On the Space of Completely Continuous Transformations.

By

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Let $\mathfrak{L}_2(\beta)$ be a class of complex valued set functions $\phi(E)$, which are absolutely continuous with respect to a function of normal sets $\beta(E)$, and $\int_A |D_{\beta(E)}\phi(a)|^2 d\beta(E)$ exist. Then, as in the case of real valued set functions, $\mathfrak{L}_2(\beta)$ is a Hilbert space.(1) In this paper, I will consider the linear transformations T, defined for all set functions in $\mathfrak{L}_2(\beta)$, such that $\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2$ is convergent, $\{\psi_\nu\}$ being a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. And I say that such transformations are completely continuous. I define the norm $\|T\|$ of T and the inner product (T, S) of two transformations T and S, as follows:

$$\|T\| = \left\{\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2\right\}^{\frac{1}{2}},$$

$$(T, S) = \sum_{\nu=1}^{\infty}(T\psi_\nu, S\psi_\nu).$$

I then proceed to investigate the properties of the space of completely continuous transformations.

Next, I find the characteristic functions of the completely continuous Hermitian transformations, and I discover that all completely continuous transformations can be expanded with respect to an orthogonal system of transformations. From these expansions, it follows that all completely continuous transformations T are expressed in the following integral form

$$T\phi(E) = \int_A D_{\beta(E')}\,\mathfrak{K}(E, a')\,D_{\beta(E')}\,\phi(a')\,d\beta(E'), \tag{1}$$

$\mathfrak{K}(E, E')$ belonging to $\mathfrak{L}_2(\beta\beta)$(2). But all transformations of these integral

(1) Hence, all considerations in this paper, from sec. 3 to sec. 17, hold also for an abstract Hilbert space.

(2) $\mathfrak{L}_2(\beta\beta)$ is a class of functions of two sets $\mathfrak{K}(E, E')$, which are absolutely continuous with respect to $\beta(E)\beta(E')$ and $\int_A\int_A |D_{\beta(E)}D_{\beta(E')}\mathfrak{K}(a, a')|^2\, d\beta(E)\, d\beta(E')$ exist.

forms are completely continuous. Hence I have the theorem:

In order that a linear transformation T, defined for all set functions ϕ in $\mathfrak{L}_2(\beta)$, may be expressed in integral form (1), it is necessary and sufficient that T should be completely continuous.

Space of Complex Set Functions.

1. Let $\beta(E)$ be a completely additive, non-negative function of normal sets defined in a metric space R which is compact in itself, and be uniformly monotone(1) at a β-normal set A. Let $\phi(E)$ be a completely additive, complex valued set function, defined at any β-normal subset of A, and absolutely continuous with respect to $\beta(E)$.

If we put

$$\phi(E)=\xi(E)+i\eta(E),$$

where $\xi(E)$ and $\eta(E)$ are real valued set functions, then we define the general derivative of $\phi(E)$ with respect to $\beta(E)$ as

$$D_{\beta(E)}\phi(a)=D_{\beta(E)}\xi(a)+iD_{\beta(E)}\eta(a),$$

and we have the fundamental theorem of integration and differentiation, i. e.

$$\phi(E)=\int_E D_{\beta(E)}\phi(a)d\beta(E). \tag{1}$$

Thus, if we consider two complex valued point functions, which differ only at points of sets whose β-value is zero, as indentical, then by (1) there is a one-to-one correspondence between $\phi(E)$ and its general derivative $f(a)=D_{\beta(E)}\phi(a)$. But, if we define

$$||f||=\left\{\int_A |f(a)|^2 d\beta(E)\right\}^{\frac{1}{2}},$$

$$(f, g)=\int_A f(a)\overline{g(a)}d\beta(E),$$

then the set of all complex valued point functions $f(a)$, for which $\int_A |f(a)|^2 d\beta(E)$ is finite, forms a Hilbert space(2). Hence, if we define

$$||\phi||=\left\{\int_A |D_{\beta(E)}\phi(a)|^2 d\beta(E)\right\}^{\frac{1}{2}}$$

(1) F. Maeda, this journal, **1** (1930), 3; **2** (1932), 175.

(2) J. v. Neumann, *Math. Ann.* **102** (1929), 109-111.

$$(\phi, \psi)=\int_A D_{\beta(E)}\,\phi(a)\overline{D_{\beta(E)}\psi(a)}d\beta(E),$$

the set of all set functions $\phi(E)$, for which $\int_A |D_{\beta(E)}\phi(a)|^2 d\beta(E)$ is finite, forms an isomorph Hilbert space. If we denote this space by $\mathfrak{L}_2(\beta)$, then $\mathfrak{L}_2(\beta)$ has all the properties of the abstract Hilbert space which has been considered in detail by J. v. Neumann(1). Alternately, we can prove this as in the case of real valued set functions.(2)

2. But the space of complex valued set functions has, as in the case of real valued set functions, a peculiar property, which does not hold in the space of complex valued point functions. For, by the Schwarzian inequality

$$\left|\int_E D_{\beta(E)}\phi(a)d\beta(E)\right|^2 \leqq \beta(E)\int_E |D_{\beta(E)}\phi(a)|^2 d\beta(E),$$

hence

$$|\phi(E)|^2 \leqq \beta(E)||\phi||^2 \tag{2}$$

for all β-normal subsets E of A.

Therefore, if $\{\phi_n(E)\}$ converges strongly to $\phi(E)$, i. e.

$$\lim_{n\to\infty} ||\phi_n-\phi||=0,$$

we have, by (2)

$$\lim_{n\to\infty} |\phi_n(E)-\phi(E)|=0,$$

hence

$$\lim_{n\to\infty} \phi_n(E)=\phi(E)$$

for all β-normal subsets E of A. That is, $\{\phi_n(E)\}$ converges to $\phi(E)$ in the ordinary sense. But this property cannot hold generally in the space of point functions.

Space of Completely Continuous Transformations.

3. A transformation T is defined as an operator which transforms any set function ϕ in $\mathfrak{L}_2(\beta)$ to a set function $T\phi$ in $\mathfrak{L}_2(\beta)$. We define the following operations on transformations:

$$S\pm T, \quad cT, \quad TS, \quad T^m,$$

(1) J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, (1932) 18–31.

(2) Cf. my previous paper "On the Space of Real Set Functions," (this journal, 3 (1933) 3–7), where "linear manifolds" is equivalent to "closed linear manifolds" in Neumann's sense.

(S, T are transformations, c is a complex number, $m=0, 1, 2, \ldots$)

$$(S\pm T)\phi=S\phi\pm T\phi, \qquad (cT)\phi=cT\phi^{(1)}$$
$$(TS)\phi=T(S\phi), \qquad T^0=1^{(2)},$$
$$T^m=TT^{m-1}, \qquad (m=1, 2, \ldots.)$$

for all set functions in $\mathfrak{L}_2(\beta)$.

A transformation T is called *linear*, when

$$T(\phi+\psi)=T\phi+T\psi.$$

A linear transformation T^*, which satisfies

$$(T\phi, \psi)=(\phi, T^*\psi),$$

for all set functions ϕ and ψ in $\mathfrak{L}_2(\beta)$, is called the *adjoint* transformation of T. Then

$$(TS)^*=S^*T^* \quad \text{and} \quad T^{**}=T.$$

If a number M exists such that for all set functions ϕ in $\mathfrak{L}_2(\beta)$,

$$\|T\phi\|\leqq M\|\phi\|$$

then T is said to be *limited*.

