Theory of Vector Valued Set Functions. II.

By

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In continuation of the work recorded in the preceding paper,[1] I investigated the unitary equivalence of the resolutions of identity. My theorems are analogous in content to those which Stone proved in his treatise.[2] But he treated the resolution of identity $E(\lambda)$ in connection with the corresponding self-adjoint transformation. Here I investigate the properties of the resolution of identity $E(U)$ from the standpoint of the vector valued set functions.

In part I, the variable U of the vector valued set function $\mathfrak{q}(U)$ is a Borel subset of a Borel set V in a metric space S which is half compact.[3] And I have assumed the uniform monotonity of the base $\sigma(U)$ of $\mathfrak{q}(U)$, in order that we may use the fundamental theorem between the integral and the derivative with respect to $\sigma(U)$.[4] But, O. Nikodym proved the fundamental theorem in the abstract space,[5] i. e. when $\phi(U)$ is absolutely continuous with respect to $\mu(U)$, there exists a point function $f(\lambda)$ which satisfies the following relation:

$$\phi(U) = \int_U f(\lambda) d\mu(U) .$$

Hence, if we say $f(\lambda)$ as the derivative of $\phi(U)$ with respect to $\mu(U)$, we have the fundamental theorem in the abstract space. Such a consideration is unsatisfactory from the standpoint of the theory of deri-

(1) F. Maeda, "Theory of Vector Valued Set Functions" this volume, 57-91. I shall refer to this paper as part I.

(2) M.H. Stone, *Linear Transformations in Hilbert Space*, (1932), 242.

(3) Cf. sec. 1, part I.

(4) Cf. footnote (3) of sec. 1, part I.

(5) O. Nikodym, Fund. Math., **15** (1930), 131–179.

vatives of set functions.[1] But, since the fundamental theorem holds without any restriction, it is very useful in the application of the theory of set functions. Hence all theorems in the preceding paper (part I) and also in this paper (part II) hold in the separable metric space,[2] without the restriction of the uniform monotonity. In this paper I first give a note with respect to the derivatives of set functions in the abstract space.

Derivatives of Set Functions in Abstract Space.

26.[3] Let V be an abstract set, and $\mathfrak{K}$ be a closed family (σ-Körper) of sets in V, which contains the set V. Let $f(\lambda)$ be a real valued point function defined in V. When the set $V[f(\lambda) > \alpha]$[4] belongs to $\mathfrak{K}$ for any value of α, then it is said that $f(\lambda)$ is measurable ($\mathfrak{K}$).

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(1) I have investigated the derivatives of set functions in previous papers; "On the General Derivatives of the Set Functions", this journal **1** (1931), 1–24; and "On the Definition and the Approximate Continuity of the General Derivatives", this journal **2** (1932), 33–53. In the first of these papers, some parts are incomplete. Here I give corrections:

Delete the last 7 lines of p. 4 and the first 7 lines of p. 5, and substitute the following: "To prove this, consider $\bar{\varphi}\{\bar{U}(a, \rho)\}$ as a function of the point a, ρ being constant. Let $\{a_\nu\}$ be any sequence of points which converges to a, and let δ_ν be the distance between a_ν and a. Then $\bar{U}(a, \rho+\delta_\nu) \geqq \bar{U}(a_\nu, \rho)$. Hence $\bar{\varphi}\{\bar{U}(a, \rho+\delta_\nu)\} \geqq \bar{\varphi}\{\bar{U}(a_\nu, \rho)\}$ for any ν. But, since $\operatorname*{Lim}_{\nu\to\infty} \bar{U}(a, \rho+\delta_\nu) = \bar{U}(a, \rho)$, we have $\lim_{\nu\to\infty} \bar{\varphi}\{\bar{U}(a, \rho+\delta_\nu)\} = \bar{\varphi}\{\bar{U}(a, \rho)\}$. Therefore, we have $\bar{\varphi}\{\bar{U}(a, \rho)\} \geqq \overline{\lim}_{\nu\to\infty} \bar{\varphi}\{\bar{U}(a_\nu, \rho)\}$. That is, $\bar{\varphi}\{\bar{U}(a, \rho)\}$ is upper semi-continuous at the point a. Therefore, $\dfrac{\bar{\varphi}\{\bar{U}(a, \rho)\}}{\bar{\varphi}\{\bar{U}(a, \lambda\rho)\}}$ is a Baire's function. Since $\operatorname*{Lim}_{\sigma\to\rho+} \bar{U}(a, \sigma) = \bar{U}(a, \rho)$, we have $\lim_{\sigma\to\rho+} \dfrac{\bar{\varphi}\{\bar{U}(a, \sigma)\}}{\bar{\varphi}\{\bar{U}(a, \lambda\sigma)\}} = \dfrac{\bar{\varphi}\{\bar{U}(a, \rho)\}}{\bar{\varphi}\{\bar{U}(a, \lambda\rho)\}}$. Hence, as Carathéodory did (*Vorlesungen über reelle Funktionen*, zweite Aufl. (1927), 483–484), we can prove that $l(a, \lambda)$ is a Baire's function."

Delete lines 16–21 of p. 10, and substitute the following: "Put $\overline{\lim}_{\rho\to 0} \dfrac{\Phi\{\bar{U}(a, \rho)\}}{\varphi\{\bar{U}(a, \rho)\}} = \bar{D}_\varphi^{(s)}\Phi$ and $\underline{\lim}_{\rho\to 0} \dfrac{\Phi\{\bar{U}(a, \rho)\}}{\varphi\{\bar{U}(a, \rho)\}} = \underline{D}_\varphi^{(s)}\Phi$, then $\bar{D}_\varphi^{(s)}\Phi$ and $\underline{D}_\varphi^{(s)}\Phi$ are also symmetric derivatives. As in sec. 2, we can prove that $\bar{D}_\varphi^{(s)}\Phi$ and $\underline{D}_\varphi^{(s)}\Phi$ are Baire's functions."

(2) The separability of S is required in the proof of the separability of $\aleph_2(\sigma)$. Cf. F. Maeda, this jourdal, **3** (1933), 5–7.

(3) The number of the section follows No. 25 of part I.

(4) $V[f(\lambda) > \alpha]$ means the set of all points λ of V, where $f(\lambda) > \alpha$.

Let $\mu(U)$ be a completely additive, non-negative set function defined in $\mathfrak{R}$. Then we can define the integral of $f(\lambda)$

$$\int_U f(\lambda)d\mu(U)$$

over any set U in $\mathfrak{R}$, which has the same properties as the Lebesgue integral.(1) Of course, when $\int_V f(\lambda)d\mu(U)$ is finite,

$$\phi(U) = \int_U f(\lambda)d\mu(U) \tag{1}$$

is a completely additive finite set function defined in $\mathfrak{R}$, which is absolutely continuous with respect to $\mu(U)$.

The converse problem is solved by O. Nikodym:(2) When $\phi(U)$ is absolutely continuous with respect to $\mu(U)$, then there exists a measurable ($\mathfrak{R}$) point function $f(\lambda)$ which satisfies (1). S. Saks gave another proof of this theorem.(3) He constructed $f(\lambda)$ as follows: we can divide V into the sum of sets contained in $\mathfrak{R}$, such that

$$V = V_1^{(p)} + V_2^{(p)} + \dots + V_n^{(p)} + \dots ,$$

where

$$2^{-p}(n-1)\mu(U) \leqq \phi(U) \leqq 2^{-p}n\mu(U) \tag{2}$$

for all subsets U of $V_n^{(p)}$ contained in $\mathfrak{R}$. Now put

$$f_p(\lambda) = 2^{-p}(n-1) \quad \text{when} \quad \lambda \in V_n^{(p)} \quad (n = 1, 2, \dots)$$

for any positive integer p. Then $\{f_p(\lambda)\}$ converges uniformly to a measurable ($\mathfrak{R}$) point function $f(\lambda)$, which satisfies (1).

