On Parallel Displacements in an n-dimensional Space to which N-dimensional General Vector Spaces are Attached.

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Introduction.

We begin by defining in § 1 the most general parallel displacement in the general manifold and obtain the condition for the existence of the *symmetrized coordinate system* in this space. The vector space at every point is denoted by $V(x, y)$ where each vector is represented by a point. In § 2 and § 3 we consider the case in which this parallel displacement admits of the coordinate transformations of the *base* space and *vector* space. In § 4 and § 5 we discuss a parallel displacement of linear elements and curves in $V(x, y)$ arising from our parallel displacement. In the case of the curve, we shall see how systems of curves are displaced by the parallel displacement and shall specialize the problem for the case in which the systems of curves become γ-ple linear systems. And thus we define the projective and affine groups from the general point of view.

Finally in § 6, taking each curve in $V(x, y)$ as an element, we consider the *parallel displacement of curves*.

§ 1. Let us suppose that a vector space $(x^1, x^2, \ldots\ldots, x^N)$ of N-dimensions is attached to each point of the base space $(y^1, y^2, \ldots\ldots y^n)$ of n-dimensions. We express this vector space by $V(x, y)$ and the base space by $Z(y)$.

According to the method which Cartan introduced when he considered his parallel displacement, we here consider a set of all the coordinate transformations in the vector space which make certain properties in $V(x, y)$ invariant

$$\bar{x}^\alpha = f^\alpha(x^\beta, a^p, y^i)^{(1)} \quad \begin{pmatrix} \alpha, \beta, \ldots\ldots = 1, 2, \ldots\ldots, N, \\ i, j \ldots\ldots = 1, 2, \ldots\ldots, n, \\ p, q, \ldots\ldots = 1, 2, \ldots\ldots, r, \end{pmatrix}. \quad (1)$$

We express this set by $G\,(x,\,a,\,y)$.

Then the general expression for the parallel displacement between the vector spaces at y^i and $y^i + dy^i$, which makes the given properties invariant, is given by the following infinitesimal transformation:

$$\begin{aligned} \delta x^\alpha &= \xi^\alpha_p(x^\beta, y^i)\delta a^p = \xi^\alpha_p(x^\beta, y^i)\pi^p\,|[\,y^i, dy^j : \theta^R\,]| \\ &= \Gamma^\alpha\,|[\,x^\beta, y^i, dy^j : \theta^R\,]|^{(2)}\,, \end{aligned} \quad (2)$$

where the form of π^p depends upon the choice of the coordinates x^α etc, and its valus is generally determined by the curve $\theta^i(t)$[3] joining two points y^i and $y^i + dy^i$. Hereafter we briefly denote by $P(x,\,y)$ the parallelism expressed by (2).

Now, we consider the case there is a coordinate system in the vector space by which the formula representing each given property at each point in $Z(y)$ comes to have the same expression. If we take this coordinate system $'x^\alpha$ related to the equation $'x^\alpha = g^\alpha(x, y)$, the functions f^α and ξ^α_p in (1) and (2) must be independent of y^i in these coordinates. We call such a transformation the *symmetrizing transformation* and the coordinate system *symmetrized coordinate system* for the given properties.

In the special case where the vector space $V(x, y)$ and its given property at each point of the base space are equivalent, there always, exists at least one symmetrized coordinate system.

Next, when $G\,(x,\alpha,y)$ is given, the condition for the existence of a symmetrized coordinate system is

$$f^\alpha\Big(x^\beta('x^\gamma, y^i),\, a^p, y^j\Big) = h^\alpha('x^\gamma, a^p)\,,$$

i.e.
$$\frac{df^\alpha}{dy^i} = \frac{\partial f^\alpha}{\partial y^i} + \frac{\partial f^\alpha}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^i} = 0\,,$$

i.e.
$$\frac{\partial x^\alpha}{\partial y^i} = \frac{-\partial f^\beta}{\partial y_i}\frac{\partial x^\alpha}{\partial \bar{x}^\beta} = F^\alpha_i(x^\beta, y^i)\,. \quad (3)$$

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(1) The formulae (1) and (2) are different from those given by other writers.

(2) For the general manifold, cf. R. König, Jahresb. D.M.V. **28** (1920); **41** (1932); E. Cartan, Bull. Sci. Math. **48** (1924) 294-320; Ann. de E. Nol. **40** (1923) 393-390; J. L. Vanderslice, Amer. Journ. of Math, **56** (1934).

(3) Cf. K. Morinaga, this Journal, **4** (1934) 134.

Consequently from (3) the condition for the existence of the symmetrizing transformation

$$'x^\alpha = g^\alpha(x^\beta\,,\; y^i)\,,$$

is that

$$\frac{\partial}{\partial a^p}\left(\frac{\partial f^\beta}{\partial y^i}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right) = 0$$

and (3) is completely integrable, $'x^\alpha$ being obtained as integral constants from differential equation (3). The condition for this integrability is that

$$\left.\begin{aligned} &\frac{\partial}{\partial y^{[l}}\left(\frac{\partial f^\alpha}{\partial y^{k]}}\frac{\partial x^\beta}{\partial \bar{x}^\alpha}\right) - \frac{\partial}{\partial x^\gamma}\left(\frac{\partial x^\beta}{\partial \bar{x}^\alpha}\frac{\partial f^\alpha}{\partial y^{[k}}\right)\frac{\partial x^\gamma}{\partial \bar{x}^{|\varepsilon|}}\frac{\partial f^\varepsilon}{\partial y^{l]}} = R^\beta_{kl} = 0\,, \\ &\frac{\partial}{\partial a^p}\left(\frac{\partial f^\alpha}{\partial y^k}\frac{\partial x^\beta}{\partial \bar{x}^\alpha}\right) = 0 \end{aligned}\right\}. \qquad (4)$$

N.B. In this parallelism, the base space and the vector space are related to each other only by the correspondence of the linear element

$$dx^\alpha = \Gamma^\alpha\,|\,[x^\beta, y^i, dy^j:\; \theta^k]\,|$$

in $V(x, y)$ to a linear element dy^i in the base space at y^i. If we regard the linear element dx^α at each point of $V(x, y)$, which corresponds to a linear element dy^i, as *a* parallel vector in $V(x, y)$, we obtain a distaut parallelism and also a system of distant parallelism by changing y^i.