4. Let T be a limited linear transformation defined for all set functions in $\mathfrak{L}_2(\beta)$, then T has its adjoint transformation $T^{*(3)}$. Let $\{\psi_\nu\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. If $\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2$ converges to a finite number, say l, then l is a constant number independent of the system $\{\psi_\nu\}^{(4)}$. For, let $\{\phi_\mu\}$ be another complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Expand $T\psi_\nu$ with respect to the system $\{\phi_\mu\}$, then

$$T\psi_\nu=\sum_{\mu=1}^{\infty}(T\psi_\nu, \phi_\mu)\phi_\mu$$

and

$$\|T\psi_\nu\|^2=\sum_{\mu=1}^{\infty}|(T\psi_\nu, \phi_\mu)|^2.^{(5)}$$

Hence, $\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|(T\psi_\nu, \phi_\mu)|^2$ converges to l. But

$$\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|(T\psi_\nu, \phi_\mu)|^2=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}|(T\psi_\nu, \phi_\mu)|^2$$
$$=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}|(T^*\phi_\mu, \psi_\nu)|^2=\sum_{\mu=1}^{\infty}\|T^*\phi_\mu\|^2. \tag{1}$$

(1) When $T\phi=0$ for all set functions ϕ, then I shall write $T=0$.

(2) Transformation 1 means that $1\phi=\phi$.

(3) Cf. F. Riesz, *Acta Litterarum, Szeged*, 5 (1930), 29.

(4) This consideration is due to J. v. Neumann, *Quantenmechanik*, loc. cit., 96.

(5) ibid., 28.

Therefore l is independent of the system $\{\psi_\nu\}$.

When $\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2$ converges to a finite number, T is called a *completely continuous* transformation. I will describe the positive value $\sqrt{\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2}$ as the *norm* of T and denote it by the symbol $\|T\|$. That is,

$$\|T\|^2=\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2.$$

By (1) we have

$$\|T\|=\|T^*\|.$$

When T and S are completely continuous; then cT and $T+S$ are likewise so, and

$$\|cT\|=|c|\cdot\|T\|,$$

$$\|T+S\|\leqq\|T\|+\|S\|.^{(1)} \tag{2}$$

5. *Let T be a completely continuous transformation, then*

$$\|T\phi\|\leqq\|T\|\cdot\|\phi\|$$

for any set function ϕ in $\mathfrak{L}_2(\beta)$.

We can always find a complete normalized orthogonal system $\{\psi_\nu\}$ in $\mathfrak{L}_2(\beta)$, so that $\psi_1=\frac{\phi}{\|\phi\|}$. Then, since $\|T\|^2=\sum_{\nu=1}^{\infty}\|T\psi_\nu\|^2$,

$$\|T\psi_1\|\leqq\|T\|.$$

Therefore

$$\|T\phi\|\leqq\|T\|\cdot\|\phi\|.$$

Hence $\|T\|=0$ when and only when $T=0$, that is, $T\phi=0$ for all set functions ϕ in $\mathfrak{L}_2(\beta)$.

If T and S are completely continuous, then TS is likewise so, and

$$\|TS\|\leqq\|T\|\cdot\|S\|.$$

Let $\{\psi_\nu\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then, since by the preceding theorem

$$\|TS\psi_\nu\|\leqq\|T\|\cdot\|S\psi_\nu\|,$$

$$\sum_{\nu=1}^{\infty}\|TS\psi_\nu\|^2\leqq\|T\|^2\sum_{\nu=1}^{\infty}\|S\psi_\nu\|^2=\|T\|^2\cdot\|S\|^2.$$

(1) By Minkowski's inequality $\{\Sigma|a_i+b_i|^2\}^{\frac{1}{2}}\leqq\{\Sigma|a_i|^2\}^{\frac{1}{2}}+\{\Sigma|b_i|^2\}^{\frac{1}{2}}$.

Hence, $\sum_{\nu=1}^{\infty} \| TS\psi_\nu \|^2$ is finite, and

$$\| TS \| \leqq \| T \| \cdot \| S \|.$$

6. If we define the *distance* between two completely continuous transformations T and S by $\| T-S \|$, then by sec. 5

(α) $\qquad \| T-T \| = 0,$

(β) $\qquad \| T-S \| = \| S-T \| > 0 \quad$ when $T \neq S$,

and by sec. 4 (2)

(γ) $\qquad \| T-R \| \leqq \| T-S \| + \| S-R \|.$

Hence, *the set of all completely continuous transformations, defined for all set functions in* $\mathfrak{L}_2(\beta)$, *is a metric space.* I denote this space by Ω.

Let $\{T_\nu\}$ be a sequence of transformations in Ω; if, in Ω a transformation T exists so that

$$\lim_{\nu\to\infty} \| T_\nu - T \| = 0,$$

then I will say that $\{T_\nu\}$ *converges strongly* to T, and denote it by

$$[\lim_{\nu\to\infty}] T_\nu = T.$$

Since

$$\| (T_\nu - T)\phi \| \leqq \| T_\nu - T \| \cdot \| \phi \|,$$

if

$$[\lim_{\nu\to\infty}] T_\nu = T,$$

then $\{T_\nu \phi\}$ *converges strongly* $T\phi$, *i. e.*

$$[\lim_{\nu\to\infty}] T_\nu \phi = T\phi$$

for all set functions ϕ *in* $\mathfrak{L}_2(\beta)$.

If $\{T_\nu\}$ *be a sequence of transformations in* Ω, *which converges strongly to* T, *then*

$$[\lim_{\nu\to\infty}] S T_\nu = ST$$

and

$$[\lim_{\nu\to\infty}] T_\nu S = TS$$

for any transformations S *in* Ω.

For,

$$\| ST_\nu - ST \| = \| S(T_\nu - T) \| \leqq \| S \| \cdot \| T_\nu - T \|,$$

and

$$\| T_\nu S - TS \| = \| (T_\nu - T)S \| \leqq \| T_\nu - T \| \cdot \| S \|.$$

Let $\{T_n\}$ be a sequence of transformations in Ω, and $\{c_n\}$ be a sequence of complex numbers. If a transformation T of Ω exists so that

$$\underset{n\to\infty}{[\lim]}\{c_1T_1+c_2T_2+\ldots\ldots+c_nT_n\}=T,$$

then I will say that the series

$$c_1T_1+c_2T_2+\ldots\ldots+c_nT_n+\ldots\ldots$$

converges strongly to T, and denote it by

$$T=c_1T_1+c_2T_2+\ldots\ldots+c_nT_n+\ldots\ldots$$

In this case, by the first of the above theorems

$$T\phi=c_1T_1\phi+c_2T_2\phi+\ldots\ldots+c_nT_n\phi+\ldots\ldots$$

for any set function ϕ in $\mathfrak{L}_2(\beta)$, the series being strongly convergent; and by the second theorem,

$$ST=c_1ST_1+c_2ST_2+\ldots\ldots+c_nST_n+\ldots\ldots,$$
$$TS=c_1T_1S+c_2T_2S+\ldots\ldots+c_nT_nS+\ldots\ldots.$$

7. *The space Ω is complete*, i. e. if

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\| T_m-T_n \|=0, \tag{1}$$

then, a transformation T exists in Ω, so that

$$\underset{n\to\infty}{[\lim]}T_n=T.$$

For, let ϕ be any set function in $\mathfrak{L}_2(\beta)$, then since

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\| (T_m-T_n)\phi \|=0,$$

$\mathfrak{L}_2(\beta)$ being complete, a set function ϕ^* exists in $\mathfrak{L}_2(\beta)$ so that

$$\underset{n\to\infty}{[\lim]}T_n\phi=\phi^*. \tag{2}$$

Denote the transformation which transforms ϕ to ϕ^* by T, i. e.

$$T\phi=\phi^*, \tag{3}$$

then T is obviously a linear transformation.

Next, I will show that T belongs to Ω, and

$$\underset{n\to\infty}{[\lim]}\, T_n=T.$$

Let $\{\psi_\nu\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then, by (2) and (3)

$$\underset{n\to\infty}{[\lim]} T_n\psi_\nu = T\psi_\nu \tag{4}$$

for all values of ν.

By (1), an integer N exists so that for all values of m and n greater than N

$$\sum_{\nu=1}^{p} \| T_m\psi_\nu - T_n\psi_\nu \|^2 \leqq \| T_m - T_n \|^2 < \varepsilon^2 \tag{5}$$

for a given positive number ε. But, since

$$\sum_{\nu=1}^{p} \| \lim_{m\to\infty} T_m\psi_\nu - T_n\psi_\nu \|^2 = \lim_{m\to\infty} \sum_{\nu=1}^{p} \| T_m\psi_\nu - T_n\psi_\nu \|^2,$$

from (4) and (5), we have

$$\sum_{\nu=1}^{p} \| T\psi_\nu - T_n\psi_\nu \|^2 < \varepsilon^2 \tag{6}$$

for any value of p.