We may call $f(\lambda)$ the *derivative* of $\phi(U)$ with respect to $\mu(U)$, at the point λ,(4) and denote it by $D_{\mu(U)}\phi(\lambda)$.

27. Now I will prove a theorem which is useful in the application of the theory of set functions.

Let $\beta(U)$ be another completely additive, non-negative set function defined in $\mathfrak{R}$. If

$$\mu(U) = \int_U g(\lambda)d\beta(U) ,$$

(1) Cf. S. Saks, *Théorie de l'intégrale*, (1933), 247–263.

(2) Nikodym, *loc. cit.*, 179.

(3) Cf. Saks, *loc. cit.*, 255.

(4) Cf. *ibid.*, 257.

then $$\int_U f(\lambda)d\mu(U) = \int_U f(\lambda)g(\lambda)d\beta(U) ,$$

when $\int_V f(\lambda)d\mu(U)$ *is finite.*

To prove this, it is sufficient to consider the case where $f(\lambda)$ is non-negative. First, assume that $f(\lambda)$ is bounded, and

$$f(\lambda) < M \qquad \text{for all} \quad \lambda \epsilon V.$$

Put $$\phi(U) = \int_U f(\lambda)d\mu(U) ,$$

and define $V_n^{(p)}$ and $f_p(\lambda)$ as in the preceding section. Then

$$f_p(\lambda) = 2^{-p}(n-1) < M , \tag{1}$$

and $\{f_p(\lambda)\}$ converges uniformly to $f^*(\lambda)$ which is equal to $f(\lambda)$ except the points of set H where $\mu(H) = 0$. Next, divide V into the sum of sets contained in $\mathfrak{K}$, such that

$$V = W_1^{(p)} + W_2^{(p)} + \dots + W_m^{(p)} + \dots$$

where $$2^{-p}(m-1)\beta(U) \leqq \mu(U) \leqq 2^{-p}m\beta(U) \tag{2}$$

for all subsets U of $W_m^{(p)}$ contained in $\mathfrak{K}$. And put

$$g_p(\lambda) = 2^{-p}(m-1) \qquad \text{when} \quad \lambda \epsilon W_m^{(p)} \qquad (m = 1, 2, \dots)$$

for any positive integer p. Then $\{g_p(\lambda)\}$ converges uniformly to $g^*(\lambda)$ which is equal to $g(\lambda)$ almost everywhere (β).

Now $$\int_U f_p(\lambda)g_p(\lambda)d\beta(U) = \sum_{m,n} 2^{-2p}(m-1)(n-1)\beta(UV_n^{(p)}W_m^{(p)})$$

$$\leqq \sum_n 2^{-p}(n-1)\mu(UV_n^{(p)}) \qquad \text{by (2),}$$

$$\leqq \phi(U) \qquad \text{by (2) of the preceding section,}$$

$$\leqq \sum_n 2^{-p} n\mu(UV_n^{(p)}) \qquad \text{,, \quad ,,}$$

$$\leqq \sum_{m,n} 2^{-2p} mn\beta(UV_n^{(p)}W_m^{(p)}) \quad \text{by (2),}$$

$$= \sum_{m,n} 2^{-2p} \{(m-1)(n-1) + (m-1) + (n-1) + 1\}\beta(UV_n^{(p)}W_m^{(p)})$$

$$\leqq \int_U f_p(\lambda)g_p(\lambda)d\beta(U) + 2^{-p}\mu(U) + 2^{-p}M\beta(U) + 2^{-2p}\beta(U)$$

by (1) and (2).

Hence $$\phi(U) = \lim_{p\to\infty}\int_U f_p(\lambda)g_p(\lambda)d\beta(U)\ .$$

But, since the monotone increasing sequence $\{f_p(\lambda)g_p(\lambda)\}$ converges to $f^*(\lambda)g^*(\lambda)$, we have

$$\phi(U) = \int_U f^*(\lambda)g^*(\lambda)d\beta(U)\ .$$ [1]

Since $$\mu(H) = 0\ ,$$

$$g^*(\lambda) = 0 \qquad \text{almost everywhere } (\beta) \text{ in } H\ .$$

But $$f(\lambda) = f^*(\lambda) \qquad \text{except the points of } H\ ,$$

and $$g(\lambda) = g^*(\lambda) \qquad \text{almost everywhere } (\beta) \text{ in } V,$$

we have $$f(\lambda)g(\lambda) = f^*(\lambda)g^*(\lambda) \qquad ,, \qquad ,,$$

Consequently, $$\phi(U) = \int_U f(\lambda)g(\lambda)d\beta(U)\ .$$

Next, assume that $f(\lambda)$ is non-bounded. Put

$$f^M(\lambda) = f(\lambda) \qquad \text{when} \quad f(\lambda) \leqq M\ ,$$
$$= M \qquad \text{when} \quad f(\lambda) > M\ .$$

Then, since $f^M(\lambda)$ is bounded, we have

$$\int_U f^M(\lambda)d\mu(U) = \int_U f^M(\lambda)g(\lambda)d\beta(U)\ .$$

But, since $\int_U f(\lambda)d\mu(U)$ is finite, and the monotone increasing sequence $\{f^M(\lambda)g(\lambda)\}$ converges to $f(\lambda)g(\lambda)$, we have

$$\int_U f(\lambda)d\mu(U) = \int_U f(\lambda)g(\lambda)d\beta(U)\ .$$ [1]

Thus, we have proved the theorem.

Put $$\phi(U) = \int_U f(\lambda)d\mu(U) = \int_U f(\lambda)g(\lambda)d\beta(U)\ ,$$

then $$f(\lambda)g(\lambda) = D_{\beta(U)}\phi(\lambda) \qquad \text{almost everywhere } (\beta) \text{ in } V,$$

and $$f(\lambda) = D_{\mu(U)}\phi(\lambda) \qquad \text{except the points of } H\ ,$$

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(1) Saks, *loc. cit.*, 254.

where $\mu(H)=0$. But

$$g(\lambda)=D_{\beta(U)}\mu(\lambda) \qquad \text{almost everywhere } (\beta) \text{ in } V,$$
$$=0 \qquad ,, \qquad ,, \qquad \text{in } H.$$

Hence, *we have the following relation:*

$$D_{\beta(U)}\phi(\lambda)=D_{\mu(U)}\phi(\lambda)D_{\beta(U)}\mu(\lambda) \quad \textit{almost everywhere } (\beta) \textit{ in } V.$$

Unitary Equivalence of Resolutions of Identity.

28. *Let* $\mathfrak{q}_1(U)$ *and* $\mathfrak{q}_2(U)$ *be two completely additive vector valued set functions generated by* $\mathrm{E}(U)$, *and* $\mathrm{E}(U)\mathfrak{b}_2=\mathfrak{q}_2(U)$. *Then a necessary and sufficient condition that* $\mathfrak{M}(\mathfrak{q}_1)=\mathfrak{M}(\mathfrak{q}_2)$ *is that*

$$\sigma_1(U)\sim\sigma_2(U)\,,\text{ (1)}$$

and $\mathfrak{b}_2$ *belongs to* $\mathfrak{M}(\mathfrak{q}_1)$.(2)

First, the condition is necessary. For, when $\mathfrak{M}(\mathfrak{q}_1)=\mathfrak{M}(\mathfrak{q}_2)$, it is evident that $\mathfrak{b}_2$ belongs to $\mathfrak{M}(\mathfrak{q}_1)$. Hence, by sec. 11, $\mathfrak{b}_2$ may be expressed in the following form

$$\mathfrak{b}_2=\int_V D_{\sigma_1(U)}\xi(\lambda)d\mathfrak{q}_1(U)\,.$$

Therefore, by sec. 16 $\quad \mathfrak{q}_2(U)=\int_U D_{\sigma_1(U)}\xi(\lambda)d\mathfrak{q}_1(U)\,.$

Hence, we have $\quad \sigma_2(U)\prec\sigma_1(U)\,.$

Similarly, $\quad \sigma_1(U)\prec\sigma_2(U)\,.$

Consequently, $\quad \sigma_1(U)\sim\sigma_2(U)\,.$

Next, the condition is sufficient. For, since $\mathfrak{b}_2$ belongs to $\mathfrak{M}(\mathfrak{q}_1)$, it is expressed as follows:

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(1) In this paper, I use the following notations, which are used by Stone (*loc. cit.*, 214): When $\sigma_2(U)$ is absolutely continuous with respect to $\sigma_1(U)$, we write $\sigma_1(U)\succ\sigma_2(U)$, and when $\sigma_1(U)\succ\sigma_2(U)$, $\sigma_2(U)\succ\sigma_1(U)$ both hold, we write $\sigma_1(U)\sim\sigma_2(U)$.

