On the Space of real Set Functions.

By

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For a real point function $f(a)$ which belongs to the class $L_2(\beta)$, that is, $[f(a)]^2$ is integrable with respect to a set function $\beta(E)$ over a set A, the norm $||f||$ of $f(a)$ is defined as

$$||f||=\left[\int_A [f(a)]^2 d\beta(E)\right]^{\frac{1}{2}},$$

and the inner product (f, g) of two point functions $f(a)$ and $g(a)$ of $L_2(\beta)$ is defined as

$$(f, g)=\int_A f(a)\,g(a)\,d\beta(E).$$

Then it is well-known that the set of all point functions of $L_2(\beta)$ is a Hilbert space, when two point functions, which differ at points of sets whose β-value is zero, are considered as identical.

Now let us say that a real set function $\phi(E)$, which is absolutely continuous with respect to $\beta(E)$, belongs to $L_2(\beta)$ when its general derivative $D_{\beta(E)}\phi(a)$ belongs to $L_2(\beta)$. By the fundamental theorem of the integration and differentiation, a unique general derivative $D_{\beta(E)}\phi(a)$ exists almost everywhere (β) and

$$\phi(E)=\int_E f(a)\,d\beta(E),$$

where $$f(a)=D_{\beta(E)}\phi(a).$$

Therefore, if we define the norm $||\phi||$ of $\phi(E)$ as

$$||\phi||=||f||,$$

and the inner product (ϕ, ψ) of $\phi(E)$ and $\psi(E)$ as

$$(\phi, \psi)=(f, g)$$

where $$g(a)=D_{\beta(E)}\psi(a),$$

the space of real point functions of $L_2(\beta)$ and the space of real set functions of $L_2(\beta)$ are isomorph.

But the space of real set functions has a peculier property. In the case of point functions, the convergence of $\{f_n(a)\}$ to $f(a)$ in the geometrical sense, that is, the strong convergence of $\{f_n(a)\}$ to $f(a)$, does not imply the ordinary convergence of $\{f_n(a)\}$ to $f(a)$ at all points a, even when we neglect points of sets whose β-value is zero. But in the case of set functions, when $\{\phi_n(E)\}$ converges strongly to $\phi(E)$, then $\{\phi_n(E)\}$ converges to $\phi(E)$ at all sets E. Concerning the point functions, the expansion of $f(a)$ with respect to a complete orthogonal system does not generally converge to $f(a)$. But in the case of set functions, the expansion of $\phi(E)$ with respect to a complete orthogonal system always converges to $\phi(E)$.

In chapter I of this paper, I define the linear combination and linear dependency of a denumerably infinite system of set functions $\{\phi_n(E)\}$ as follows: When

$$\phi(E)=c_1\phi_1(E)+c_2\phi_2(E)+\dots\dots+c_n\phi_n(E)+\dots\dots$$

the series being strongly convergent, then $\phi(E)$ is a linear combination of $\phi_1(E), \phi_2(E), \dots\dots, \phi_n(E), \dots\dots$ And when $\phi(E)=0$, then $\phi_1(E), \phi_2(E), \dots\dots, \phi_n(E), \dots\dots$ is linearly dependent. On the basis of these definitions I investigate the properties of the space of set functions.

The weak convergence of the sequence of set functions may be defined as in the case of point functions. But this convergence is a conception barely compatible with the metric property of the space of functions. Hence, instead of weak convergence, I introduce Courant's conception of asymptotic dimension, and investigate the properties of the bounded sequence of set functions.

In chapter II, I apply the properties of the space of set functions to the solutions of integral equations and find all the characteristic functions of a real symmetric kernel with respect to a charactristic constant simultaneously. The real symmetric kernel can always be expanded with respect to the orthogonal system of characteristic functions. And the solutions of the integral equation with real symmetric kernel are obtained by the method of undetermined coefficients.

Chapter I. Space of Real Set Functions.

Metric Space $L_2(\beta)$.