§ 2. In this section we shall consider a special case of point(1) paralleism—say $\bar{P}(x, y)$, which is expressed as

$$\delta x^\alpha = \xi^\alpha_p(x, y)\delta a^p = \xi^\alpha_p(x, y)\Phi^p_i(y)dy^i = \eta^\alpha_i(x, y)dy^i\,. \qquad (5)$$

This parallel displacement may be classified into three cases with respect to coordinate transformations $T(y)$ in the base space $Z(y)$, expressed in infinitesimal form,

$$'y^i = y^i + \zeta^i(y)dt\,.$$

Case I. Where the parallel displacement remains unchanged when the coordinate transformation of $Z(y)$ is applied, i.e. the case in which

(1) Cf. K. Morinaga, this Journal, **4**, (1934) 131.

the variation δx^α of any vector component x^α by the parallel displacement from a point y^i to $y^i + {}^i dy^i$ is equal to that by the parallel displacement of the same vector from $y^i + \zeta^i(y)dt$ to $y^i + dy^i + \zeta^i(y+dy)\,dt$. In this case we say that the paralldlism *admits* $T(y)$ *directly.*

Case II. When the two variations $\delta x^\alpha s$ by the parallel displacement of the same x^α in the same direction dy^i at two points y^i and $'y^i = y^i + \zeta^i(y)dt$ become equal by means of a suitable transformation of the linear element at $'y^i$ such that

$$\overline{d'y^i} = d'y^i + \theta^i_j('y)d'y^j dt \, .$$

In this case we say that $\bar{P}(x, y)$ *admits* $T(y)$ *indirectly.*

Case *I* is a special case of *II* when $\theta^i_j = \dfrac{\partial \zeta^i(y)}{\partial y^j}$.

Case III. When neither *I* nor *II* occurs, i.e. the general case.

The expressions for the condition that $\bar{P}(x, y)$ admits $T(y)$ directly and indirectly, are, by calculation, as follows,

$$\zeta^j(y)\frac{\partial \eta^\alpha_i(x, y)}{\partial y^j} + \eta^\alpha_j(x, y)\frac{\partial \zeta^j(y)}{\partial y^i} = 0 \, , \tag{6}$$

and

$$\zeta^j(y)\frac{\partial \eta^\alpha_i(x, y)}{\partial y^j} + \eta^\alpha_j(x, y)\theta^j_i(y) = 0 \, . \tag{7}$$

In the paticular case, when $T(y)$ consists of *a* γ_1-parameter group:

$$\zeta^j(y)dt = \zeta^j_a(y)d\tau^a \quad (a = 1, 2, \ldots\ldots, r_1)$$

the system of equations

$$\zeta^j_a(y)\frac{\partial f(\eta, y, x)}{\partial y^j} + \frac{\partial \zeta^j_a(y)}{\partial y^k}\eta_j\frac{\partial f(\eta, y, x)}{\partial \eta_k} = 0$$

forms a complete system. Hence when, and only when, the rank of matrix

$$\left\| \frac{\partial \varphi^\lambda_{l+s}}{\partial y^i}\zeta^{\prime}_\lambda , \ldots\ldots , \frac{\partial \varphi^\lambda_{l+s}}{\partial y^i}\zeta^n_\lambda \right\|$$

is smaller than n, the equations (6) have solutions for η_i, where we assume that the unconnected operators of

$$\zeta_a^i \frac{\partial}{\partial y^i} \quad (a = 1, 2, \ldots, r_1)$$

are

$$\zeta_1^i \frac{\partial}{\partial y^i}, \quad \ldots, \quad \zeta_l^i \frac{\partial}{\partial y^i}$$

and the others are expressed by

$$\zeta_{l+s}^i \frac{\partial}{\partial y^i} = \sum_{\bar{a}} \varphi_{l+s}^{\bar{a}} \zeta_{\bar{a}}^i \frac{\partial}{\partial y^i} \quad (\bar{a} = 1, 2, \ldots l; \; s = l+1, \ldots r_1)$$

So we can obtain the parallel displacement formed by η_i^α which admits $\zeta_a^i(y)$. (directly and indirectly).

As a special case, when the transformation, which η_i^α admits of, contains an n-parameter abelian group, we see that, by suitably choosing the coordinates in $Z(y)$, all the functions η_i^αs become independent of y^i, *e. g.* $\eta_i^\alpha(x)$.

We are now in a position to get a result, *when* η_i^α (x, y) *is given such that the number of the solutions of the equation*

$$\eta_i^\alpha \frac{\partial f}{\partial x^\alpha} = 0 \qquad (i = 1, 2, \ldots n)$$

which dose not contain x^α, *is the same as the number of rank of the matrix* $\| \eta_i^\alpha \|$, *then a parallel displacement*

$$'\delta x^\alpha = {}'\eta_i^\alpha(x, y) dy^i$$

which indirectly admits all the transformations of which the parallel displacement formed by η_i^α *admits, is related by the following equation to the original* η_i^α

$$'\eta_i^\alpha = F_\beta^\alpha(\phi_c^w, y) \eta_i^\beta(x, y) \,. \tag{8}$$

Proof. If the unconnected sets of $\eta_i^\alpha \frac{\partial}{\partial y^i}$ $(\alpha = 1, 2, \ldots\ldots N)$ are

$$\eta_i^1 \frac{\partial}{\partial y^i}, \quad \ldots, \quad \eta_i^l \frac{\partial}{\partial y^i},$$

we can take the coordinate system y^i so that

$$\eta_{l+'w}^a = 0 \quad \text{and} \quad \eta_i^{l+w} = \sum_a \phi_a^w \eta_i^a \quad \begin{pmatrix} a = 1, 2, \ldots, l \\ w = l+1, \ldots, N \\ 'w = l+1, \ldots, n \end{pmatrix}. \tag{9}$$

Therefore, regarding ζ^j and θ^i_j as unknown functions the equation (7) becomes

$$\left.\begin{aligned} &\zeta^j(y)\frac{\partial\phi^w_a}{\partial y^j}=0\,,\quad \zeta^j(y)\frac{\partial\eta^c_b}{\partial y^j}\bar{\eta}^a_c \overset{(1)}{=} -\theta^a_b\,,\\ &\theta^{l+'w}_a(y)=\text{arbitraly}\end{aligned}\right\}. \qquad (7)'$$