(2) Cf. Stone, *loc. cit.*, 244.

$$\mathfrak{b}_2 = \int_V D_{\sigma_1(U)}\xi(\lambda) d\mathfrak{q}_1(U) ,$$

where $$\xi(U) = \big(\mathfrak{b}_2 , \mathfrak{q}_1(U)\big) . \tag{1}$$

Hence $$\mathfrak{q}_2(U) = \int_U D_{\sigma_1(U)}\xi(\lambda) d\mathfrak{q}_1(U)$$

and $$\sigma_2(U) = \int_U | D_{\sigma_1(U)}\xi(\lambda) |^2 d\sigma_1(U) . \tag{2}$$

Let $E(U)\mathfrak{b}_1 = \mathfrak{q}_1(U)$, and let $\mathfrak{a}$ be the component of $\mathfrak{b}_1$ contained in $\mathfrak{M}(\mathfrak{q}_2)$. Then

$$\mathfrak{a} = \int_V D_{\sigma_2(U)}\zeta(\lambda) d\mathfrak{q}_2(U) , \tag{3}$$

where $$\zeta(U) = \big(\mathfrak{b}_1 , \mathfrak{q}_2(U)\big) .$$

But $$\zeta(U) = \big(\mathfrak{b}_1 , E(U)\mathfrak{b}_2\big) = \big(E(U)\mathfrak{b}_1 , \mathfrak{b}_2\big) = \big(\mathfrak{q}_1(U), \mathfrak{b}_2\big) .$$

Hence, form (1) $$\zeta(U) = \overline{\xi(U)} .$$

Then, by sec. 27 $D_{\sigma_2(U)}\zeta(\lambda) D_{\sigma_1(U)}\sigma_2(\lambda) = \overline{D_{\sigma_1(U)}\xi(\lambda)}$

almost everywhere (σ_1). Hence, by (2), we have

$$D_{\sigma_2(U)}\zeta(\lambda) D_{\sigma_1(U)}\xi(\lambda) = 1$$

almost everywhere (σ_1). Therefore, from (3) and (2)

$$\| \mathfrak{a} \|^2 = \int_V | D_{\sigma_2(U)}\zeta(\lambda) |^2 d\sigma_2(U) = \int_V | D_{\sigma_2(U}\ \zeta(\lambda) D_{\sigma_1(U)}\xi(\lambda) |^2 d\sigma_1(U)$$
$$= \int_V d\sigma_1(U) = \| \mathfrak{b}_1 \|^2 .$$

Consequently, $$\mathfrak{a} = \mathfrak{b}_1 .$$

That is, $\mathfrak{b}_1$ belongs to $\mathfrak{M}(\mathfrak{q}_2)$. Hence $\mathfrak{q}_1(U) = E(U)\mathfrak{b}_1$ belongs to $\mathfrak{M}(\mathfrak{q}_2)$.

Therefore $$\mathfrak{M}(\mathfrak{q}_1) \leqq \mathfrak{M}(\mathfrak{q}_2) .$$

But, by the assumption $\mathfrak{b}_2$ belongs to $\mathfrak{M}(\mathfrak{q}_1)$, therefore

$$\mathfrak{M}(\mathfrak{q}_2) \leqq \mathfrak{M}(\mathfrak{q}_1) .$$

Consequently, $$\mathfrak{M}(\mathfrak{q}_1) = \mathfrak{M}(\mathfrak{q}_2) .$$

29. Let $\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots, \mathfrak{q}_i(U), \ldots\}$ be an orthogonal system, generated by a resolution of identity $E(U)$. When $\sum_i \sigma_i(U)$ converges to a finite value, say $\sigma(U)$, for all Borel subsets U of V, then by sec. 20, $\sum_i \mathfrak{q}_i(U)$ converges strongly to a completely additive vector valued set function, say $\mathfrak{q}(U)$, with base $\sigma(U)$, which is generated by $E(U)$. If $\mathfrak{f}$ is orthogonal to all $\mathfrak{M}(\mathfrak{q}_i)$, then

$$\left(\mathfrak{f}, \mathfrak{q}_i(U)\right) = 0 \qquad \text{for all } U, \qquad (i = 1, 2, \ldots).$$

Hence
$$\left(\mathfrak{f}, \mathfrak{q}(U)\right) = 0 \qquad \text{for all } U.$$

Therefore, $\mathfrak{f}$ is orthogonal to $\mathfrak{M}(\mathfrak{q})$. Consequently

$$\mathfrak{M}(\mathfrak{q}) \leqq \mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2) \oplus \ldots \oplus \mathfrak{M}(\mathfrak{q}_i) \oplus \ldots . \tag{1}$$

Now, *if* $\quad V = V_0 + V_1 + V_2 + \ldots + V_i + \ldots,$

and
$$\mathfrak{q}_j(V_i) = 0 \quad \textit{when} \quad i \neq j \quad \begin{pmatrix} i = 0, 1, 2, \ldots \\ j = 1, 2, 3, \ldots \end{pmatrix}, \tag{2}$$

then
$$\mathfrak{M}(\mathfrak{q}) = \mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2) \oplus \ldots \oplus \mathfrak{M}(\mathfrak{q}_i) \oplus \ldots .$$

From (2),
$$\mathfrak{q}(UV_j) \,[=]\, \sum_i \mathfrak{q}_i(UV_j) = \mathfrak{q}_j(UV_j),$$
and
$$\mathfrak{q}_j(U) \,[=]\, \sum_i \mathfrak{q}_j(UV_i) = \mathfrak{q}_j(UV_j).$$
Hence
$$\mathfrak{q}(UV_j) = \mathfrak{q}_j(U).$$

Let $\mathfrak{f}$ be any vector which is orthogonal to $\mathfrak{M}(\mathfrak{q})$. Then

$$\left(\mathfrak{f}, \mathfrak{q}(UV_j)\right) = 0 \qquad \text{for all } U.$$

Hence
$$\left(\mathfrak{f}, \mathfrak{q}_j(U)\right) = 0 \qquad \text{for all } U.$$

That is, $\mathfrak{f}$ is orthogonal to $\mathfrak{M}(\mathfrak{q}_j)$. But, this relation holds for all j, we have $\quad \mathfrak{M}(\mathfrak{q}) \geqq \mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2) \oplus \ldots \oplus \mathfrak{M}(\mathfrak{q}_j) \oplus \ldots .$

Consequently, from (1), we have the required result.