1. Let $\beta(E)$ be a completely additive, non-negative function of normal sets defined in a metric space R which is compact in itself, and let it be uniformly monotone[1] at a β-normal set A. If a completely additive, real set function $\phi(E)$, defined at any β-normal subset of A, is absolutely continuous wlth respect to $\beta(E)$, and

$$\int_A [D_{\beta(E)}\phi(a)]^2 d\beta(E)$$

is finite, then I say that $\phi(E)$ *belongs to the class or space* $L_2(\beta)$. I define the norm of $\phi(E)$ as

$$\left[\int_A [D_{\beta(E)}\phi(a)]^2 d\beta(E)\right]^{\frac{1}{2}}$$

and denote it by $||\phi||$. Then, by the Schwarzian inequality

$$\left[\int_E D_{\beta(E)}\phi(a)\, d\beta(E)\right]^2 \leqq \beta(E)\int_E [D_{\beta(E)}\phi(a)]^2 d\beta(E),$$

therefore, we have

$$|\phi(E)| \leqq ||\phi|| \, \{\beta(E)\}^{\frac{1}{2}} \tag{1}$$

for all β-normal subsets E of A,

If $\phi(E), \phi_1(E)$ and $\phi_2(E)$ belong to $L_2(\beta)$, then $c\phi(E)$ and $\phi_1(E)+\phi_2(E)$ also belong to $L_2(\beta)$, i.e., *the space* $L_2(\beta)$ *is linear*. For

$$||c\phi|| = |c| \cdot ||\phi||,$$

and

$$||\phi_1+\phi_2|| \leqq ||\phi_1|| + ||\phi_2||,\text{[2]} \tag{2}$$

where c is a real number.

Let the distance between two set functions $\phi_1(E)$ and $\phi_2(E)$ be defined as $||\phi_1-\phi_2||$, then *the space* $L_2(\beta)$ *is metric*, since

(i) $||\phi_1-\phi_1||=0,$

(ii) $||\phi_1-\phi_2||=||\phi_2-\phi_1||>0$ when $\phi_1 \neq \phi_2$,

for if $||\phi_1-\phi_2||=0$, then by the definition $D_{\beta(E)}\phi_1(a)-D_{\beta(E)}\phi_2(a)$ is zero almost everywhere (β), therefore $\phi_1(E)=\phi_2(E)$.

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(1) F. Maeda, this Journal, **1** (1931), 4.

(2) By Minkowski's inequality $[\sum(a_i+b_i)^2]^{\frac{1}{2}} \leqq [\sum a_i^2]^{\frac{1}{2}} + [\sum b_i^2]^{\frac{1}{2}}$.

(iii) $$\|\phi_1-\phi_2\|+\|\phi_2-\phi_3\|\geqq\|\phi_1-\phi_3\|$$

by (2).

2. In the metric space $L_2(\beta)$, the convergence of the sequence of elements shall be defined as follows:

Let $\{\phi_n\}$ be a sequence of elements of the metric space $L_2(\beta)$; if an element ϕ of $L_2(\beta)$ exists so that

$$\lim_{n\to\infty}\|\phi_n-\phi\|=0, \tag{1}$$

then the sequence $\{\phi_n\}$ converges to ϕ.

But this definition of convergence is based on the conception of the distance between two elements. Therefore this convergence must be distinguished from the ordinary convergence of the sequence $\{\phi_n\}$ of set functions, i.e.

$$\lim_{n\to\infty}\phi_n(E)=\phi(E). \tag{2}$$

(2) means that at any set E, the sequence of real numbers $\{\phi_n(E)\}$ converges to a real number $\phi(E)$.

To keep the distinction between these two convergences, when the sequence $\{\phi_n\}$ satifies (1), we say that $\{\phi_n\}$ *converges strongly to* ϕ. In this paper I will denote this convergence by the symbol

$$[\lim_{n\to\infty}]\,\phi_n(E)=\phi(E).$$

But, since, by (1) of sec. 1

$$|\phi_n(E)-\phi(E)|\leqq\|\phi_n-\phi\|\,\{\beta(E)\}^{\frac{1}{2}},$$

if $$[\lim_{n\to\infty}]\,\phi_n(E)=\phi(E),$$

then $$\lim_{n\to\infty}\phi_n(E)=\phi(E).$$

This property is peculiar to the sequence of set functions, for, in the case of point functions, a strongly convergent sequence does not necessarily converge in the ordinary sense.

