{}^{\nu} - mA_{\lambda}^{\nu}M_{\mu\omega} + \frac{2}{n-1}A_{[\mu}^{\nu}R_{\omega]\lambda} - \frac{2m}{n-1}M_{\lambda[\mu}A_{\omega]}^{\nu}.$$

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