30. Let
$$\{\mathfrak{q}^{(1)}(U), \mathfrak{q}^{(2)}(U), \ldots, \mathfrak{q}^{(i)}(U), \ldots\} \tag{1}$$
be an orthogonal system in $\mathfrak{H}$. When

$$\sigma^{(1)}(U) \succ \sigma^{(2)}(U) \succ \ldots \succ \sigma^{(i)}(U) \succ \ldots,$$

then (1) is called an *ordered orthogonal system*. In this connection, we have the following theorem:

Let E(U) *be a resolution of identity. Then there exists a complete ordered orthogonal system in* $\mathfrak{H}$, *generated by* E(U).[1]

By sec. 18, we have a complete orthogonal system

$$\left\{\mathfrak{q}_1(U),\ \mathfrak{q}_2(U),\ \ldots\ldots,\ \mathfrak{q}_3(U),\ \ldots\ldots\right\} \tag{1}$$

in $\mathfrak{H}$, generated by E(U). We may assume that $\sum_i \sigma_i(U)$ converges. For, if $\sum_i \sigma_i(U)$ does not converge, take a sequence $\{c_i\}$ of positive numbers such that $\sum_i c_i^2 \sigma_i(U)$ converges; and take the complete orthogonal system

$$\left\{c_1\mathfrak{q}_1(U),\ c_2\mathfrak{q}_2(U),\ \ldots\ldots,\ c_i\mathfrak{q}_i(U),\ \ldots\ldots\right\}$$

instead of (1).

By the so-called regularizing transposition,[2] we replace (1) by an orthogonal system

$$\left\{\mathfrak{q}^{(1)}(U),\ \mathfrak{q}_2^{(1)}(U),\ \mathfrak{q}_3^{(1)}(U),\ \ldots\ldots,\ \mathfrak{q}_i^{(1)}(U),\ \ldots\ldots\right\}$$

generated by E(U), which has the following properties:

(α) $\mathfrak{M}(\mathfrak{q}^{(1)})$ contains $\mathfrak{M}(\mathfrak{q}_1)$ and certain subsets of $\mathfrak{M}(\mathfrak{q}_2)$, $\mathfrak{M}(\mathfrak{q}_3)$, ... ;

(β) $\mathfrak{M}(\mathfrak{q}_i^{(1)}) \subseteqq \mathfrak{M}(\mathfrak{q}_i) \qquad (i = 2, 3, \ldots\ldots)$;

(γ) $\mathfrak{M}(\mathfrak{q}^{(1)}) \oplus \mathfrak{M}(\mathfrak{q}_2^{(1)}) \oplus \ldots\ldots \oplus \mathfrak{M}(\mathfrak{q}_i^{(1)}) \ldots\ldots$
$= \mathfrak{M}(\mathfrak{q}_1) \oplus \mathfrak{M}(\mathfrak{q}_2) \oplus \ldots\ldots \oplus \mathfrak{M}(\mathfrak{q}_i) \oplus \ldots\ldots$;

(δ) $\sigma^{(1)}(U) \succ \sigma_i^{(1)}(U) \qquad (i = 2, 3, \ldots\ldots)$.

The regularizing transposition can be described as follows: Each function of (1) is to be resolved into two components $\mathfrak{p}_i^{(1)}(U)$ and $\mathfrak{q}_i^{(1)}(U)$ such that

$$\mathfrak{q}_i(U) = \mathfrak{p}_i^{(1)}(U) + \mathfrak{q}_i^{(1)}(U),$$

(1) Cf. Stone, *loc. cit.*, 250.

(2) I am indebted for this process to Stone (*ibid.*), but I have simplified the proof.

the closed linear manifolds $\mathfrak{M}(\mathfrak{p}_i^{(1)})$ and $\mathfrak{M}(\mathfrak{q}_i^{(1)})$ are orthogonal, and

$$\mathfrak{M}(\mathfrak{q}_i) = \mathfrak{M}(\mathfrak{p}_i^{(1)}) \oplus \mathfrak{M}(\mathfrak{q}_i^{(1)}) .$$

This resolution is accomplished by induction. We first put

$$\mathfrak{p}_1^{(1)}(U) = \mathfrak{q}_1(U), \qquad \mathfrak{q}_1^{(1)}(U) = 0 .$$

When the mutually orthogonal vector valued set functions $\mathfrak{p}_1^{(1)}(U)$, $\mathfrak{q}_1^{(1)}(U)$, $\mathfrak{p}_2^{(1)}(U)$, $\mathfrak{q}_2^{(1)}(U)$, ..., $\mathfrak{p}_{i-1}^{(1)}(U)$, $\mathfrak{q}_{i-1}^{(1)}(U)$ have been defined, we put

$$\mathfrak{r}_i(U) = \sum_{n=1}^{i-1} \mathfrak{p}_n^{(1)}(U) , \qquad \rho_i(U) = \| \mathfrak{r}_i(U) \|^2$$

and

$$\tau_i(U) = \rho_i(U) + \sigma_i(U) .$$

Let W_i $(i > 1)$ be the set of all points λ of V where the Baire's function $D_{\tau_i(U)}\, \rho_i(\lambda) = 0$. And put

$$\mathfrak{p}_i^{(1)}(U) = \mathfrak{q}_i(UW_i) , \qquad \mathfrak{q}_i^{(1)}(U) = \mathfrak{q}_i(U - UW_i) . \tag{2}$$

Thus we have a sequence of Borel sets

$$W_2,\ W_3,\ \ldots .,\ W_i,\ \ldots . .$$

Since $\rho_p(W_p) = 0$, it must be that

$$\rho_i(W_p) = 0 \qquad \text{when} \qquad i < p .$$

Hence $D_{\tau_i(U)}\, \rho_i(\lambda) = 0$ almost everywhere (τ_i) in W_p, when $i < p$. Hence, if we put

$$W_i' = W_i \dotplus W_{i+1} \dotplus \ldots . ,$$

$D_{\tau_i(U)}\, \rho_i(\lambda) = 0$ almost everywhere (τ_i), therefore, of course, almost everywhere (σ_i), in W_i'. Hence

$$\mathfrak{q}_i(W_i' - W_i) = 0 .$$

Therefore, from (2)

$$\mathfrak{p}_i^{(1)}(U) = \mathfrak{q}_i(UW_i') , \qquad \mathfrak{q}_i^{(1)}(U) = \mathfrak{q}_i(U - UW_i') .$$

And
$$W_i' \geqq W_{i+1}' \qquad (i = 2, 3, \ldots .) .$$

Put
$$V_0 = W_2' W_3' \ldots . W_i' \ldots . , \qquad V_1 = V - W_2' ,$$

$$V_i = W_i' - W_{i+1}' \qquad (i = 2, 3, \ldots .).$$

Then $$V = V_0 + V_1 + \dots + V_i + \dots$$

and $$\mathfrak{p}_j^{(1)}(V_i) = 0 \quad \text{when} \quad i \neq j \quad \begin{pmatrix} i = 0, 1, 2, \dots \\ j = 1, 2, \dots \end{pmatrix}. \tag{3}$$

For, since $$\mathfrak{r}_i(W_i') = 0 ,$$

we have $$\mathfrak{p}_j^{(1)}(W_i') = 0 \quad \text{when} \quad j < i .$$

Hence $$\mathfrak{p}_j^{(1)}(V_i) = 0 \quad \text{when} \quad j < i ,$$

and $$\mathfrak{p}_j^{(1)}(V_0) = 0 .$$

Since $$V_i \subseteqq V - W_j' \quad \text{when} \quad 0 < i < j ,$$

and $$\mathfrak{p}_j^{(1)}(U) = \mathfrak{q}_j(UW_j') ,$$

we have $$\mathfrak{p}_j^{(1)}(V_i) = 0 \quad \text{when} \quad 0 < i < j .$$

Since $$\mathfrak{p}_i^{(1)}(U) = \int_U h_{(W_i)}(\lambda) d\mathfrak{q}_i(U) ,$$

$$\mathfrak{q}_i^{(1)}(U) = \int_U \left\{1 - h_{(W_i)}(\lambda)\right\} d\mathfrak{q}_i(U) ,$$

by sec. 16, $\mathfrak{p}_i^{(1)}(U)$ and $\mathfrak{q}_i^{(1)}(U)$ are generated by $E(U)$, and are orthogonal. And $\mathfrak{M}(\mathfrak{p}_i^{(1)})$, $\mathfrak{M}(\mathfrak{q}_i^{(1)})$ are contained in $\mathfrak{M}(\mathfrak{q}_i)$. Hence

$$\mathfrak{M}(\mathfrak{q}_i) \geqq \mathfrak{M}(\mathfrak{p}_i^{(1)}) \oplus \mathfrak{M}(\mathfrak{q}_i^{(1)}) .$$