Since $\| T\psi_\nu \| \leqq \| T_n\psi_\nu \| + \| T\psi_\nu - T_n\psi_\nu \|,$

$$\left\{ \sum_{\nu=1}^{p} \| T\psi_\nu \|^2 \right\}^{\frac{1}{2}} \leqq \left[\sum_{\nu=1}^{p} \{ \| T_n\psi_\nu \| + \| T\psi_\nu - T_n\psi_\nu \| \}^2 \right]^{\frac{1}{2}};$$

hence by Minkowski's inequality

$$\left\{ \sum_{\nu=1}^{p} \| T\psi_\nu \|^2 \right\}^{\frac{1}{2}} \leqq \left\{ \sum_{\nu=1}^{p} \| T_n\psi_\nu \|^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{\nu=1}^{p} \| T\psi_\nu - T_n\psi_\nu \| \right\}^{\frac{1}{2}}.$$

But since $\sum_{\nu=1}^{\infty} \| T_n\psi_\nu \|^2 = \| T_n \|^2$, by (6)

$$\left\{ \sum_{\nu=1}^{p} \| T\psi_\nu \|^2 \right\}^{\frac{1}{2}} \leqq \| T_n \| + \varepsilon.$$

But this inequality holds for any value of p, therefore $\sum_{\nu=1}^{\infty} \| T\psi_\nu \|^2$ converges to a finite value. That is, T belongs to Ω. And from (6), as p tends to ∞,

$$\| T - T_n \| < \varepsilon$$

for $n > N$. Therefore

$$\underset{n\to\infty}{[\lim]}\, T_n = T.$$

Volterra's Solution of Functional Equations.

8. The functional equation

$$\phi = \psi + T\phi, \tag{1}$$

where ψ is a given set function in $\mathfrak{L}_2(\beta)$, and ϕ an unknown set function, is a generalization of Fredholm's integral equation of the second kind. When $\|T\|<1$, we have a solution which corresponds to Volterra's solution of the ordinary integral equation.

Instead of interated kernels, consider the iterated transformations. By sec. 5, we have

$$\|T^2\|\leqq\|T\|\cdot\|T\|=\|T\|^2.$$

Generally

$$\|T^n\|\leqq\|T\|\cdot\|T^{n-1}\|\leqq\|T\|^n.$$

Let

$$T_n=T+T^2+\ldots.+T^n,$$

Then, since $\|T\|<1$, we have for $m>n$

$$\|T_m-T_n\|\leqq\|T^{n+1}\|+\|T^{n+2}\|+\ldots+\|T^m\|$$

$$\leqq\|T\|^{n+1}\frac{1}{1-\|T\|},$$

Hence

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\|T_m-T_n\|=0,$$

therefore, Ω being complete, there exists a transformation S in Ω so that

$$[\lim_{n\to\infty}]\,T_n=-S\,;$$

that is,

$$-S=T+T^2+\ldots.+T^n+\ldots.,$$

the series being strongly convergent. By sec. 6, we have

$$-S-T=T(T+T^2+\ldots.+T^{n-1}+\ldots.)=-TS,$$

$$=(T+T^2+\ldots.+T^{n-1}+\ldots.)T=-ST.$$

Hence, we have the so-called reciprocal property between T and S:

$$S+T=TS=ST. \qquad (2)$$

From (1) we have

$$S\phi=S\psi+ST\phi,$$

then, from (2)

$$S\phi=S\psi+S\phi+T\phi,$$

that is,

$$S\psi+T\phi=0,$$

Combining with (1), we have

$$\phi=\psi-S\psi. \qquad (3)$$

Hence, if (1) has a solution in $\mathfrak{L}_2(\beta)$, it is given by (3). Using (2), we can easily verify that (3) satisfies (1).

To shew that (1) has only one solution, it is sufficient that the solution of

$$\phi = T\phi \tag{4}$$

is a null function. From (4), it must be that

$$\|\phi\| \leqq \|T\| \cdot \|\phi\|.$$

But, since $\|T\| < 1$, we have

$$\|\phi\| = 0,$$

that is,

$$\phi = 0.$$

So we get the conclusion:

In the functional equation

$$\phi = \psi + T\phi, \tag{5}$$

if ψ *and* T *belong to* $\mathfrak{L}_2(\beta)$ *and* Ω *respectively, and*

$$\|T\| < 1,$$

then this equation has one and only one solution belonging to $\mathfrak{L}_2(\beta)$ *and this solution is given by*

$$\phi = \psi - S\psi. \tag{6}$$

Normalized Orthogonal System of Transformations.

9. Let $\{T_n\}$ be a finite, or infinite sequence of transformations in Ω. If a transformation T in Ω is expressed as

$$T = c_1 T_1 + c_2 T_2 + \dots + c_n T_n + \dots,$$

the series being strongly convergent,(1) then I say that T is a *linear combination* of the elements of $\{T_n\}$. And if a sequence $\{c_n\}$ of complex numbers exists, not all of them being zero, so that the series

$$c_1 T_1 + c_2 T_2 + \dots + c_n T_n + \dots$$

converges strongly to 0, then I say that the elements of $\{T_n\}$ are *linearly dependent*.

A subset Γ of Ω is called a *linear manifold* when all the linear combinations of the elements of Γ belong also to Γ. Linear manifolds are closed, for if

(1) Finite series are considered as strongly convergent.

$$[\lim_{n\to\infty}]\, T_n = T$$

then T is a linear combination of $T_1, T_2-T_1, \ldots, T_n-T_{n-1}, \ldots$. If there exists a sequence $\{T_n\}$ of transformations belonging to Γ, so that all the elements of Γ are linear combinations of $\{T_n\}$, then $\{T_n\}$ is called the *fundamental sequence* of the linear manifold Γ.

10. Let S and T be any linear transformations in Ω, and $\{\psi_\nu\}$ be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Now I will shew that

$$\sum_{\nu=1}^{\infty}(S\psi_\nu, T\psi_\nu)$$

converges to a finite number, which is independent of the system $\{\psi_\nu\}$.

Let $\{\phi_\mu\}$ be another complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then, since by Hölder's inequality

$$\sum_{\mu=1}^{\infty}|(S\psi_\nu, \phi_\mu)(\phi_\mu, T\psi_\nu)| \leqq \sqrt{\sum_{\mu=1}^{\infty}|(S\psi_\nu, \phi_\mu)|^2\sum_{\mu=1}^{\infty}|(\phi_\mu, T\psi_\nu)|^2}$$
$$= ||S\psi_\nu||\cdot||T\psi_\nu||,$$

and

$$\sum_{\nu=1}^{\infty}||S\psi_\nu||\cdot||T\psi_\nu|| \leqq \sqrt{\sum_{\nu=1}^{\infty}||S\psi_\nu||^2\sum_{\nu=1}^{\infty}||T\psi_\nu||^2}$$
$$= ||S||\cdot||T||,$$

$\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}|(S\psi_\nu, \phi_\mu)(\phi_\mu, T\psi_\nu)|$ converges to a finite value less than $||S||\cdot||T||$. Hence

$$\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}(S\psi_\nu, \phi_\mu)(\phi_\mu, T\psi_\nu)$$

and

$$\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}(S\psi_\nu, \phi_\mu)(\phi_\mu, T\psi_\nu)$$

converge to the same finite number, say l. Since

$$S\psi_\nu = \sum_{\mu=1}^{\infty}(S\psi_\nu, \phi_\mu)\phi_\mu,$$

we have

$$\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}(S\psi_\nu, \phi_\mu)(\phi_\mu, T\psi_\nu) = \sum_{\nu=1}^{\infty}(S\psi_\nu, T\psi_\nu),$$

hence

$$l = \sum_{\nu=1}^{\infty}(S\psi_\nu, T\psi_\nu).$$

But

$$\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}(S\psi_\nu, \phi_\mu)(\phi_\mu, T\psi_\nu)=\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}(T^*\phi_\mu, \psi_\nu)(\psi_\nu, S^*\phi_\mu)$$

$$=\sum_{\mu=1}^{\infty}(T^*\phi_\mu, S^*\phi_\mu); \tag{1}$$

hence, l is a number independent of the system $\{\psi_\nu\}$.