Taking as $\zeta^i(y)$ all the independent solutions of $(7)'$, substituting them in (7) and putting $'\eta_i$ for η^a_i, we can deduce the equations for $'\eta_c$:

$$\left.\begin{aligned} &\frac{\partial'\eta_c}{\partial y^j}-{'\eta_a}\frac{\partial\eta^b_c}{\partial y^j}\bar{\eta}^a_b+\rho^a_{wc}(x,y)\frac{\partial\phi^w_b}{\partial y_j}\eta^b_a=0\,,\\ &'\eta_{l+'w}=0\end{aligned}\right\} \qquad (7)''$$

where $\rho^a_{wc}{}'(x, y)$ is any function. Therefore, we have

$$'\eta_i=\sum_a\lambda_a\eta^a_i$$

where λ_a is a solution of

$$\frac{\partial\lambda_a}{\partial y^j}+\rho^d_{wb}\frac{\partial\phi^w_c}{\partial y^j}\eta^c_d\bar{\eta}^b_a=0\,.^{(2)}$$

i.e. $$\lambda_a=F_a(\phi^w_c, x)\,.$$

In this case, if we put F_a $(a=1, 2, \ldots, l)$ and η^a_i $(\alpha=1, 2, \ldots\ldots, N;\ i=1, 2, \ldots\ldots, n)$ as K and G in König's "vector analysis,"[3] we can apply his theory to the set of our parallel displacements.

3. Now we return to the general parallel displacement $P(x, y)$. We denote the transformation in $V(x, y)$ by $T(x)$.

If $P.T.x=T.P.x$, the parallel displacement $P(x, y)$ is said to admit $T(x)$. When we write the expression of $T(x)$ in an infinitesimal form:

$$'x^\alpha=x^\alpha+\zeta^\alpha(x, y)dt\,, \qquad (10)$$

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(1) $\bar{\eta}^a_c$ are functions which satisfy $\eta^b_d\bar{\eta}^a_b=\varepsilon^a_d$

(2) From this and the 1st equation of $(7)'$, we have $\zeta^j\dfrac{\partial\lambda_a}{\partial y^j}=0$; therefore it follows that $\lambda_a=F_a(\phi^w_c, x)$.

(3) R. König, loc. cit.

the condition is expressed as follows ;

$$\left(\Gamma^{\alpha}\frac{\partial}{\partial x^{\alpha}}+dy^{i}\frac{\partial}{\partial y^{i}}\;,\quad \zeta^{\beta}(x, y)\frac{\partial}{\partial x^{\beta}}\right)=0 \tag{11}$$

§ 4. By the parallelism $P(x, y)$, when a linear element dx^{α} at x^{α} in $V(x, y)$ corresponds to a linear element $\overline{dx}$ at $\bar{x}$ in $V(x, y+dy)$, the relation between them is calculated from (2) as follows:

$$\overline{dx}^{\alpha}=dx^{\alpha}+\partial_{\beta}\Gamma^{\alpha}dx^{\beta}\,. \tag{12}$$

Therefore, when we fix x^{α}, we have a new manifold $V(dx, y, x)$ in which y^{i} represents the coordinates of point and dx^{α} the vector, and we may consider that (12) expresses a parallel displacement—say $Q(x)$ —of vector in the base space where $\partial_{\beta}\Gamma^{\alpha}$ is regarded as the coefficient of connection.(1) Thus a set of parallel displacements $Q(x)$ is obtained by changing x^{α}.

We therefore know that a non-linear parallel displacement $P(x, y)$ can be regarded as to consisting of a set of such linear parallel displacements $Q(x)$.

Therefore, in the special case, where $P(x, y)$, become $\bar{P}(x, y)$, if $\bar{P}(x, y)$ admits a coordinate transformation $T(y)$, then $T(y)$ is admitted by each linear displacement $Q(x)$, of which $\bar{P}(x, y)$ consists. Further the vector $\zeta^{\alpha}(x, y)$ of the transformation $T(x)$, which $P(x, y)$ admits, furnishes a parallel vector-field in $Q(x)$, where x^{α} is regarded as fixed.

Further, if we put instead of (12),

$$\left.\begin{aligned}\bar{x}^{\alpha}&=x^{\alpha}+\Gamma^{\alpha}\,|\,[x^{\beta}, y^{i}, dy^{j}\,;\;\theta^{k}]\,|\\ \bar{v}^{\alpha}&=v^{\alpha}+\Gamma^{\alpha}_{\beta}\,|\,[x^{\gamma}, y^{j}, dy^{j}:\theta^{k}]\,|\,v^{\beta}\end{aligned}\right\}, \tag{13}$$

where $\Gamma^{\alpha}_{\beta}\,|\,[x^{\gamma}, y^{i}, dy^{j}:\theta^{k}]\,|$ is any functional of x^{α}, y^{i}, dy^{i} and $\theta^{i}(t)$, we can define a more general parallel dispacement than that mentioned above.

In the latter case i.e. (13) we have the condition that any holonomic subspace in $V(x, y+dy)$ corresponds to a holonomic subspace in $V(x, y)$ by reason of the parallel displacement of vector $\bar{v}^{\lambda}$:

$$\Gamma^{\alpha}_{\beta}=\frac{\partial\Gamma^{\alpha}}{\partial x^{\beta}}+\delta^{\alpha}_{\beta}\Phi\,|\,[x^{\gamma}, y^{i}, dy^{j}:\theta^{k}]\,|\;, \tag{14}$$

(1) Cf. K. Morinaga, loc. cit. 134.

where $\Phi \mid [x^\beta, y^i, dy^j : \theta^k] \mid$ is any functional.

§5. Let us consider how a curve containing in $V(x, y)$ is brought into a curve in $V(x, y+dy)$ by means of $P(x, y)$.