Since $$\mathfrak{q}_i(U) = \mathfrak{p}_i^{(1)}(U) + \mathfrak{q}_i^{(1)}(U) ,$$

we have $$\mathfrak{M}(\mathfrak{q}_i) \leqq \mathfrak{M}(\mathfrak{p}_i^{(1)}) \oplus \mathfrak{M}(\mathfrak{q}_i^{(1)}) .$$

Therefore, $$\mathfrak{M}(\mathfrak{q}_i) = \mathfrak{M}(\mathfrak{p}_i^{(1)}) \oplus \mathfrak{M}(\mathfrak{q}_i^{(1)}) .$$

Since $\mathfrak{p}_i^{(1)}(U)$ belongs to $\mathfrak{M}(\mathfrak{q}_i)$ for any value of i, and $\sum_i \sigma_i(U)$ converges, by sec. 20 $\sum_i \mathfrak{p}_i^{(1)}(U)$ converges strongly to a vector valued set function, say $\mathfrak{q}^{(1)}(U)$, generated by $E(U)$. And since (3) holds, by sec. 29,

$$\mathfrak{M}(\mathfrak{q}^{(1)}) = \mathfrak{M}(\mathfrak{p}_1^{(1)}) \oplus \mathfrak{M}(\mathfrak{p}_2^{(1)}) \oplus \dots \oplus \mathfrak{M}(\mathfrak{p}_i^{(1)}) \oplus \dots .$$

Then $\left\{\mathfrak{q}^{(1)}(U), \mathfrak{q}_2^{(1)}(U), \mathfrak{q}_3^{(1)}(U), \dots, \mathfrak{q}_i^{(1)}(U), \dots\right\}$ is an orthogonal system generated by $E(U)$, which satisfies (α), (β) and (γ).

To prove that (δ) holds, I will shew that $\mathfrak{q}_i^{(1)}(U_0) = 0$ for any Borel set U_0 where $\mathfrak{q}^{(1)}(U_0) = 0$. Since

$$\mathfrak{q}_i^{(1)}(U) = \mathfrak{q}_i(U - UW_i) ,$$

when $U_0 \leqq W_i$, we have $\mathfrak{q}_i^{(1)}(U_0) = 0$. Hence we may assume that $U_0 \leqq V - W_i$. Then

$$D_{\tau_i(U)} \rho_i(\lambda) > 0 \qquad \text{in } U_0 . \tag{4}$$

And, if $\mathfrak{q}_i^{(1)}(U_0) \neq 0$,

then, since $\mathfrak{q}_i^{(1)}(U_0) = \mathfrak{q}_i(U_0)$,

we have $\tau_i(U_0) \neq 0$. (5)

Hence, from (4) and (5), we have

$$\rho_i(U_0) \neq 0 , \qquad \text{that is,} \qquad \mathfrak{r}_i(U_0) \neq 0 ,$$

which contradicts the fact: $\mathfrak{q}^{(1)}(U_0) = 0$. Therefore, it must be that

$$\mathfrak{q}_i^{(1)}(U_0) = 0 .$$

Next, consider the orthogonal system $\left\{\mathfrak{q}_2^{(1)}(U), \mathfrak{q}_3^{(1)}(U), \ldots, \mathfrak{q}_i^{(1)}(U), \ldots\right\}$ generated by $E(U)$. Then, since

$$\sigma_i^{(1)}(U) \leqq \sigma_i(U) \qquad (i = 2, 3, \ldots) ,$$

$\sum_{i=2}^{\infty} \sigma_i^{(1)}(U)$ converges. Hence by the regularizing transposition, we have an orthogonal system $\left\{\mathfrak{q}^{(2)}(U), \mathfrak{q}_3^{(2)}(U), \mathfrak{q}_4^{(2)}(U), \ldots, \mathfrak{q}_i^{(2)}(U), \ldots\right\}$ generated by $E(U)$, which satisfies the corresponding conditions (α), (β), (γ), and (δ). But

$$\mathfrak{q}^{(2)}(U) \, [=] \sum_{i=2}^{\infty} \mathfrak{p}_i^{(2)}(U) ,$$

where $\mathfrak{q}_i^{(1)}(U) = \mathfrak{p}_i^{(2)}(U) + \mathfrak{q}_i^{(2)}(U) \qquad (i = 2, 3, \ldots)$,

$\mathfrak{p}_i^{(2)}(U)$ and $\mathfrak{q}_i^{(2)}(U)$ being orthogonal. Hence

$$\sigma^{(1)}(U) \succ \sigma_i^{(1)}(U) \succ \pi_i^{(2)}(U) \qquad (i = 2, 3, \ldots) ,$$

where $\pi_i^{(2)}(U)$ is the base of $\mathfrak{p}_i^{(2)}(U)$. And we have

$$\sigma^{(1)}(U) \succ \sigma^{(2)}(U) .$$

Next, from $\left\{\mathfrak{q}_3^{(2)}(U), \mathfrak{q}_4^{(2)}(U), \ldots, \mathfrak{q}_i^{(2)}(U), \ldots\right\}$ by the regularizing transposition, we have $\left\{\mathfrak{q}^{(3)}(U), \mathfrak{q}_4^{(3)}(U), \mathfrak{q}_5^{(3)}(U), \ldots, \mathfrak{q}_i^{(3)}(U), \ldots\right\}$ and

$$\sigma^{(2)}(U) \succ \sigma^{(3)}(U) .$$

Repeating this process, we have the required ordered orthogonal system

$$\left\{\mathfrak{q}^{(1)}(U), \mathfrak{q}^{(2)}(U), \ldots, \mathfrak{q}^{(i)}(U), \ldots\right\} .$$

31. *Let* $\left\{\mathfrak{q}_1(U), \mathfrak{q}_2(U), \ldots, \mathfrak{q}_i(U), \ldots\right\}$ (1)

and $\left\{\mathfrak{q}_1'(U), \mathfrak{q}_2'(U), \ldots, \mathfrak{q}_i'(U), \ldots\right\}$

be two complete ordered orthogonal systems in $\mathfrak{H}$, *generated by* E(U), *then*

$$\sigma_i(U) \sim \sigma_i'(U) \qquad (i = 1, 2, \ldots) .$$[1]

Let $\mathfrak{b}_\nu' = \mathfrak{q}_\nu'(V)$, and expand $\mathfrak{b}_\nu'$ with respect to (1), then by sec. 12,

$$\mathfrak{b}_\nu' [=] \sum_i \int_V D_{\sigma_i(U)} \xi_i^{(\nu)}(\lambda) d\mathfrak{q}_i(U) ,$$

where $\xi_i^{(\nu)}(U) = \left(\mathfrak{b}_\nu', \mathfrak{q}_i(U)\right) \qquad (\nu, i = 1, 2, \ldots) .$

Hence, by sec. 16

$$\mathfrak{q}_\nu'(U) = \mathrm{E}(U)\mathfrak{b}_\nu' [=] \sum_i \int_U D_{\sigma_i(U)} \xi_i^{(\nu)}(\lambda) d\mathfrak{q}_i(U) . \qquad (2)$$