Therefore, $\sum_{\nu=1}^{\infty}(S\psi_\nu, T\psi_\nu)$ converges to a finite number l which is independent of the system $\{\psi_\nu\}$. I will describe this number as the *inner product* of S and T, and denote it by (S, T). Then

$$|(S, T)|\leqq||S||\cdot||T||, \tag{2}$$

and by (1)

$$(S, T)=(T^*, S^*).$$

Of course,

$$(T, T)=||T||^2.$$

From

$$(S\phi, T\phi)=\overline{(T\phi, S\phi)},$$
$$(S\phi, (T_1+T_2)\phi)=(S\phi, T_1\phi)+(S\phi, T_2\phi),$$
$$(cS\phi, T\phi)=c(S\phi, T\phi),$$

we have following relations

$$(S, T)=\overline{(T, S)}, \tag{3}$$

$$(S, T_1+T_2)=(S, T_1)+(S, T_2), \tag{4}$$
$$(cS, T)=c(S, T).$$

If

$$[\lim_{n\to\infty}]\, T_n=T,$$

then by (2)

$$|(T-T_n, S)|\leqq||T-T_n||\cdot||S||,$$

hence, by (4)

$$\lim_{n\to\infty}(T_n, S)=(T, S).$$

In other words, if

$$T=c_1T_1+c_2T_2+\ldots\ldots+c_nT_n+\ldots\ldots,$$

the series being strongly convergent, then

$$(T, S)=c_1(T_1, S)+c_2(T_2, S)+\ldots\ldots+c_n(T_n, S)+\ldots\ldots.$$

11. If

$$(T, S)=0$$

then, by (3) of the preceding section,

$$(S, T)=0.$$

In this case, I will say that the two linear transformations T and S are *orthogonal*.

Let $\{T_n\}$ be a sequence of linear transformations belonging to Ω. If

$$(T_i, T_j)=0 \quad \text{for} \quad i \neq j,$$
$$=1 \quad \text{for} \quad i=j,$$

then $\{T_n\}$ is said to be a *normalized orthogonal system*, as in the case of sequence of set functions.

An orthogonal system $\{T_n\}$ is said to be *complete* in a linear manifold Γ of linear transformations, if no transformation S of Γ, the null transformation excepted, exists so that

$$(S, T_n)=0$$

for all values of n.

Then as in the case of set functions,(1) we have the following theorems:

The necessary and sufficient condition that a normalized orthogonal system $\{T_n\}$ should be complete in a linear manifold Γ is that $\{T_n\}$ is a fundamental system of Γ.

If $\{T_n\}$ is complete in Γ, any transformation T in Γ may be expanded as follows:

$$T=c_1T_1+c_2T_2+\dots\dots+c_nT_n+\dots\dots,$$

where

$$c_n=(T, T_n),$$

and

$$\| T \|^2=\sum_{n=1}^{\infty} | c_n |^2.$$

Conversely, if $\{c_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} | c_n |^2$ is convergent, then there exists a unique linear transformation T in Γ, so that

$$T=c_1T_1+c_2T_2+\dots\dots+c_nT_n+\dots\dots.^{(2)}$$

And, if S and T be any two transformations in Γ, then

$$(S, T)=\sum_{n=1}^{\infty}(S, T_n)(T_n, T).^{(3)}$$

(1) This journal, **3** (1933) 10-11.

(2) This corresponds to the Riesz-Fischer theorem.

(3) This corresponds to Parseval's theorem.

Hence, if we consider $\{c_n\}$ as the coordinates of the linear transformation T, then Γ is a Hilbert space.

Expansion of Completely Continuous Hermitian Transformation.

12. A linear transformation H, which satisfies the relation

$$H = H^*,$$

is called a *Hermitian transformation*.

An example of a completely continuous Hermitian transformation is the following projecting transformation. Let $\mathfrak{L}$ be a linear manifold of p dimensions in $\mathfrak{L}_2(\beta)$, and

$$\eta_1, \eta_2, \ldots\ldots, \eta_p$$

be a complete normalized orthogonal system in $\mathfrak{L}$. Then any set function ϕ in $\mathfrak{L}_2(\beta)$ can be decomposed into two components, one in $\mathfrak{L}$ and the other orthogonal to $\mathfrak{L}$; i. e.

$$\phi = \phi' + \phi'', \tag{1}$$

where $$\phi' = (\phi, \eta_1)\eta_1 + (\phi, \eta_2)\eta_2 + \ \ldots + (\phi, \eta_p)\eta_p \tag{2}$$

and $$\phi'' = \phi - \phi'.$$

A linear transformation which transforms ϕ into ϕ', is called a *projecting transformation*[1] of $\mathfrak{L}$, and is denoted by $P_{\mathfrak{L}}$, i. e.

$$P_{\mathfrak{L}}\phi = \phi' \tag{3}$$

for any set function ϕ in $\mathfrak{L}_2(\beta)$.

Let

$$\eta_1, \eta_2, \ldots\ldots, \eta_p, \xi_1, \xi_2, \ldots\ldots, \xi_n, \ldots\ldots$$

be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$, then since

$$P_{\mathfrak{L}}\xi_n = 0 \qquad (n = 1, 2, \ldots\ldots),$$

we have

$$\| P_{\mathfrak{L}} \|^2 = \sum_{i=1}^{p} \| P_{\mathfrak{L}}\eta_i \|^2 = \sum_{i=1}^{p} \| \eta_i \|^2 = p,$$

hence $P_{\mathfrak{L}}$ belongs to Ω, and its norm is equal to the square root of the number of the dimension of $\mathfrak{L}$.

$P_{\mathfrak{L}}$ is a Hermitian transformation. For, decompose any set functions ϕ and ψ in $\mathfrak{L}_2(\beta)$ in two components as (1), i. e.

$$\phi = \phi' + \phi'',$$
$$\psi = \psi' + \psi''.$$

(1) Cf. J. v. Neumann, *loc. cit.*, 40.

Then, since $(\phi', \psi'')=0$ and $(\phi'', \psi')=0$,

$$(P_{\mathfrak{L}}\phi, \psi)=(\phi', \psi'+\psi'')=(\phi', \psi'),$$
$$(\phi, P_{\mathfrak{L}}\psi)=(\phi'+\phi'', \psi')=(\phi', \psi').$$

That is,

$$(P_{\mathfrak{L}}\phi, \psi)=(\phi, P_{\mathfrak{L}}\psi).$$

Let two linear manifolds $\mathfrak{L}$ and $\mathfrak{M}$ of finite dimensions, say p and q, be orthogonal, i. e. any set function in $\mathfrak{L}$ is orthogonal to any set function in $\mathfrak{M}$. And let

$$\eta_1, \eta_2, \ldots, \eta_p,$$
$$\xi_1, \xi_2, \ldots, \xi_q,$$

be complete normalized orthogonal systems in $\mathfrak{L}$ and $\mathfrak{M}$ respectively. Then we can take

$$\eta_1, \eta_2, \ldots, \eta_p, \xi_1, \xi_2, \ldots, \xi_q, \chi_1, \chi_2, \ldots, \chi_n, \ldots$$

as a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then, since

$$P_{\mathfrak{M}}\eta_i=0 \qquad (i=1, 2, \ldots, p),$$
$$P_{\mathfrak{L}}\xi_i=0 \qquad (i=1, 2, \ldots, q),$$
$$P_{\mathfrak{L}}\chi_i=0 \qquad (i=1, 2, \ldots),$$

$$(P_{\mathfrak{L}}, P_{\mathfrak{M}})=\sum_{i=1}^{p}(P_{\mathfrak{L}}\eta_i, P_{\mathfrak{M}}\eta_i)+\sum_{i=1}^{q}(P_{\mathfrak{L}}\xi_i, P_{\mathfrak{M}}\xi_i)+\sum_{i=1}^{\infty}(P_{\mathfrak{L}}\chi_i, P_{\mathfrak{M}}\chi_i)=0.$$

That is, $P_{\mathfrak{L}}$ and $P_{\mathfrak{M}}$ are orthogonal.