3. *The metric space $L_2(\beta)$ is complete.*

Let $\{\phi_n\}$ be a sequence of set functions of $L_2(\beta)$, and

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\|\phi_m-\phi_n\|=0.$$

Then $\{D_{\beta(E)}\phi_n(a)\}$ is a sequence of point functions of $L_2(\beta)$ and

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \| D_{\beta(E)}\phi_m(a) - D_{\beta(E)}\phi_n(a) \| = 0.$$

Then, as J. v. Neumann did,[1] we can find a point function $f(a)$ of $L_2(\beta)$ so that

$$\lim_{n\to\infty} \| D_{\beta(E)}\phi_n(a) - f(a) \| = 0. \qquad (1)$$

Let

$$\phi(E) = \int_E f(a)\, d\beta(E)$$

then (1) becomes

$$\lim_{n\to\infty} \| \phi_n - \phi \| = 0.$$

That is, $\{\phi_n\}$ converges strongly to a set function of $L_2(\beta)$.

4. *The metric space $L_2(\beta)$ is separable.*[2]

Consider a sequence of rational numbers

$$\ldots . < \rho_{-n} < \ldots . < \rho_{-2} < \rho_{-1} < \rho_0 < \rho_1 < \rho_2 < \ldots . < \rho_n < \ldots .$$

so that

$$\lim_{n\to\infty} \rho_{-n} = -\infty, \quad \lim_{n\to\infty} \rho_n = +\infty,$$

and

$$\rho_n - \rho_{n-1} < \varepsilon$$

for any value of n.

For a set function $\phi(E)$ of $L_2(\beta)$, consider a sequence of sets

$$A_i = A\,(\rho_i \leqq D_{\beta(E)}\phi(a) < \rho_{i+1}), \quad (i = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots)$$

and let us define a set function $\phi_\varepsilon(E)$ so that

$$\phi_\varepsilon(a) = \int_E f_\varepsilon(a)\, d\beta(E)$$

where

$$f_\varepsilon(a) = \rho_i$$

when a is a point of A_i. Then since

$$\| \phi_\varepsilon \|^2 = \int_A [f_\varepsilon(a)]^2 d\beta(E) \leqq \int_A [D_{\beta(E)}\phi(a)]^2 d\beta(E) + \eta = \| \phi \|^2 + \eta,$$

η being a small number, $\phi_\varepsilon(E)$ belongs to $L_2(\beta)$ and

$$\| \phi - \phi_\varepsilon \|^2 = \int_A [D_{\beta(E)}\phi(a) - f_\varepsilon(a)]^2 d\beta(E) \leqq \varepsilon^2 \beta(A). \qquad (1)$$

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(1) J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik*, (1932), 32–34.

(2) I have modified the proof of J. v. Neumann in respect of point functions. (*Math. Ann.* **102** (1929), 109–111.)

Now

$$\phi_\varepsilon(E)=\sum_{i=-\infty}^{\infty}\int_{A_iE} f_\varepsilon(a)\,d\beta(E)=\sum_{i=-\infty}^{\infty}\rho_i\beta(A_iE),$$

hence,

$$\|\phi_\varepsilon(E)-\sum_{i=-t}^{t}\rho_i\beta(A_iE)\|^2=\sum_{|i|>t}\int_{A_i}[f_\varepsilon(a)]^2d\beta(E)=\sum_{|i|>t}\rho_i^2\beta(A_i)\,;$$

but $$\sum_{i=-\infty}^{\infty}\rho_i^2\beta(A_i)=\int_A[f_\varepsilon(a)]^2d\beta(E)=\|\phi_\varepsilon\|^2,$$

if we make t sufficiently large,

$$\|\phi_\varepsilon(E)-\sum_{i=-t}^{t}\rho_i\beta(A_iE)\|<\varepsilon. \tag{2}$$

Then by (1) and (2) the set $L_2'(\beta)$ of functions of the form

$$\sum_{i=1}^{s} r_i\beta(B_iE)$$

is dense in $L_2(\beta)$, where s is an integer, r_i is a rational number, and B_i is a β-normal subset of A. Of course, $\sum_{i=1}^{s} r_i\beta(B_iE)$ belongs to $L_2(\beta)$.