When $\nu = 1$, we have

$$\sigma_1'(U) = \sum_i \int_U | D_{\sigma_i(U)} \xi_i^{(1)}(\lambda) |^2 d\sigma_i(U) .$$

But $\sigma_1(U) \succ \sigma_2(U) \succ \ldots \succ \sigma_i'U) \succ \ldots ;$

hence, we have $\sigma_1'(U) \prec \sigma_1(U) .$

Similarly we have $\sigma_1(U) \prec \sigma_1'(U) .$

Consequently, $\sigma_1(U) \sim \sigma_1'(U) .$

Next we shall shew that if

$$\sigma_i(U) \sim \sigma_i'(U) \qquad (i = 1, 2, \ldots, n) ,$$

(1) Cf. Stone, *loc. cit.*, 258.

then $$\sigma_{n+1}(U) \sim \sigma'_{n+1}(U) .$$

Assume that the relation

$$\sigma_{n+1}(U) \succ \sigma'_{n+1}(U)$$

does not hold. Then a Borel set W exists such that

$$\sigma_{n+1}(W) = 0 \qquad \text{and} \qquad \sigma'_{n+1}(W) \neq 0 .$$

Then a Borel subset W' of W exists, where $\sigma'_{n+1}(W') \neq 0$, such that

$$D_{\sigma_1(U)}\sigma'_{n+1}(\lambda) > 0 \text{ [1]} \tag{3}$$

at all points of W'.

Now $$D_{\sigma_1(U)}\sigma'_i(\lambda) > 0 \tag{4}$$

almost everywhere (σ_1) in W', for all values of $i \leqq n$. For, if

$$D_{\sigma_1(U)}\sigma'_i(\lambda) = 0$$

at all points of a Borel subset W_0' of W', where $\sigma_1(W_0') \neq 0$, then

$$\sigma'_i(W'_0) = 0 .$$

Since $\sigma'_{n+1}(U) \prec \sigma'_i(U)$, we have

$$\sigma'_{n+1}(W'_0) = 0 .$$

Hence $$D_{\sigma_1(U)}\sigma'_{n+1}(\lambda) = 0$$

almost everywhere (σ_1) in W_0', which contradicts (3).

Let U be any Borel subset of W'. Then, since

$$\sigma_i(U) = 0 \qquad (i = n+1, n+2, \ldots\ldots) ,$$

we have, from (2),

$$\sigma'_\nu(U) = \sum_{i=1}^{n}\int_U |D_{\sigma_1(U)}\xi_i^{(\nu)}(\lambda)|^2 d\sigma_i(U)$$

$$= \int_U \sum_{i=1}^{n} |D_{\sigma_1(U)}\xi_i^{(\nu)}(\lambda)|^2 D_{\sigma_1(U)}\sigma_i(\lambda) d\sigma_1(U) ,$$

---

(1) Since $\sigma'_1(U) \succ \sigma'_{n+1}(U)$ and $\sigma'_1(U) \sim \sigma_1(U)$, we have $\sigma_1(U) \succ \sigma'_{n+1}(U)$.

and $$0=\left(\mathfrak{q}'_\mu(U),\mathfrak{q}'_\nu(U)\right)=\sum_{i=1}^n\int_U D_{\sigma_i(U)}\xi_i^{(\mu)}(\lambda)\overline{D_{\sigma_i(U)}\xi_i^{(\nu)}(\lambda)}d\sigma_i(U)$$
$$=\int_U\sum_{i=1}^n D_{\sigma_i(U)}\xi_i^{(\mu)}(\lambda)\overline{D_{\sigma_i(U)}\xi_i^{(\nu)}(\lambda)}D_{\sigma_1(U)}\sigma_i(\lambda)\,d\sigma_1(U)$$

when $\mu\neq\nu$. Hence

$$\left.\begin{aligned}D_{\sigma_1(U)}\sigma'_\nu(\lambda)&=\sum_{i=1}^n|D_{\sigma_i(U)}\xi_i^{(\nu)}(\lambda)|^2D_{\sigma_1(U)}\sigma_i(\lambda)\,,\\0&=\sum_{i=1}^n D_{\sigma_i(U)}\xi_i^{(\mu)}(\lambda)\overline{D_{\sigma_i(U)}\xi_i^{(\nu)}(\lambda)}D_{\sigma_1(U)}\sigma_i(\lambda)\quad(\mu\neq\nu)\,,\end{aligned}\right\}\tag{5}$$

almost everywhere (σ_1) in W', where $\mu,\nu=1,2,\ldots\ldots$

Since (3) and (4) hold, by means of the abbreviations

$$A_{\nu i}=D_{\sigma_i(U)}\xi_i^{(\nu)}(\lambda)\left\{\frac{D_{\sigma_1(U)}\sigma_i(\lambda)}{D_{\sigma_1(U)}\sigma'_\nu(\lambda)}\right\}^{\frac{1}{2}},$$

$$x_i=\overline{D_{\sigma_i(U)}\xi_i^{(n+1)}(\lambda)}\left\{\frac{D_{\sigma_1(U)}\sigma_i(\lambda)}{D_{\sigma_1(U)}\sigma'_{n+1}(\lambda)}\right\}^{\frac{1}{2}},$$

where $i,\nu=1,2,\ldots\ldots,n$, (5) reduces to the system

$$\left.\begin{aligned}&\sum_{i=1}^n A_{\mu i}\bar{A}_{\nu i}=\delta_{\mu\nu}\,,\\ \text{and}\quad&\sum_{i=1}^n A_{\nu i}x_i=0\,,\qquad\sum_{i=1}^n|x_i|^2=1\,,\end{aligned}\right\}\tag{6}$$

where $\mu,\nu=1,2,\ldots\ldots,n$.

If we interpret $(x_1,x_2,\ldots\ldots,x_n)$ as a point in n-dimensional Euclidean space and $\{A_{\nu i}\}$ as the matrix of a linear transformation T in this space, then (6) signifies that the point $(x_1,x_2,\ldots\ldots,x_n)$ is at unit distance from $(0,0,\ldots,0)$, that the transformation T is isometric, and T carries $(x_1,x_2,\ldots\ldots,x_n)$ into $(0,0,\ldots\ldots,0)$. But this is absurd.

Hence, it must be that $\quad\sigma_{n+1}(U)\succ\sigma'_{n+1}(U)$.

Similarly, we have $\quad\sigma'_{n+1}(U)\succ\sigma_{n+1}(U)$.

Therefore, $\quad\sigma_{n+1}(U)\sim\sigma'_{n+1}(U)$.

Consequently, by mathematical induction

$$\sigma_i(U)\sim\sigma'_i(U)$$

for all values of i.