If we denote the linear manifold, whose fundamental system is composed by only one set function η_i by $[\eta_i]$, then

$$P_{[\eta_i]}\phi=(\phi, \eta_i)\eta_i.$$

And

$$(P_{[\eta_i]}, P_{[\eta_j]})=1, \qquad \text{when } i=j,$$
$$=0, \qquad \text{when } i\neq j.$$

Hence, $P_{[\eta_1]}, P_{[\eta_2]}, \ldots, P_{[\eta_p]}$ form a normalized orthogonal system. And, from (2) and (3), we have the expansion of $P_{\mathfrak{L}}$ with respect to this normalized orthogonal system, i. e.

$$P_{\mathfrak{L}}=P_{[\eta_1]}+P_{[\eta_2]}+\cdots+P_{[\eta_p]}.$$

In what follows, I intend to expand completely continuous Hermitian transformations with respect to the orthogonal system of projecting transformations.

13. Let λ be a number; if, in $\mathfrak{L}_2(\beta)$, the null function excepted, there exists a set function η which satisfies the relation

$$\eta=\lambda H\eta, \tag{1}$$

then λ is called a *characteristic constant* of H. And, when λ is a characteristic constant, all set functions in $\mathfrak{L}_2(\beta)$, which satisfy (1), are called *characteristic functions* of H with respect to λ. Then, we can easily prove that all characteristic constants of H are real, and any two characteristic functions of H with respect to different characteristic constants are orthogonal.

For any completely continuous Hermitian transformation, there exists at least one characteristic constant.[1]

By sec. 5, for any set function ϕ in $\mathfrak{L}_2(\beta)$

$$\|H\phi\|\leqq\|H\|\cdot\|\phi\|,$$

therefore, there exists a finite upper bound, say l, of $\frac{\|H\phi\|}{\|\phi\|}$ for all set functions ϕ in $\mathfrak{L}_2(\beta)$. If H has a characteristic constant λ and

$$\eta=\lambda H\eta,$$

then

$$\frac{\|H\eta\|}{\|\eta\|}=\frac{1}{|\lambda|};$$

therefore, the absolute value of λ can not be less than $\frac{1}{l}$. In what follows, I will show that at least one value of $\frac{1}{l}$ or $-\frac{1}{l}$ is a characteristic constant of H.

As in the previous paper, we can shew that the lower bound of $\|\psi-\frac{1}{l}H\psi\|$ or $\|\psi+\frac{1}{l}H\psi\|$ for any normalized set function ψ in $\mathfrak{L}_2(\beta)$, is zero.

Denote the value, $\frac{1}{l}$ or $-\frac{1}{l}$, by λ, such that the lower bound of

$$\|\psi-\lambda H\psi\|$$

for any normalized set function ψ, is zero. Then there exists a sequence $\{\psi_\nu\}$ which satisfies

$$\lim_{\nu\to\infty}\|\psi_\nu-\lambda H\psi_\nu\|=0.$$

(1) Since all the considerations about the sequence of linear manifolds in the space of real valued set functions hold likewise in the space of complex valued set functions without any essential change, the proof of this theorem is almost identical with that of my previous paper (this journal, **3** (1933), 30-33). Hence, I will here give only an outline of the proof.

I will call such a sequence $\{\psi_\nu\}$, the minimal sequence of $\|\psi-\lambda H\psi\|$.

Let $\{\psi_\nu\}$ be of p asymptotic dimensions, then there exists an associated sequence $\{\mathfrak{L}_n\}$ of linear manifolds of p dimensions.[1] Let

$$\eta_{n_1}, \eta_{n_2}, \ldots, \eta_{n_p}$$

be the normalized orthogonal system in $\mathfrak{L}_n$, then as in the previous paper,

$$\lim_{n\to\infty}\|\eta_{n_i}-\lambda H\eta_{n_i}\|=0. \qquad (i=1, 2, \ldots, p)$$

Since $\|\eta_{n_i}\|=1$ for any value of n, then

$$\lim_{n\to\infty}\|H\eta_{n_i}\|=\frac{1}{|\lambda|}. \qquad (i=1, 2, \ldots, p)$$

Since $\|H\|^2\geqq\sum_{i=1}^{p}\|H\eta_{n_i}\|^2$ for any value of n, we have

$$\|H\|^2\geqq\frac{p}{\lambda^2}.$$

That is, the asymptotic dimensions of minimal sequences of $\|\psi-\lambda H\psi\|$ can not be greater than $\lambda^2\|H\|^2$. But for any minimal sequence $\{\psi_\nu\}$, the asymptotic dimensions are $\geqq 1$; for, $\|\psi_\nu\|=1$ for all values of ν.

Let p be the greatest asymptotic dimensions of all the minimal sequences of $\|\psi-\lambda H\psi\|$; then as in the previous paper, there exists a linear manifold $\mathfrak{L}$ of p dimensions, composed of all the characteristic functions of H with respect to λ. I will describe this linear manifold $\mathfrak{L}$ as the *characteristic manifold* of H with respect to λ.

14. In the preceding section, we find a characteristic constant of H which has the least absolute value. We will denote this characteristic constant by $\lambda^{(1)}$, and the characteristic manifold of H with respect to $\lambda^{(1)}$ by $\mathfrak{L}^{(1)}$, and let its dimension be p_1.

Since two characteristic functions of H with respect to different characteristic constants are orthogonal, in order to find characteristic functions of H with respect to λ, different from $\lambda^{(1)}$, it is sufficient to consider the set functions which are orthogonal to $\mathfrak{L}^{(1)}$.

Let

$$H_1=H-\frac{1}{\lambda^{(1)}}P_{\mathfrak{L}^{(1)}}, \qquad (1)$$

then, since H and $P_{\mathfrak{L}^{(1)}}$ are Hermitian transformations, H_1 is likewise so. Decompose any set function ϕ in $\mathfrak{L}_2(\beta)$ into two components

(1) Cf. my previous paper (this journal, **3** (1933), 21.)

$$\phi = \phi' + \phi'',$$

where ϕ' is contained in $\mathfrak{L}^{(1)}$, and ϕ'' is orthogonal to $\mathfrak{L}^{(1)}$. Then

$$H_1\phi = H(\phi' + \phi'') - \frac{1}{\lambda^{(1)}} P_{\mathfrak{L}^{(1)}}(\phi' + \phi'')$$

$$= \frac{1}{\lambda^{(1)}}\phi' + H\phi'' - \frac{1}{\lambda^{(1)}}\phi'.$$

Hence

$$H_1\phi = H\phi''. \tag{2}$$

If ϕ is a characteristic function of H with respect to λ different to $\lambda^{(1)}$, then ϕ must be orthogonal to $\mathfrak{L}^{(1)}$, that is, $\phi = \phi''$. Hence, by (2) ϕ is also a characteristic function of H_1 with respect to the same characteristic constant λ.

Next, let ϕ be a characteristic function of H_1 with respect to λ, then, since by (2) H_1 transforms all set functions in $\mathfrak{L}^{(1)}$ into null function, ϕ must be orthogonal to $\mathfrak{L}^{(1)}$. That is, $\phi = \phi''$. Then by (2) ϕ is also a characteristic function of H with respect to the same characteristic constant λ.

Therefore, to find all the characteristic constants and corresponding characteristic manifolds of H, different to $\lambda^{(1)}$ and $\mathfrak{L}^{(1)}$, it is sufficient to consider all the characteristic constants and corresponding characteristic manifolds of H_1.