If we regard that a vector field $v^\lambda(x, y)$ in $V(x, y)$ represents a system of curves expressed by

$$\frac{dx^1}{v^1} = \frac{dx^2}{v^2} = \dots = \frac{dx^N}{v^N}$$

and by the parallel displacement $P(x, y)$, this system of curves is brought into a system of curves in $V(x, y+dy)$, whose equation is

$$\frac{dx^1}{\bar{v}^1} = \frac{dx^2}{\bar{v}^2} = \dots = \frac{dx^N}{\bar{v}^N}{}^{(1)},$$

then we easily see from (12), that the following relations hold

$$\bar{v}^\alpha(x, y+dy) = v^\alpha(x, y) + \left\{ v^\beta \frac{\partial \Gamma^\alpha}{\partial x^\beta} - \frac{\partial v^\alpha}{\partial x^\beta} \Gamma^\beta \right\}^{(2)}. \tag{15}$$

Therefore, in order that, in $V(z, y+dy)$, a system of curves $\bar{v}^\alpha (x,$ $y+dy)$ shoud become identical with the given system of curves $v^\alpha(x,$ $y+dy)$, the following formula gives the condition:

$$\left(v^\alpha \frac{\partial}{\partial x^\alpha}, \Gamma^\beta \frac{\partial}{\partial x^\beta} + dy^i \frac{\partial}{\partial y^i} \right) = A \mid [x^\alpha, y^i, dy^j : \theta^k] \mid v^\beta \frac{\partial}{\partial x^\beta} \tag{16}$$

where $A \mid [x^\alpha, y^i, dy^j : \theta] \mid$ is any functional.

As seen above, from a system of curves in $V(x, y)$ represented by $v^\lambda(x, y)$, a set of systems of curves are obtained as the point y^i moves through the whole base space; so we may call such a set of systems of curves a field of systems of curves. And also we may say that (16) represents the condition that $v^\lambda(x, y)$ makes a field of parallel systems of curves.

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(1) Here $\bar{v}^\lambda(x, y+dy)$ is a vector at x^α in $V(x, y+dy)$ which is parallel to the vector at point $x^\alpha - \delta x^\alpha$ in $V(x, y)$.

(2) If we use (13) expressing the more general parallel displacement than (12) deduced from (2), instead of (15) we have $\bar{v}^\alpha(x, y+dy) = v^\alpha(x, y) + \left\{ v^\beta \Gamma^\alpha_\beta - \frac{\partial v^\alpha}{\partial x^\beta} \Gamma^\beta \right\}$

Next we take a field of r_1-independent systems of curves $\underset{1}{v}^\alpha, \ldots\ldots \underset{\gamma_1}{v}^\alpha$ and consider a linear system of curves $\sum_{a=1}^{\gamma_1} p^a \underset{a}{v}$ where p^a is any function of x^α and y^i. We denote this linear system by $(\underset{1}{v}, \ldots\ldots, \underset{\gamma_1}{v}; x, y)$. Then the condition that $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y)$ should be brought into $(v, \ldots\ldots, v; x, y+dy)$ by meahs of $P(x, y)$ is obtained from (16) as follows:

$$\left(\underset{a}{v}^\alpha \frac{\partial}{\partial x^\alpha}, \Gamma^\beta \frac{\partial}{\partial x^\beta} + dy^i \frac{\partial}{\partial y^i}\right) = \sum_b p_a^b \underset{b}{v}^\beta \frac{\partial}{\partial x^\beta} \tag{17}$$

where $p_a^b \,|\, [x^\alpha, y^i, dy^j : \theta^k] \,|$ is any functional.

Now, in the linear system of curves $\sum_a p^a \underset{a}{v}^d$, let p^a belong to a given function "corpus" M, and express this linear system by $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y : M)$. Then the condition that $(\underset{1}{v}, \ldots\ldots \underset{\gamma_1}{v}; x, y : M)$ should be brought into $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y+dy : M)$ by means of $P(x, y)$ (*i.e.* (15)) is easily calculated as follows:

$$\left.\begin{aligned} &\text{if} \quad p^a < M \\ &\text{then} \quad \Gamma^\alpha \frac{\partial p^a}{\partial x^\alpha} + dy^i \frac{\partial p^a}{\partial y^j} < M, \\ &\left(\underset{a}{v}^\alpha \frac{\partial}{\partial x^\alpha}, \Gamma^\beta \frac{\partial}{\partial x^\beta} + dy^i \frac{\partial}{\partial y^i}\right) = \sum_b p_a^b \underset{b}{v}^\alpha \frac{\partial}{\partial x^\alpha} \end{aligned}\right\} \tag{18}$$

where $p_a^b \,|\, [x^\alpha, y^i, dy^j : \theta^k] \,| < M$, θ^i and dy^i being regarded as parameters.

In this section we have seen how a system of curves is transferred by $P(x, y)$. Let us, now, see how each curve in a system of curves is transferred. Actually, in order that any curve in the system of curves $\sum_a p^a \underset{a}{v}^\alpha$ be transferred to a curve belonging to the linear system $(\underset{1}{v}, \ldots\ldots \underset{\gamma_1}{v}; x, y; M)$ by means of $P(x, y)$, it is necessary and sufficient that the transferred system $\overline{\sum_a p^a \underset{a}{v}^\alpha}$ should have the following form:

$$\overline{\sum_a p^a \underset{a}{v}^\alpha} = \rho \sum_a{}' p^a \underset{a}{v}^\alpha$$

where ρ is a proportional factor and $'p^a\,|\,[x^\alpha, y^i, dy^j : \theta^k]\,|$ is a function which belongs to M along(1) the curve $\sum_a p^a \underset{a}{v}^\alpha$, θ^k and dy^i being regarded as parameters.

The above condition is rewritten in the following form

$$\left(\sum_a p^a \underset{a}{v}^d \frac{\partial}{\partial x^\alpha},\ \Gamma^\beta \frac{\partial}{\partial x^\beta} + dy^i \frac{\partial}{\partial y^i}\right) = \sum_a (Ap^a + R^a) \underset{a}{v}^\alpha \frac{\partial}{\partial x^\alpha} \tag{19}$$

or

$$\bar{\delta}\, p^a \underset{a}{v}^\alpha = \sum_a (Ap^a + R^a)\, \underset{a}{v}^{\alpha\,(2)} \tag{19$'$}$$

where A is a functional and $R^a\,|\,[x^d, y^i, dy^j : \theta^k]\,|$ a function which belongs to M along the curve $\sum_a p^a \underset{a}{v}^\alpha$.

Even in the case where each curve of the system v^λ and of the system w^λ is transferred respectively into $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v} : x, y : M)$, a curve of the system $pv^\lambda + qw^\lambda$ $(p, q < M)$ is not always transferred into the linear syslhem.