32. *Let* $\mathbf{E}_1(U)$ *and* $\mathbf{E}_2(U)$ *be two resolutions of identity. And let* $\mathfrak{q}_1(U)$ *and* $\mathfrak{q}_2(U)$ *be completely additive vector valued set functions generated by* $\mathbf{E}_1(U)$ *and* $\mathbf{E}_2(U)$ *respectively. There exists an isometric transformation* $\mathbf{V}$ *with domain* $\mathfrak{M}(\mathfrak{q}_1)$ *and range* $\mathfrak{M}(\mathfrak{q}_2)$, *such that*

$$\mathbf{V}\mathbf{E}_1(U)\mathbf{V}^{-1}\mathfrak{f} = \mathbf{E}_2(U)\,\mathfrak{f}$$

for all vectors $\mathfrak{f}$ *in* $\mathfrak{M}(\mathfrak{q}_2)$, *if and only if*

$$\sigma_1(U) \sim \sigma_2(U)\ .\text{[1]} \tag{1}$$

First, the condition (1) is necessary. For, put

$$\mathfrak{a}_1 = \mathbf{V}^{-1}\mathfrak{b}_2\ ,$$

where $\mathfrak{b}_2$ is a vector such that

$$\mathbf{E}(U)\mathfrak{b}_2 = \mathfrak{q}_2(U)\ .$$

Then
$$\begin{aligned} \|\mathbf{E}_1(U)\mathfrak{a}_1\|^2 &= \|\mathbf{E}_1(U)\mathbf{V}^{-1}\mathfrak{b}_2\|^2 = \|\mathbf{V}\mathbf{E}_1(U)\mathbf{V}^{-1}\mathfrak{b}_2\|^2 \\ &= \|\mathbf{E}_2(U)\mathfrak{b}_2\|^2 = \sigma_2(U)\ . \end{aligned} \tag{2}$$

But, since $\mathfrak{a}_1$ is a vector in $\mathfrak{M}(\mathfrak{q}_1)$, $\mathfrak{a}_1$ can be expressed in the form

$$\mathfrak{a}_1 = \int_V D_{\sigma_1(U)}\xi(\lambda)d\mathfrak{q}_1(U)\ .$$

Hence
$$\mathbf{E}_1(U)\mathfrak{a}_1 = \int_U D_{\sigma_1(U)}\xi(\lambda)d\mathfrak{q}_1(U)\ ;$$

therefore, by (2)
$$\sigma_2(U) = \int_U |D_{\sigma_1(U)}\xi(\lambda)|^2 d\sigma_1(U)\ .$$

Consequently,
$$\sigma_2(U) \prec \sigma_1(U)\ .$$

Similarly,
$$\sigma_1(U) \prec \sigma_2(U)\ .$$

Therefore,
$$\sigma_1(U) \sim \sigma_2(U)\ .$$

Next, the condition (1) is sufficient. For, if we put

$$\mathfrak{b}'_1 = \int_V \left\{D_{\sigma_1(U)}\sigma_2(\lambda)\right\}^{\frac{1}{2}} d\mathfrak{q}_1(U)\ ,$$

then $\mathfrak{b}'_1$ belongs to $\mathfrak{M}(\mathfrak{q}_1)$. Let

$$\mathfrak{q}'_1(U) = \mathbf{E}_1(U)\mathfrak{b}'_1\ ,$$

(1) Cf. Stone, *loc. cit.*, 244.

then $$\sigma_1'(U) = \int_U D_{\sigma_1(U)}\,\sigma_2(\lambda)d\sigma_1(U) = \sigma_2(U)\ .$$

Therefore, since $\sigma_1'(U) \sim \sigma_1(U)$ and $\mathfrak{b}_1'$ belongs to $\mathfrak{M}(\mathfrak{q}_1)$, by sec. 28

$$\mathfrak{M}(\mathfrak{q}_1') = \mathfrak{M}(\mathfrak{q}_1)\ .$$

$\mathfrak{M}(\mathfrak{q}_1')$ and $\mathfrak{M}(\mathfrak{q}_2)$ are isomorph to $\mathfrak{L}_2(\sigma_1')$ and $\mathfrak{L}_2(\sigma_2)$ respectively.(1) But, since $\sigma_1'(U) = \sigma_2(U)$, we have $\mathfrak{L}_2(\sigma_1') = \mathfrak{L}_2(\sigma_2)$. Hence, we can difine an isometric transformation $\mathbf{V}$ with domain $\mathfrak{M}(\mathfrak{q}_1')$ and range $\mathfrak{M}(\mathfrak{q}_2)$, such that, if $\mathfrak{f}$ be a vector in $\mathfrak{M}(\mathfrak{q}_2)$ and expressed in the form

$$\mathfrak{f} = \int_V D_{\sigma_2(U)}\,\xi(\lambda)d\mathfrak{q}_2(U)\ ,$$

then $$\mathbf{V}^{-1}\mathfrak{f} = \int_V D_{\sigma_1'(U)}\,\xi(\lambda)d\mathfrak{q}_1'(U)\ .$$

But $$\mathbf{E}_2(U)\mathfrak{f} = \int_U D_{\sigma_2(U)}\,\xi(\lambda)d\mathfrak{q}_2(U) = \int_V D_{\sigma_2(U)}\,\xi(\lambda)h_{(U)}(\lambda)d\mathfrak{q}_2(U)\ ,$$

$$\mathbf{E}_1(U)\mathbf{V}^{-1}\mathfrak{f} = \int_U D_{\sigma_1'(U)}\,\xi(\lambda)d\,\mathfrak{q}_1'(U) = \int_V D_{\sigma_1'(U)}\,\xi(\lambda)\,h_{(U)}(\lambda)d\mathfrak{q}_1'(U)\ .$$

Hence, by the definition of $\mathbf{V}$, we have

$$\mathbf{V}^{-1}\mathbf{E}_2(U)\mathfrak{f} = \mathbf{E}_1(U)\mathbf{V}^{-1}\mathfrak{f}\ ,$$

that is, $$\mathbf{E}_2(U)\mathfrak{f} = \mathbf{V}\mathbf{E}_1(U)\mathbf{V}^{-1}\mathfrak{f}$$

for any vector $\mathfrak{f}$ in $\mathfrak{M}(\mathfrak{q}_2)$.

33. *Let* $\{\mathfrak{q}_1^{(1)}(U),\ \mathfrak{q}_2^{(1)}(U),\ \ldots\ldots,\ \mathfrak{q}_i^{(1)}(U),\ \ldots\ldots\}$ *and* $\{\mathfrak{q}_1^{(2)}(U),\ \mathfrak{q}_2^{(2)}(U),\ \ldots\ldots,\ \mathfrak{q}_i^{(2)}(U),\ \ldots\ldots\}$ *be two complete ordered orthogonal systems generated by* $\mathbf{E}^{(1)}(U)$ *and* $\mathbf{E}^{(2)}(U)$ *respectively. There exists a unitary transformation* $\mathbf{U}$ *such that*

$$\mathbf{U}\mathbf{E}^{(1)}(U)\mathbf{U}^{-1}\mathfrak{f} = \mathbf{E}^{(2)}(U)\,\mathfrak{f} \qquad (1)\ .$$

for all vectors $\mathfrak{f}$ *in* $\mathfrak{H}$, *if and only if*

$$\sigma_i^{(1)}(U) \sim \sigma_i^{(2)}(U) \qquad (i = 1,\ 2,\ \ldots\ldots)\ .^{(2)} \qquad (2)$$

(1) Cf. sec. 11.
(2) Cf. Stone, *loc. cit.*, 263.

First, the condition (2) is necessary. For, put

$$\mathfrak{a}_i^{(2)} = U\mathfrak{b}_i^{(1)} \tag{3}$$

where $\mathfrak{b}_i^{(1)}$ is a vector such that

$$\mathbf{E}^{(1)}(U)\mathfrak{b}_i^{(1)} = \mathfrak{q}_i^{(1)}(U) .$$

And put
$$\mathfrak{p}_i^{(2)}(U) = \mathbf{E}^{(2)}(U)\mathfrak{a}_i^{(2)} , \tag{4}$$

and
$$\rho_i^{(2)}(U) = \| \mathfrak{p}_i^{(2)}(U) \|^2 .$$

Then
$$\left(\mathfrak{p}_i^{(2)}(U), \mathfrak{p}_j^{(2)}(U')\right) = \left(\mathbf{E}^{(2)}(U)\mathfrak{a}_i^{(2)}, \mathbf{E}^{(2)}(U')\mathfrak{a}_j^{(2)}\right)$$
$$= \left(U^{-1}\mathbf{E}^{(2)}(U)\mathfrak{a}_i^{(2)}, U^{-1}\mathbf{E}^{(2)}(U')\mathfrak{a}_j^{(2)}\right),$$

by (1) and (3)
$$= \left(\mathbf{E}^{(1)}(U)\mathfrak{b}_i^{(1)}, \mathbf{E}^{(1)}(U')\mathfrak{b}_j^{(1)}\right) = \left(\mathfrak{q}_i^{(1)}(U), \mathfrak{q}_j^{(1)}(U')\right).$$