Let

$$\eta_1, \eta_2, \ldots, \eta_{p_1}$$

be the complete normalized orthogonal system in $\mathfrak{L}^{(1)}$, and

$$\eta_1, \eta_2, \ldots, \eta_{p_1}, \xi_1, \xi_2, \ldots, \xi_n, \ldots$$

be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$, then since

$$H_1\eta_i = 0 \qquad (i = 1, 2, \ldots, p_1),$$

and

$$P_{\mathfrak{L}^{(1)}}\xi_i = 0 \qquad (i = 1, 2, \ldots, n, \ldots),$$

we have

$$(H_1, P_{\mathfrak{L}^{(1)}}) = \sum_{i=1}^{p_1}(H_1\eta_i, P_{\mathfrak{L}^{(1)}}\eta_i) + \sum_{i=1}^{\infty}(H_1\xi_i, P_{\mathfrak{L}^{(1)}}\xi_i)$$

$$= 0.$$

That is, H_1 and $P_{\mathfrak{L}^{(1)}}$ is orthogonal. Hence by (1)

$$\|H_1\|^2 = \|H\|^2 - \frac{1}{\lambda^{(1)2}}\|P_{\mathfrak{L}^{(1)}}\|^2 = \|H\|^2 - \frac{p_1}{\lambda^{(1)2}}.$$

If

$$\| H \|^2 - \frac{p_1}{\lambda^{(1)2}} = 0,$$

then

$$H = \frac{1}{\lambda^{(1)}} P_{\mathfrak{L}^{(1)}}$$

But, if

$$\| H \|^2 - \frac{p_1}{\lambda^{(1)2}} > 0,$$

by the method of preceding section, we find a characteristic constant $\lambda^{(2)}$ of H_1 which has the least value, and a characteristic manifold $\mathfrak{L}^{(2)}$ of p_2 dimensions. Then $|\lambda^{(1)}| \leqq |\lambda^{(2)}|$ and

$$\frac{p_2}{\lambda^{(2)2}} \leqq \| H_1 \|^2 = \| H \|^2 - \frac{p_1}{\lambda^{(1)2}}.$$

Now put

$$H_2 = H_1 - \frac{1}{\lambda^{(2)}} P_{\mathfrak{L}^{(2)}},$$

and apply the above method to H_2 instead of H_1. And so on.

If an integer n exists so that

$$\| H \|^2 - \frac{p_1}{\lambda^{(1)2}} - \frac{p_2}{\lambda^{(2)2}} - \cdots - \frac{p_n}{\lambda^{(n)2}} = 0,$$

then

$$H = \frac{1}{\lambda^{(1)}} P_{\mathfrak{L}^{(1)}} + \frac{1}{\lambda^{(2)}} P_{\mathfrak{L}^{(2)}} + \cdots + \frac{1}{\lambda^{(n)}} P_{\mathfrak{L}^{(n)}}$$

is the required result.

But if

$$\| H \|^2 - \frac{p_1}{\lambda^{(1)2}} - \frac{p_2}{\lambda^{(2)2}} - \cdots - \frac{p_n}{\lambda^{(n)2}} > 0$$

for any value of n, then, since the infinite series

$$\frac{p_1}{\lambda^{(1)2}} + \frac{p_2}{\lambda^{(2)2}} + \cdots + \frac{p_n}{\lambda^{(n)2}} + \cdots$$

converges,

$$\frac{1}{\lambda^{(1)}} P_{\mathfrak{L}^{(1)}} + \frac{1}{\lambda^{(2)}} P_{\mathfrak{L}^{(2)}} + \cdots + \frac{1}{\lambda^{(n)}} P_{\mathfrak{L}^{(n)}} + \cdots$$

converges strongly to a Hermitian transformation, say S, in Ω. But as in the previous paper,[1] we can prove that S is H. And we have the

(1) This journal, **3** (1933), 37.

expansion of H with respect to the orthogonal system of projecting transformations:

$$H=\frac{1}{\lambda^{(1)}}P_{\mathfrak{L}^{(1)}}+\frac{1}{\lambda^{(2)}}P_{\mathfrak{L}^{(2)}}+\ldots\ldots+\frac{1}{\lambda^{(n)}}P_{\mathfrak{L}^{(n)}}+\ldots\ldots^{(1)} \qquad (3)$$

and

$$\|H\|^2=\frac{p_1}{\lambda^{(1)2}}+\frac{p_2}{\lambda^{(2)2}}+\cdots\quad+\frac{p_n}{\lambda^{(n)2}}+\cdots\cdot$$

Let

$$\eta_1^{(n)},\ \eta_2^{(n)},\ \ldots\ldots,\ \eta_{p_n}^{(n)}$$

be a normalized orthogonal system of $\mathfrak{L}^{(n)}$. And put

$$\begin{aligned}&\eta_1=\eta_1^{(1)}, \quad \eta_2=\eta_2^{(1)}, \quad \ldots\ldots, \ \eta_{p_1}=\eta_{p_1}^{(1)},\\ &\eta_{p_1+1}=\eta_1^{(2)}, \quad \eta_{p_1+2}=\eta_2^{(2)}, \quad \ldots\ldots, \ \eta_{p_1+p_2}=\eta_{p_2}^{(2)},\\ &\ldots\ldots\ \ldots\ldots\ldots\ldots\\ &\ldots\ldots\ \ldots\ldots\ldots\ldots\end{aligned}$$

and

$$\begin{aligned}&\lambda_1=\lambda_2=\ldots\ldots\ldots=\lambda_{p_1}=\lambda^{(1)},\\ &\lambda_{p_1+1}=\lambda_{p_1+2}=\ldots\ldots=\lambda_{p_1+p_2}=\lambda^{(2)},\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

Then, by sec. 12, $\{P_{[\eta_n]}\}$ is a normalized orthogonal system. Since

$$\begin{aligned}&P_{\mathfrak{L}^{(1)}}=P_{[\eta_1]}+P_{[\eta_2]}+\ldots\ldots\ldots+P_{[\eta_{p_1}]},\\ &P_{\mathfrak{L}^{(2)}}=P_{[\eta_{p_1+1}]}+P_{[\eta_{p_1+2}]}+\ldots\ldots+P_{[\eta_{p_1+p_2}]},\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\end{aligned}$$

by (3) *we have the following expansion of H with respect to the normalized orthogonal system of projecting transformations*:

$$H=\frac{1}{\lambda_1}P_{[\eta_1]}+\frac{1}{\lambda_2}P_{[\eta_2]}+\ldots\ldots+\frac{1}{\lambda_n}P_{[\eta_n]}+\ldots\ldots,$$

and

$$\|H\|^2=\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\ldots\ldots+\frac{1}{\lambda_n^2}+\ldots\ldots,$$

where

$$P_{[\eta_n]}\phi=(\phi,\ \eta_n)\eta_n.$$

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(1) Using this expansion of H, we can find the solutions of $\psi=H\phi$ and $\phi=\psi+\lambda H\phi$ by the method of undertermined coefficients. (Cf. *ibid.*, 40–42.)

Expansion of Completely Continuous Transformation.

15. Let T be a completely continuous transformation. Then, since

$$(TT^*)^* = T^{**}T^* = TT^*,$$
$$(T^*T)^* = T^*T^{**} = T^*T,$$

TT^* and T^*T are Hermitian transformations. Let α be any characteristic constant of TT^*, and μ be a non-null characteristic function of TT^* with respect to α, i. e.

$$\mu = \alpha TT^*\mu. \qquad (1)$$

Then, since

$$(TT^*\mu,\ \mu) = (T^*\mu,\ T^*\mu) > 0,$$

and

$$(TT^*\mu,\ \mu) = \frac{1}{\alpha}(\mu,\ \mu),$$

α must be positive. Denote $\sqrt{\alpha}$ by λ, Put

$$\nu = \lambda T^*\mu, \qquad (2)$$

then by (1)

$$\mu = \lambda T\nu. \qquad (3)$$

Eliminate μ from (2) and (3),

$$\nu = \lambda^2 T^*T\nu. \qquad (4)$$

That is, λ^2 is also a characteristic constant of T^*T, and the characteristic function of T^*T with respect to λ^2 is ν, connected with μ by the relation (2) or (3). Conversely, all characteristic constants of T^*T are also characteristic constants of TT^*.