For; From the assumption and (19)′ we have

$$\left.\begin{aligned} \bar{\delta}\, \underset{a}{v}^\alpha &= A_a \underset{a}{v}^\alpha + \sum_c R^c_a \underset{c}{v}^\alpha, \\ \bar{\delta}\, \underset{b}{v}^\alpha &= A_b \underset{b}{v}^\alpha + \sum_c R^c_b \underset{c}{v}^\alpha \end{aligned}\right\} \tag{E}$$

where A_a and A_b are functionals; $R^c_a\ |\,[x^\alpha, y^i, dy^j : \theta^k]\,|$ and $R^c_b\ |\ [x^d, y^i, dy^j : \theta^k]\,|$ belong to M along the curves of $\underset{a}{v}^\lambda$ and $\underset{b}{v}^\lambda$, respectively. So we have from (E),

$$\begin{aligned} \bar{\delta}(p_a \underset{a}{v}^\alpha + p_b \underset{b}{v}^\alpha) = \sum_{\bar{c}} (p_a R^{\bar{c}}_a + p_b R^{\bar{c}}_b) \underset{\bar{c}}{v}^\alpha + \Bigg\{ & p_a A_a + p_b R^a_b - p_a A_b - \frac{(p_a)^2}{p_b} R^b_a \\ & + \left(\Gamma^\beta \frac{\partial p_b}{\partial x^\beta} + dy^i \frac{\partial p_b}{\partial y^i}\right) \frac{p_a}{p_b} - \left(\Gamma^\beta \frac{\partial p_a}{\partial x^\beta} + dy^i \frac{\partial p_a}{\partial y^i}\right) \Bigg\} \underset{a}{v}^\alpha \\ & + \left\{ A_b + \frac{p_a}{p_b} R_a - \left(\Gamma^\beta \frac{\partial p_b}{\partial x^\beta} + dy_i \frac{\partial p_b}{\partial y_i}\right) \frac{1}{p_b} \right\} (p_a \underset{a}{v}^\alpha + p_b \underset{b}{v}^\alpha), \\ & (\bar{c} \neq ab)\ . \end{aligned} \tag{20}$$

(1) The term "a function f belong to M along a curve c" means that f takes the same value along c as a certain function $\varphi(x, y)$ belonging to M although f and $\varphi(x, y)$ are not the same at other points.

(2) $\bar{\delta} v^\lambda$ denotes $\bar{v}^\lambda(x, y + dy) - v^\lambda(x, y + dy)$

Hence we know that a curve of the system $p_a \underset{a}{v}^d + p_b \underset{b}{v}^\alpha$ cannot, in general, be transferred into the linear system $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y: M)$ unless the first and second terms both belong to M along the curve.

Since every constant belongs to M, we get by certain calculations from (20) the following equations as the necessary condition that any curves of the linear system $(\underset{1}{v}, \ldots\ldots\ldots, \underset{r_1}{v}; x, y: M)$ be transferred into the system $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y: M)$:

$$\delta \underset{a}{v}^\alpha = A_a \underset{a}{v}^\alpha + \sum \underset{c}{R}{}_a^c \underset{c}{v}^\alpha \qquad (a, c = 1, 2, \ldots . r_1), \tag{21}$$

where A_a and R_a^c satisfy the following equations

$$\left.\begin{aligned} &\frac{\partial R_a^{\bar{c}}}{\partial x^\alpha} v^\alpha + \frac{\partial R_b^{\bar{c}}}{\partial x^\alpha} \underset{b}{v}^\alpha = 0\,,\text{[1]} \\ &\frac{\partial A_a - A_b}{\partial x^\alpha} v^\alpha - \frac{\partial R_a^b}{\partial x^\alpha} \underset{b}{v}^\alpha = 0\,, \\ &\text{and } R_a^c \text{ belongs to } M \text{ along } \underset{a}{v}^\alpha \text{ (only)}, \quad \begin{pmatrix} a,\ b,\ \bar{c} = 1,\ 2,\ \ldots,\ r_1; \\ a,\ b \neq \bar{c} \end{pmatrix} \end{aligned}\right\} \tag{21$'$}$$

[F]. *In the case where M forms a number corpus, we can verify by a calculation not here shown that the condition (21) is sufficient.*

In the general case we have not sucsseeded in determining the general expression of the functional $R^\alpha \,|\, [x^\alpha, y^i, dy^j : \theta] \,|$ belonging to M along a curve of linear system $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y: M)$. So we will continue our discussion with the special case in which M consists of functions which satisfy

$$\underset{a}{v}^\alpha \frac{\partial f}{\partial x^\alpha} = 0 \qquad (a = 1, 2, \ldots, r_1).\text{[2]}$$

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(1) Only for $a, b, c, \ldots$, even when the same index appears twice in a term this term dose not stand for the sum of the terms obtained by giving the index each of its r_1 values.

(2) If M consists of the solutions of $\underset{a}{v}^\alpha \dfrac{\partial f}{\partial x^\alpha} = 0$ $(a = 1, 2, \ldots r_1)$, we have

$$(\underset{1}{v}, \ldots, \underset{r_1}{v}; x, y: M) = (v, \ldots, v; x, y: C)$$

with respect to curves.

When the vector space is euclidean, the condition that every element of a linear system $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y : M)$ of a r_1-independent straight lines

$$\frac{dx^1}{\underset{a}{v^1}} = \frac{dx^2}{\underset{a}{v^2}} = \ldots . = \frac{dx^N}{\underset{a}{v^N}} \quad (a = 1, 2, \ldots . , r_1)$$

should be a straight line, is that M[1] consists of the common solution of

$$\underset{a}{v^\alpha} \frac{\partial f}{\partial x^\alpha} = 0 \qquad (a = 1, 2, \ldots . , r_1) .$$