Hence
$$\left(\mathfrak{p}_i^{(2)}(U), \mathfrak{p}_j^{(2)}(U')\right) = 0 \qquad \text{when} \qquad i \neq j ,$$

and
$$\rho_i^{(2)}(U) = \sigma_i^{(1)}(U) . \tag{5}$$

Therefore,
$$\rho_1^{(2)}(U) \succ \rho_2^{(2)}(U) \succ \cdots\cdots \succ \rho_i^{(2)}(U) \succ \cdots\cdots .$$

That is,
$$\left\{\mathfrak{p}_1^{(2)}(U), \mathfrak{p}_2^{(2)}(U), \ldots\ldots, \mathfrak{p}_i^{(2)}(U), \ldots\ldots\right\} \tag{6}$$

is an ordered orthogonal system generated by $\mathbf{E}^{(2)}(U)$.

Let $\mathfrak{f}$ be any vector in $\mathfrak{H}$, and let $U^{-1}\mathfrak{f}$ be expanded with respect to

$$\left\{\mathfrak{q}_1^{(1)}(U), \mathfrak{q}_2^{(1)}(U), \ldots\ldots, \mathfrak{q}_i^{(1)}(U), \ldots\ldots\right\},$$

then
$$U^{-1}\mathfrak{f} [=] \sum_i \int_V D_{\sigma_i^{(1)}(U)}\, \xi_i(\lambda) d\mathfrak{q}_i^{(1)}(U) \tag{7}$$

where
$$\xi_i(U) = \left(U^{-1}\mathfrak{f}, \mathfrak{q}_i^{(1)}(U)\right) \qquad (i = 1, 2, \ldots\ldots).$$

But
$$\left(U^{-1}\mathfrak{f}, \mathfrak{q}_i^{(1)}(U)\right) = \left(\mathfrak{f}, U\mathbf{E}^{(1)}(U)\mathfrak{b}_i^{(1)}\right),$$

by (3)
$$= \left(\mathfrak{f}, U\mathbf{E}^{(1)}(U)U^{-1}\mathfrak{a}_i^{(2)}\right),$$

by (1)
$$= \left(\mathfrak{f}, \mathbf{E}^{(2)}(U)\mathfrak{a}_i^{(2)}\right).$$

Hence
$$\xi_i(U) = \left(\mathfrak{f}, \mathfrak{p}_i^{(2)}(U)\right) \qquad (i = 1, 2, \ldots\ldots). \tag{8}$$

And from (7) $\mathfrak{f}\,[=]\sum_i U\int_V D_{\sigma_i^{(1)}(U)}\xi_i(\lambda)d\mathfrak{q}_i^{(1)}(U)$,

by sec. 5 $[=]\sum_i\int_V D_{\sigma_i^{(1)}(U)}\xi_i(\lambda)dU\mathbf{E}^{(1)}(U)\mathfrak{b}_i^{(1)}$,

by (1) $[=]\sum_i\int_V D_{\sigma_i^{(1)}(U)}\xi_i(\lambda)d\mathbf{E}^{(2)}(U)U\mathfrak{b}_i^{(1)}$,

by (3), (4) and (5) $[=]\sum_i\int_V D_{\rho_i^{(2)}(U)}\xi_i(\lambda)d\mathfrak{p}_i^{(2)}(U)$.

Combining with (8), by sec. 12, we see that (6) is complete in $\mathfrak{H}$. Hence (6) is a complete ordered orthogonal system in $\mathfrak{H}$, generated by $\mathbf{E}^{(2)}(U)$. Consequently, by sec. 31

$$\rho_i^{(2)}(U)\sim\sigma_i^{(2)}(U) \qquad (i=1,2,\ \ldots\ldots).$$

Hence, by (5) $\qquad \sigma_i^{(1)}(U)\sim\sigma_i^{(2)}(U) \qquad (i=1,2,\ \ldots\ldots).$

Next, the condition (2) is sufficient. For, since (2) holds, by sec. 32, for any value of i, there exists an isometric transformatiou V_i with domain $\mathfrak{M}(\mathfrak{q}_i^{(1)})$ and range $\mathfrak{M}(\mathfrak{q}_i^{(2)})$, such that

$$V_i\mathbf{E}^{(1)}(U)V_i^{-1}\mathfrak{f}_i=\mathbf{E}^{(2)}(U)\mathfrak{f}_i \tag{9}$$

for all vectors $\mathfrak{f}_i$ in $\mathfrak{M}(\mathfrak{q}_i^{(2)})$. Let $\mathfrak{g}$ be any vector in $\mathfrak{H}$, and $\mathfrak{g}_i$ be its component contained in $\mathfrak{M}(\mathfrak{q}_i^{(1)})$. Thus

$$\mathfrak{g}\,[=]\,\mathfrak{g}_1+\mathfrak{g}_2+\ldots\ldots+\mathfrak{g}_i+\ldots\ldots. \tag{10}$$

Since $V_i\mathfrak{g}_i$ belongs to $\mathfrak{M}(\mathfrak{q}_i^{(2)})$,

$$V_1\mathfrak{g}_1+V_2\mathfrak{g}_2+\ldots\ldots+V_i\mathfrak{g}_i+\ldots\ldots.$$

converges strongly to a vector, say $U\mathfrak{g}$. Then the domain and range of U are $\mathfrak{H}$. Let $\mathfrak{h}$ be another vector in $\mathfrak{H}$, and as (10) let

$$\mathfrak{h}\,[=]\,\mathfrak{h}_1+\mathfrak{h}_2+\ldots\ldots+\mathfrak{h}_i+\ldots\ldots.$$

Then, since $\qquad (U\mathfrak{g},U\mathfrak{h})=\sum_i(V_i\mathfrak{g}_i,V_i\mathfrak{h}_i)=\sum_i(\mathfrak{g}_i,\mathfrak{h}_i)=(\mathfrak{g},\mathfrak{h})$,

U is a unitary transformation. And

$$\mathbf{E}^{(2)}(U)U\mathfrak{g}\,[=]\,\mathbf{E}^{(2)}(U)V_1\mathfrak{g}_1+\mathbf{E}^{(2)}(U)V_2\mathfrak{g}_2+\ldots\ldots+\mathbf{E}^{(2)}(U)V_i\mathfrak{g}_i+\ldots\ldots.$$

Since the component of $\mathbf{E}^{(1)}(U)\mathfrak{g}$ contained in $\mathfrak{M}(\mathfrak{q}_i^{(1)})$ is $\mathbf{E}^{(1)}(U)\mathfrak{g}_i$, we have

$$UE^{(1)}(U)\mathfrak{g}\ [=]\ V_1E^{(1)}(U)\mathfrak{g}_1+V_2E^{(1)}(U)\mathfrak{g}_2+\ldots\ldots+V_iE^{(1)}(U)\mathfrak{g}_i+\ldots\ldots.$$

But, from (9) $\quad V_iE^{(1)}(U)\mathfrak{g}_i = E^{(2)}(U)V_i\mathfrak{g}_i \qquad (i = 1, 2, \ldots\ldots).$

Consequently $\quad E^{(2)}(U)U\mathfrak{g} = UE^{(1)}(U)\mathfrak{g}\ .$

Let $\quad U\mathfrak{g} = \mathfrak{f}\ ,$

then $\quad E^{(2)}(U)\mathfrak{f} = UE^{(1)}(U)U^{-1}\mathfrak{f}\ ;$

that is, (1) holds.