Let

$$\mu_1,\ \mu_2,\ \ldots\ldots,\ \mu_n,\ \ldots\ldots \qquad (5)$$

and

$$\nu_1,\ \nu_2,\ \ldots\ldots,\ \nu_n,\ \ldots\ldots \qquad (5')$$

be the normalized orthogonal system of all characteristic functions of TT^* and T^*T respectively, and their corresponding characteristic constants be

$$\lambda_1^2,\ \lambda_2^2,\ \ldots\ldots,\ \lambda_n^2,\ \ldots\ldots^{(1)}$$

Denote the linear manifolds, which have (5) and (5′) for their fundamental systems, by $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

Since, by (2)

(1) Cf. Vivanti-Schwank, *Lineare Integralgleichungen*, (1929) 188.

$$(T\phi,\ \mu_n)=(\phi,\ T^*\mu_n)=\frac{1}{\lambda}(\phi,\ \nu_n) \qquad (n=1,\ 2,\ \ldots.),$$

if $T\phi=0$, then ϕ is orthogonal to $\mathfrak{N}$.

Conversely, if ϕ is orthogonal to $\mathfrak{N}$, then ϕ is orthogonal to all the characteristic functions of T^*T, hence $T^*T\phi=0$. But

$$\|T\phi\|^2=(T\phi,\ T\phi)=(T^*T\phi,\ \phi),$$

therefore,

$$T\phi=0.$$

Hence, *the necessary and sufficient condition that*

$$T\phi=0 \tag{6}$$

is that ϕ is orthogonal to $\mathfrak{N}$.

We have a similar theorem for T^*.

16. Let

$$\nu_1,\ \nu_2,\ \ldots.,\ \nu_i,\ \ldots.,\ \chi_1,\ \chi_2,\ \ldots.,\ \chi_i,\ \ldots.$$

be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then

$$\|T\|^2=\sum_{i=1}^{\infty}\|T\nu_i\|^2+\sum_{i=1}^{\infty}\|T\chi_i\|^2,$$

but, by (6) of the preceding section,

$$T\chi_i=0 \qquad (i=1,\ 2,\ \ldots.).$$

Hence

$$\|T\|^2=\sum_{i=1}^{\infty}\|T\nu_i\|^2=\sum_{i=1}^{\infty}(T\nu_i,\ T\nu_i)=\sum_{i=1}^{\infty}(T^*T\nu_i,\ \nu_i),$$

then by (4) of the preceding section,

$$\|T\|^2=\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}(\nu_i,\ \nu_i)=\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}.$$

Therefore, $\sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}$ *converges to* $\|T\|^2$.

17. Decompose any set function ϕ into two components:

$$\phi=\phi'+\phi'',$$

where $\phi'=\sum_{n=1}^{\infty}(\phi,\ \nu_n)\nu_n$, and ϕ'' is orthogonal to $\mathfrak{N}$. Then, since by sec. 15 (6) $T\phi''=0$, we have

$$T\phi=T\phi'=\sum_{n=1}^{\infty}(\phi,\ \nu_n)T_n\ \nu_n,$$

hence, by sec. 15 (3)

$$T\phi=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}(\phi,\ \nu_n)\ \mu_n. \tag{1}$$

Similarly, we have

$$T^*\psi=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}(\psi,\ \mu_n)\nu_n. \tag{2}$$

Denote the transformation which transforms ϕ into $(\phi,\ \nu_n)\mu_n$ by Q_n, i. e.

$$Q_n\phi=(\phi,\ \nu_n)\,\mu_n. \tag{3}$$

Then

$$(Q_n\phi,\ \psi)=(\phi,\ \nu_n)(\mu_n,\ \psi)=(\phi,\ (\psi,\ \mu_n)\nu_n),$$

that is,

$$Q_n^*\psi=(\psi,\ \mu_n)\nu_n. \tag{4}$$

Let

$$\nu_1,\ \nu_2,\ \ldots,\ \nu_i,\ \ldots,\ \chi_1,\ \chi_2,\ \ldots,\ \chi_i,\ \ldots$$

be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$. Then

$$(Q_m,\ Q_n)=\sum_{i=1}^{\infty}(Q_m\nu_i,\ Q_n\nu_i)+\sum_{i=1}^{\infty}(Q_m\chi_i,\ Q_n\chi_i)$$

$$=\sum_{i=1}^{\infty}((\nu_i,\ \nu_m)\mu_m,\ (\nu_i,\ \nu_n)\mu_n)+\sum_{i=1}^{\infty}((\chi_i,\ \nu_m)\,\mu_m,\ (\chi_i,\ \nu_n)\mu_n),$$

hence

$$\begin{aligned}(Q_m,\ Q_n)&=1 \qquad \text{when} \qquad m=n,\\ &=0 \qquad \text{when} \qquad m\neq n.\end{aligned}$$

And similarly,

$$\begin{aligned}(Q_m^*,\ Q_n^*)&=1 \qquad \text{when} \qquad m=n,\\ &=0 \qquad \text{when} \qquad m\neq n.\end{aligned}$$

Hence $\{Q_n\}$ and $\{Q_n^*\}$ are normalized orthogonal systems. Since, by sec. 16, $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}$ converges,

$$\frac{1}{\lambda_1}Q_1+\frac{1}{\lambda_2}Q_2+\ldots+\frac{1}{\lambda_n}Q_n+\ldots$$

and

$$\frac{1}{\lambda_1}Q_1^*+\frac{1}{\lambda_2}Q_2^*+\ldots+\frac{1}{\lambda_n}Q_n^*+\ldots$$

are strongly convergent. Hence, by (1) (2) (3) (4), we have the conclusion:

The completely continuous transformations T and T^* can always be expanded with respect to the normalized orthogonal systems $\{Q_n\}$ and $\{Q_n^*\}$ respectively; i. e.

$$T=\frac{1}{\lambda_1}Q_1+\frac{1}{\lambda_2}Q_2+\ldots+\frac{1}{\lambda_n}Q_n+\ldots$$

$$T^* = \frac{1}{\lambda_1} Q_1^* + \frac{1}{\lambda_2} Q_2^* + \dots + \frac{1}{\lambda_n} Q_n^* + \dots,$$

and $$\| T \|^2 = \| T^* \|^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \dots + \frac{1}{\lambda_n^2} + \dots$$

where $$Q_n \phi = (\phi, \nu_n) \mu_n,$$

$$Q_n^* \phi = (\phi, \mu_n) \nu_n.$$

Integral Forms of Completely Continuous Transformations.

18. Let H be a completely continuous Hermitian transformation. Then, by sec. 14, H can be expanded with respect to the normalized orthogonal system of projecting transformations, i. e.

$$H = \frac{1}{\lambda_1} P_{[\eta_1]} + \frac{1}{\lambda_2} P_{[\eta_2]} + \dots + \frac{1}{\lambda_n} P_{[\eta_n]} + \dots, \tag{1}$$

and $$\| H \|^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \dots + \frac{1}{\lambda_n^2} + \dots$$

where $$P_{[\eta_n]} \phi = (\phi, \eta_n) \eta_n. \tag{2}$$

Since (2) may be written in the integral form

$$P_{[\eta_n]} \phi(E) = \eta_n(E) \int_A D_{\beta(E')} \phi(a') \overline{D_{\beta(E')} \eta_n(a')} d\beta(E')$$

$$= \int_A D_{\beta(E')} \mathfrak{K}^{(n)}(E, a') D_{\beta(E')} \phi(a') d\beta(E'),$$

where $$\mathfrak{K}^{(n)}(E, E') = \eta_n(E) \overline{\eta_n(E')},$$

$P_{[\eta_n]}$ is a linear transformation with kernel $\eta_n(E) \overline{\eta_n(E')}$.