The projective group or affine group is defined as a transformation group which transforms any element of the system $(\underset{1}{v}, \ldots . . \underset{N}{v}, V; x, y: C)$[2] defined above into the same system. So if we denote the fundamental operator of the group

$$\xi^\alpha \frac{\partial}{\partial x^\alpha} ,$$

the condition that the group is projective is obtained from [F] as follows

$$\left(\underset{a}{v^\beta} \frac{\partial}{\partial x^\beta} , \xi^\beta \frac{\partial}{\partial x^\beta}\right) = \left(\rho_a \underset{a}{v^\alpha} + \sum_b g_a^b \underset{b}{v^\alpha}\right) \frac{\partial}{\partial x^\alpha} . \tag{22}$$

where $\rho_a(x, y)$ and $g_a^b(x, y)$ satisfy following relations

$$\left.\begin{aligned} &\frac{\partial g_a^c}{\partial x^\alpha} \underset{b}{v^\alpha} + \frac{\partial g_b^c}{\partial x^\alpha} \underset{a}{v^\alpha} = 0 , \\ &\frac{\partial \rho_a - \rho_b}{\partial x^\alpha} \underset{a}{v^\alpha} - \frac{\partial g_a^b}{\partial x^\alpha} \underset{b}{v^\alpha} = 0 , \quad (a, b, c = 1, 2, \ldots . , N) \end{aligned}\right\} , \tag{22$'$}$$

; and in the affine case we have

$$\left(\underset{a}{v^\beta} \frac{\partial}{\partial x^\beta} , \xi^\beta \frac{\partial}{\partial x^\beta}\right) = \sum_b g_a^b \underset{b}{v^\alpha} \frac{\partial}{\partial x^\alpha} \quad (a, b = 1, 2, \ldots . , N) \tag{23}$$

where g_a^b is any common solution of

$$\underset{c}{v^\beta} \frac{\partial f}{\partial x^\beta} = 0 \qquad (c = 1, 2, \ldots . , N)$$

(1) We can easily see this by so choosing a Cartesian coordinate system in the vector space that $'\underset{a}{v^\alpha} = \delta_a^\alpha \ (a = 1, 2, \ldots , r_1)$.

(2) In this case M becomes a number corpus C.

i. e, constant.

Now we attempt to define the general projective and affine transformation group in $V(x, y)$ by extending the idea above mentioned. First, we give r_1-independent systems of curves $\underset{1}{v}^\alpha \ldots\ldots, \underset{r_1}{v}^\alpha$ and assume that M is a function corpus which satisfies

$$\underset{a}{v}^\beta \frac{\partial f}{\partial x^\beta} = 0 \qquad (a = 1, 2, \ldots\ldots, r_1)$$

The condition that the linear system $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}; x, y : M)$ is changed internally by means of the transformation $\xi^\alpha \frac{\partial}{\partial x^\alpha}$, is obtained from (22) and foot-note on page (23), as follows

$$\left(\underset{a}{v}^\beta \frac{\partial}{\partial x^\beta}, \ \xi^\beta \frac{\partial}{\partial x^\beta}\right) = \left(A_a \underset{a}{v}^\alpha + \sum_b R_a^b \underset{b}{v}^\alpha\right) \frac{\partial}{\partial x^\alpha} \qquad (a, b = 1, 2, \ldots\ldots, r_1) \tag{24}$$

where A_a and R_a^b may be any solutions of the equations,

$$\left.\begin{aligned} &\frac{\partial R_a^c}{\partial x^\alpha} \underset{b}{v}^\alpha + \frac{\partial R_b^c}{\partial x^\alpha} \underset{a}{v}^\alpha = 0 \, , \\ &\frac{\partial A_a - A_b}{\partial x^\alpha} \underset{a}{v}^\alpha - \frac{\partial R_a^b}{\partial x^\alpha} \underset{a}{v}^\alpha = 0 \, . \quad (a, b, c = 1, 2, \ldots\ldots, r_1) \end{aligned}\right\} . \tag{24'}$$

If we take ξ^α such that they satisfy (24), the transformation group

$$\delta f = \sum_l \xi_l^\alpha \frac{\partial f}{\partial x^\alpha} \delta \tau^l \qquad (l = 1, 2, \ldots\ldots, r_2)$$

may define a generalized projective group, where each curve in $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}, x, y : M)$ is regarded as a straight line. Specially if we put $A_a = 0$ in (24) and restrict g_a^b to the common solutions of the equations

$$\underset{a}{v}^\alpha \frac{\partial f}{\partial x^\alpha} = 0 \quad (a = 1, 2, \ldots\ldots, r_1) \, ,$$

then the functions ξ^α *taken from* (*24*) *are considered as defining a generalized affine group, each curve in* $(\underset{1}{v}, \ldots\ldots, \underset{\gamma_1}{v}\,;\, x, y : M)$ *being regarded as a straight line.*

Next, if we put $\Gamma^\alpha \frac{\partial}{\partial x^\alpha} + dy^i \frac{\partial}{\partial x^i}$ for $\xi^\alpha \frac{\partial}{\partial x^\alpha}$ in (24), the result above obtained are interpreted in the case of parallel displacement instead of transformation of coordinates. Thus we able to define a *generalized projective and affine displacements.*

N. B. 1. When the vector spaces at every point are equivalent, the linear system $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v}\,;\, x, y : M)$ is expressed in a form indepedent of y^i in a symmetrized coordinate system. In this case, the infinitesimal operator $\xi^\alpha \frac{\partial}{\partial x^\alpha}$ of the generalized projective (affine) group furnishes from (2) the coefficient of connection Γ^α of the generalized projective (affine) parallel displacement.

Specially, when ξ^α_p in (2) forms a group and a symmetrized coordinate system exist in the vector space, ξ^α_p becomes the general symbol of the general affine transformation in which the trajectories ξ^α_p may be considered as straight lines, and then the coefficient of connection Γ^α in (2) defines the generalized affine parallel displacement.

N. B. 2. From (20) and the result obtained by substituting $\Gamma^\alpha \frac{\partial}{\partial x^\alpha} + dy \frac{\partial}{\partial x^i}$ for $\xi^\alpha \frac{\partial}{\partial x^\alpha}$ in (24) we can easily verify that in the case of a generalized projective (affine) parallel displacement, the function corpus M is invariant for $\Gamma^\alpha \frac{\partial}{\partial x^\alpha} + dy^i \frac{\partial}{\partial y^i}$.

§6. Generalizing the idea of parallel displacement just mentioned, we consider a parallelism in a manifold which has each curve *as an element in* the vector spaces.

When a system of curves $v^\lambda(x, y)$ in $V(x, y)$ corresponds to a system of curves $\bar{v}^\lambda$ in $V(x, y+dy)$ along a curve $y^i = \theta^i(t)$ which joins y^i and $y^i = dy^i$, $\bar{v}^\lambda$ may be written in the most general form:

$$\bar{v}^\alpha = F^\alpha \,|\, [x^\beta, y^i, dy^j, : \theta^k, v^\lambda] \,| \tag{25}$$

or

$$\bar{v}^\alpha = v^\alpha + F^\alpha_1 \,|\, [x^\beta, y^i\, dy^j : \theta^k, v^\lambda] \,|, \tag{25'}$$

where F^λ is a general analytical functional of v^α, and we express this parallelism by P_f.

N. B. If

$$'v^{\alpha} = H^{\alpha} \mid [x^{\beta}, y^{i} : v^{\alpha}, h(s)] \mid$$ (1)

is the set of functional transformation of $v^{\lambda}(x, y)$ which makes a certain given property invariant, the general parallelism which also makes the property invarient is defined by the following equation :

$$F_1^{\alpha} \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\lambda}] \mid = H^{\alpha}_{h(x)} \Phi \mid [x, y^{i}, dy^{j} : \theta^{k}] \mid + H^{\alpha}_{h(y)} \Phi \mid [y, y^{i}, dy^{j} : \theta^{k}] \mid + \int \bar{H}^{\alpha} \mid [x^{\beta}, y^{i} : v^{\lambda} ; s] \mid \Phi \mid [s, y^{i}, dy^{j} : \theta^{k}] \mid ds \quad (26)$$ (2)

where $\Phi \mid [s, y^{i}, dy^{j} : \theta^{k}] \mid$ is any functional of $\theta^{k}(t)$.

As in § 5, the condition that a curve of a system of curves v^{λ} in $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v} ; x, y : M)$ be transferred to a curve in the linear system is

$$\bar{v}^{\alpha} = F^{\alpha} \mid x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\lambda}] \mid = v^{\alpha}(x, y+dy) + A \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}] \mid v^{\alpha} + \sum_{a} \underset{a}{R^{a}} \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}] \mid \underset{a}{v^{\alpha}} \quad (27)$$

where A is any functional, and R^{a} any function belonging to M along the curve.

Now, we can proof the follorving proposition.

The most general form of the parallelism in which the cuives $v^{\alpha}(x, y)$ and $\rho(x, y)\, v^{\alpha}(x, y)$ can be treated as identical, is expressed as follows

$$\bar{v}^{\alpha} = v^{\alpha} + N^{\alpha}_{\mu} \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}] \mid v^{\mu} + v^{\alpha} G \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\gamma}] \quad (28)$$

where N^{λ}_{μ} and G are any functionals.

Proof. If a parallelism satisfies this assumption for any $v^{\alpha}(x, y)$ and $\rho(x, y)$, F_1^{μ} in (25) must have the form :

$$F_1^{\mu} \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, \rho v^{\gamma}] \mid = \rho(x, y) F_1^{\mu} \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\gamma}] \mid + v^{\mu} F_2 \mid [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, \rho, v^{\gamma}] \mid \quad (29)$$

where F_2 is any functional. Here we vary the form of function $v^{\alpha}(x, y)$ continuosly only in the neighburhood of the point x^{α}, so that $'v^{\mu_1}(x, y)$ may vanish at the point x^{α} for the index μ only. Then $F_1^{\mu} \mid [x^{\beta}, y^{i}, dy^{i}, : \theta^{k}, v^{\gamma}] \mid$ must be a linear form of v^{α}, since $\rho(x, y)$ may vary arbitrarily. So the term in F_1^{μ} having the integral sign must be of the form

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(1) $h(s)$ in H^{λ} denotes that the transformation contains any arbitrary function.

(2) $H^{\alpha}_{h(x)}$, $H^{\alpha}_{h(y)}$ and $\bar{H}^{\lambda}$ represent the symbols of infinitesimal transformation for H^{α}.

$$\int L^{\mu}_{\lambda} | [x^{\beta}, y^{i}, dy^{j}, u^{l} : \theta^{k}] | \rho(u^{l}, y) v^{\lambda}(u^{l}, y) du^{(1)}$$

and, from (29), we know that $\rho(x, y)$ must be outside the intergral sign, so we have

$$\int L^{\mu}_{\lambda} | [x^{\beta}, y^{i}, dy^{j}, u^{l} : \theta^{k}] | \rho(u^{l}, y) v^{\lambda}(u^{l}, y) du$$
$$= \rho(x, y) \int L^{\mu}_{\lambda} | [x^{\beta}, y^{i}, dy^{j} u^{l} \theta^{k}] | v^{\lambda}(u^{l}, y) du .$$

From the arbitrarity of $\rho(x, y)$, we can easily deduce that

$$L^{\mu}_{\lambda} | [x^{\beta}, y^{i}, dy^{j}, u^{l} : \theta^{k}] | \equiv 0 . \tag{30}$$

Further, in $F_1{}^{\mu} | [x^{\beta}, y^{i}, dy^{j}, \theta^{k}, \rho v^{\tau}] |$, since $\rho(x, y)$ can be made any function in the neighbourhood of a point x^{α} and since the value of the function $\frac{\partial \rho}{\partial x^{\alpha}}, \frac{\partial^2 \rho}{\partial x^{\alpha} \partial x^{\beta}}, \ldots\ldots$ which come from the terms containing $\frac{\partial \rho v^{\mu}}{\partial x^{\alpha}}, \frac{\partial \rho v^{\mu}}{\partial x^{\alpha} \partial x^{\beta}}, \ldots\ldots$ may take any value whatever without changing the value of $\rho(x, y)$ at the point x^{α}, therefore their coefficients must vanish. Putting together the results above obtained, we know that, in

$$F^{\mu}_{1} | [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\tau}] | ,$$

if we put, at the point x^{α},

$$v^{\mu}(x, y) = 0 \quad \text{(for the index } \mu \text{ only)} ,$$

there remains only a term which has the form

$$N^{\mu}_{\lambda} | [x^{\alpha}, y^{i}, dy^{j} : \theta^{k}] | v^{\lambda} .$$

Therefore the most general form of $F_1{}^{\alpha}$ satisfying the assumption can be written in the form

$$F^{\alpha}_{1} | [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\tau}] |$$
$$= N^{\alpha}_{\lambda} | [x^{\beta}, y^{i}, dy^{j} : \theta^{k}] | v^{\lambda} + v^{\alpha} G | [x^{\beta}, y^{i}, dy^{j} : \theta^{k}, v^{\tau}] | \tag{31}$$
$$(\alpha = 1, 2, \ldots, N) ,$$

(1) We express a multiple integral by a single integral sign for the sake of brebity.

and vice versa. Thus we obtain the proof of the proposition.