Now, $\{\eta_n(E) \overline{\eta_n(E')}\}$ is a normalized orthogonal system in $\mathfrak{L}_2(\beta\beta)$.[1] For

$$(\eta_m \bar{\eta}_m, \eta_n \bar{\eta}_n) = \int_A \int_A D_{\beta(E)} \eta_m(a) \overline{D_{\beta(E')} \eta_m(a')} \overline{D_{\beta(E)} \eta_n(a)} D_{\beta(E')} \eta_n(a') d\beta(E) d\beta(E')$$

$$= (\eta_m, \eta_n) \overline{(\eta_m, \eta_n)} = \begin{cases} 1 & \text{when } m = n, \\ 0 & \text{when } m \neq n. \end{cases}$$

Since

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \dots + \frac{1}{\lambda_n^2} + \dots$$

(1) Cf. my previous papers, "On the Space of Real Set Functions," (this journal, **3** (1933), 24-26) and "Repeated Integrals in Metric Space," (which recently appeared in *Tôhoku Mathematical Journal.*)

converges, therefore

$$\frac{1}{\lambda_1}\eta_1(E)\overline{\eta_1(E')}+\frac{1}{\lambda_2}\eta_2(E)\overline{\eta_2(E')}+\ldots\ldots+\frac{1}{\lambda^n}\eta_n(E)\overline{\eta_n(E')}+\ldots\ldots$$

converges strongly to a function of two sets in $\mathfrak{L}_2(\beta\beta)$, say $\mathfrak{K}(E, E')$, and

$$\|\mathfrak{K}\|^2=\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\ldots\ldots+\frac{1}{\lambda_n^2}+\ldots\ldots$$

which is equal to $\|H\|^2$.

Hence, (1) may be written as follows:

$$H\phi(E)=\int_A D_{\beta(E')}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E').$$

Thus, *we have the integral form of the completely continuous Hermitian transformation H with kernel* $\mathfrak{K}(E, E')$, and

$$\mathfrak{K}(E, E')=\overline{\mathfrak{K}(E', E)},$$
$$\|\mathfrak{K}\|=\|H\|.$$

19. In sec. 17, I have shown that a completely continuous transformation T and its adjoint transformation T^* can be expanded with respect to the normalized orthogonal systems $\{Q_n\}$ and $\{Q_n^*\}$ repectively, i. e.

$$T=\frac{1}{\lambda_1}Q_1+\frac{1}{\lambda_2}Q_2+\ldots\ldots+\frac{1}{\lambda_n}Q_n+\ldots\ldots,$$

$$T^*=\frac{1}{\lambda_1}Q_1^*+\frac{1}{\lambda_2}Q_2^*+\ldots\ldots+\frac{1}{\lambda_n}Q_n^*+\ldots\ldots,$$

and

$$\|T\|=\|T^*\|^2=\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\ldots\ldots+\frac{1}{\lambda_n^2}+\ldots\ldots,$$

where

$$Q_n\phi=(\phi, \nu_n)\mu_n,$$
$$Q_n^*\phi=(\phi, \mu_n)\nu_n,$$

that is,

$$Q_n\phi(E)=\int_A \mu_n(E)\overline{D_{\beta(E')}\nu_n(a')}D_{\beta(E')}\phi(a')d\beta(E'),$$

$$Q_n^*\phi(E)=\int_A \nu_n(E)\overline{D_{\beta(E')}\mu_n(a')}D_{\beta(E')}\phi(a')d\beta(E').$$

Since $\{\mu_n(E)\}$ and $\{\nu_n(E)\}$ are normalized orthogonal systems in $\mathfrak{L}_2(\beta)$, the sequence of functions of two sets $\{\mu_n(E)\overline{\nu_n(E')}\}$ is also a normalized orthogonal system in $\mathfrak{L}_2(\beta\beta)$. For

$$(\mu_m\bar{\nu}_m, \mu_n\bar{\nu}_n) = \int_A\int_A D_{\beta(E)}\mu_m(a)\overline{D_{\beta(E')}\nu_m(a')}\overline{D_{\beta(E)}\mu_n(a)}D_{\beta(E')}\nu_n(a')d\beta(E)d\beta(E')$$

$$= (\mu_m, \mu_n)\overline{(\nu_m, \nu_n)} = \begin{cases} 1 & \text{when } m=n, \\ 0 & \text{when } m\neq n. \end{cases}$$

Since $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}$ converges, the series

$$\frac{1}{\lambda_1}\mu_1(E)\overline{\nu_1(E')} + \frac{1}{\lambda_2}\mu_2(E)\overline{\nu_2(E')} + \dots + \frac{1}{\lambda_n}\mu_n(E)\overline{\nu_n(E')} + \dots$$

convsrges strongly to a function of two sets in $\mathfrak{L}_2(\beta\beta)$, say $\mathfrak{K}(E, E')$, and

$$\|\mathfrak{K}\|^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \dots + \frac{1}{\lambda_n^2} + \dots$$

which is equal to $\|T\|^2$. Now *we have the integral form of T with kernel $\mathfrak{K}(E, E')$, i. e.*

$$T\phi(E) = \int_A D_{\beta(E')}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E').$$

Similarly

$$T^*\phi(E) = \int_A D_{\beta(E')}\mathfrak{K}^*(E, a')D_{\beta(E')}\phi(a')d\beta(E'),$$

where

$$\mathfrak{K}^*(E, E') = \overline{\mathfrak{K}(E', E)}.$$

20. Now, I proceed to the converse problem: Are all the linear transformations

$$T\phi(E) = \int_A D_{\beta(E')}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E')$$

with kernel $\mathfrak{K}(E, E')$ belonging to $\mathfrak{L}_2(\beta\beta)$, completely continuous?

First, let T have a Hermitian kernel, i. e.

$$\mathfrak{K}(E, E') = \overline{\mathfrak{K}(E', E)}.$$

Then, as in the previous paper,(1) we find a normalized orthogonal system $\{\eta_n(E)\}$ of all characteristic functions of $\mathfrak{K}(E, E')$, and a sequence $\{\lambda_n\}$ of corresponding characteristic constants; and T transforms a set function which is orthogonal to all $\eta_n(E)$ into null function. Let

$$\eta_1(E), \eta_2(E), \dots, \eta_n(E), \dots, \chi_1(E), \chi_2(E), \dots, \chi_n(E), \dots$$

be a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$; then

(1) This journal, **3** (1933), 38.

$$\| T\|^2=\sum_{n=1}^{\infty}\| T\eta_n(E)\|^2+\sum_{n=1}^{\infty}\| T\chi_n(E)\|^2=\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}.$$

But, $\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}$ converges to $\|\mathfrak{K}\|^2$, therefore all transformations with Hermitian kernel are completely continuous.

Next, consider the general transformation T with kernel $\mathfrak{K}(E, E')$. Then

$$\mathfrak{H}_1(E, E')=\mathfrak{K}(E, E')+\overline{\mathfrak{K}(E', E)}$$

and

$$\mathfrak{H}_2(E, E')=i\{\mathfrak{K}(E, E')-\overline{\mathfrak{K}(E', E)}\}$$

are Hermitian kernels, and T may be expressed as

$$T=\frac{1}{2}(H_1-iH_2),$$

where H_1 and H_2 are transformations with kernels $\mathfrak{H}_1(E, E')$ and $\mathfrak{H}_2(E,E')$ respectively. Since H_1 and H_2 are completely continuous, by sec. 4, T is likewise so.

Combining this result with the theorems of the preceding sections, we have the following conclusion :(1)

In order that a linear transformation T, defined for all set functions in $\mathfrak{L}_2(\beta)$, may be expressed in an integral form, i. e.

$$T\phi(E)=\int_A D_{\beta(E)}\mathfrak{K}(E, a')D_{\beta(E')}\phi(a')d\beta(E')$$

with kernel $\mathfrak{K}(E,E')$ belonging to $\mathfrak{L}_2(\beta\beta)$, it is necessary and sufficient that T is completely continuous, that is, $\sum_{\nu=1}^{\infty}\| T\psi_\nu\|^2$ converges to a finite value, where $\{\psi_\nu\}$ is a complete normalized orthogonal system in $\mathfrak{L}_2(\beta)$.

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(1) From this theorem, we have a similar theorem for a linear point functional transformation

$$Tf(a)=\int_A K(a, a')f(a')d\beta(E')$$

defined for all point functions $f(a)$ in metric space, $|f(a)|^2$ being integrable with respect to $\beta(E)$. J. Radon considered the case where $f(a)$ are continuous functions in Euclidian space. (*Sitzgsber. Akad. Wiss. Wien* IIa, **128** (1919), 1100).