Next, as the sum of two systems of curves v^α and w^α we define a system of curves $v^\lambda + w^\lambda$. The general expression of parallel displacement which preserves the summation, is obtained in the following form:

$$\bar{v}^\alpha = v^\alpha + N^\alpha_\lambda \,|\, [x^\beta, y^i, dy^j : \theta^k] \,|\, v^\lambda + \int L^\alpha_\lambda \,|\, [x^\beta, y^i, dy^j : \theta^k] \,|\, v^\lambda(u^l, y) du + P^{\alpha\nu}_\lambda \,|\, [x^\beta, y^i, dy^j : \theta^k] \,|\, \frac{\partial v^\lambda}{\partial x^\nu} + P^{\alpha\nu_1\nu_2}_\lambda \,|\, [x^\beta, y^i, dy^j : \theta^k] \frac{\partial^2 v^\lambda}{\partial x^{\nu_1} \partial x^{\nu_2}} + \ldots . \tag{32}$$

We denot this parallelism by $\bar{P}_f$.

When the corpus M consists of the common solution of

$$\underset{a}{v}^\alpha \frac{\partial f}{\partial x^\alpha} = 0 , \qquad (a = 1, 2, \ldots, r_1) ,$$

the condition that every curve in the linear system $(\underset{1}{v}, \ldots \underset{r_1}{v} ; x, y : M)$ be transferred into $(\underset{1}{v}, \ldots\ldots, \underset{r_1}{v} ; x, y : M)$ by the parallel displacement (32) is, from (22),

$$\underset{a}{\bar{v}}^\alpha = \bar{F}^\alpha \,|\, x^\beta, y^i, dy^j ;\ \theta^k, v^\gamma] \,| = \underset{a}{v}^\alpha(x, y+dy) + A_a \underset{a}{v}^\alpha + \sum_b R^b_a \underset{b}{v}^\alpha , \tag{33}$$

where A_a and R^b_a are related by

$$\frac{\partial R^c_b}{\partial x^\alpha} \underset{a}{v}^\alpha + \frac{\partial R^e_a}{\partial x^\alpha} \underset{b}{v}^\alpha = 0$$

$$\frac{\partial A_a - A_b}{\partial x^\alpha} \underset{b}{v}^\alpha - \frac{\partial R^b_a}{\partial x^\alpha} \underset{a}{v}^\alpha = 0 \quad (a, b, c = 1, 2, \ldots r_1) .$$

When all the curves in this linear system of curves are regarded as straight lines, the parallelism which satisfies the above conditions gives a generalized projective displacement in $\bar{P}_f$.

And we have a proposition.

When the vector space $V(x, y)$ *at* y^i *is euclidean,*[1] *the condition that the parallel displacement (32) should transfer any straight line to a straight line is written as follows:*

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(1) In deducing the formulae (34) the Cartesian coordinate system is used.

$$\left.\begin{aligned} N^\lambda_\mu\,|\,[x^\beta, y^i, dy^j : \theta^k]\,| &= \delta^\lambda_\mu N\,|\,[x^\beta, y^i, dy^i : \theta^k]\,| + K^\lambda_{1\mu}\,|\,[y^i, dy^j : \theta^k]\,|, \\ L^\lambda_\mu\,|\,[x^\beta, y^i, dy^j, u^l : \theta^k]\,| &= \delta^\lambda_\mu L\,|\,[x^\beta, y^i, dy^i, u^l : \theta^k]\,| + K^\lambda_{2\mu}\,|\,[y^i, dy^j, u^l : \theta^k]\,|, \\ P^{\lambda\nu}_\mu\,|\,[x^\beta, y^i, py^i : \theta^k]\,| &= \delta^\lambda_\mu P^\nu\,|\,[x^\beta, y^i, dy^j : \theta^k]\,| + K^\lambda_{3\mu}\,|\,[y^i, dy^j : \theta]\,|, \text{ etc.} \end{aligned}\right\} \tag{34}$$

where N, L, P^ν, $P_{\nu_1\nu_2}$ etc. are any functional.

Proof. By putting $v^\lambda = k^\lambda(y)$ in (32), we have

$$\bar{k}^\lambda = k^\lambda + N^\lambda_\mu\,|\,[x^\beta, y^i, dy^j : \theta^k]\,|\,k^\mu + k^\mu \int L^\lambda_\mu\,|\,[x^\beta, y^i, dy^i, u^l : \theta^k]\,du$$

and on the other hand, from the assumption that the parallelism is a projective one (in common sence) in P_f, we have

$$\bar{k}^\lambda = k^\lambda + B\,|\,[x^\beta, y^i, dy^j : \theta]\,|\,k^\lambda + K^\lambda\,|\,[k^\mu, y^i, dy^j : \theta^k]\,|$$

where $B\,|\,[x^\beta, y^i\, dy^j : \theta^k]\,|$ and $K^\lambda\,|\,[k^\mu, y^j, dy^j : \theta^k]\,|$ are any functionals. So, by changing k^λ, we have

$$\begin{aligned} N^\lambda_\mu\,|\,[x^\alpha, y^i, dy^j : \theta^k] &+ \int L^\lambda_\mu\,|\,[x^\beta y^i dy^j u^l : \theta^k]\,|\,du \\ &= \delta^\lambda_\mu B\,|\,[x^\beta, y^i dy^j : \theta^k] + K^\lambda_\mu\,|\,[y^i, dy^j : \theta^k]\,| \end{aligned} \tag{35}$$

Next, applying the same process to $\rho(x, y)\, k^\lambda(y)$ as above which also represents a system of straight lines too and taking account of the arbitrarity of $\rho(x, y)$ and (35), we can easily deduce the equations (34) as the condition that a parallel displacement (32) should be projective (in common sence) in $\bar{P}_f$.

In conclusion the author wishes to express his hearty thanks to Prof. T. Iwatsuki who has given his kind guidance.

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(1) The parallel displacement (32) satisfying (34) is an affine parallel displacement in a restricted sense in P_f.