B_i being β-normal, there exists an open set O_i, so that

$$O_i\geqq B_i$$

and $$\beta(O_i-B_i)<\varepsilon.$$

Since R is compact in istself, it is separable. Then there exists a sequence of special neighbourhoods(1)

$$V_1,\ V_2,\dots,\ V_m,\dots.$$

and any open set can be expressed as the sum of these neighbourhoods, thus

$$O_i=V_{1,i}\dotplus V_{2,i}\dotplus\dots\dotplus V_{k,i}\dotplus\dots.$$

Hence, if we take a sufficiently great integer k which depends to i,

$$\beta\{O_i-B_i(V_{1,i}\dotplus V_{2,i}\dotplus\dots\dotplus V_{k,i})\}<2\varepsilon$$

for all values of i.

Then

$$\|\sum_{i=1}^{s} r_i\beta\{E(V_{1,i}\dotplus V_{2,i}\dotplus\dots\dotplus V_{k,i})\}-\sum_{i=1}^{s} r_i\beta(B_iE)\|$$

$$\leqq\sum_{i=1}^{s}|r_i|.\|\beta\{E(V_{1,i}\dotplus V_{2,i}\dotplus\dots\dotplus V_{k,i})\}-\beta(B_iE)\|$$

(1) F. Hausdorff, *Mengenlehre*, (1927), 126.

$$\leqq \sum_{i=1}^{s} |r_i| . [\beta\{O_i - B_i(V_{1,i} \dot{+} V_{2,i} \dot{+} \dots \dot{+} V_{k,i})\}]^{\frac{1}{2}}$$

$$< \sqrt{2\varepsilon} \sum_{i=1}^{s} |r_i| .$$

Hence, the set $L_2''(\beta)$ of functions of the form

$$\sum_{i=1}^{s} r_i \, \beta\{E(V_{1,i} \dot{+} V_{2,i} \dot{+} \dots \dot{+} V_{k,i})\}$$

is dense in $L_2'(\beta)$, and therefore dense in $L_2(\beta)$. But $L_2''(\beta)$ is a denumerably infinite set, and consequently $L_2(\beta)$ is separable.

5. Given a sequence $\{\phi_n\}$ of set functions of $L_2(\beta)$ and a sequence $\{c_n\}$ of real numbers, if a set function $\phi(E)$ of $L_2(\beta)$ exists so that

$$[\lim_{n\to\infty}]\{c_1\phi_1(E) + c_2\phi_2(E) + \dots + c_n\phi_n(E)\} = \phi(E),$$

then I say that the series

$$c_1\phi_1(E) + c_2\phi_2(E) + \dots + c_n\phi_n(E) + \dots \qquad (1)$$

converges strongly to $\phi(E)$. Then, by sec. 2, the series (1) converges to $\phi(E)$ in the ordinary sense; i.e.

$$\phi(E) = c_1\phi_1(E) + c_2\phi_2(E) + \dots + c_n\phi_n(E) + \dots .$$

In this case, the convergence being strong, I say that $\phi(E)$ is a *linear combination* of the elements of $\{\phi_n\}$. And if there exists a sequence $\{c_n\}$ of real numbers, not all of them being zero, so that the series (1) converges strongly to zero, then I say that the elements of $\{\phi_n\}$ are *linearly dependent*.(1)

A subset L of $L_2(\beta)$ is called a *linear manifold* if every linear combination of the elements of $\{\phi_n\}$ belongs also to L, $\{\phi_n\}$ being any denumerable subset of L. The linear manifold is closed, for if $[\lim_{n\to\infty}]\,\phi_n(E) = \phi(E)$, then $\phi(E)$ is a linear combination of $\phi_1(E), \phi_2(E) - \phi_1(E), \dots, \phi_n(E) - \phi_{n-1}(E), \dots$. If there exists a sequence $\{\psi_n\}$ of set functions belonging to L, so that every element of L is a linear combination of the elements of $\{\psi_n\}$, then $\{\psi_n\}$ is called the *fundamental system* of the linear manifold L.

(1) When the sequence $\{\phi_n\}$ has only finite elements, the definitions of linear combination and linear dependency are similar to those applying to the case of point functions. But a finite sequence may be considered as a particular case of an infinite sequence. Therefore, in many parts of this paper, I consider only the sequence of infinite elements.

Complete Normalized Orthogonal System.

6. Let $\phi(E)$ and $\psi(E)$ be any two set functions of $L_2(\beta)$, then by the Schwarzian inequality

$$\int_A D_{\beta(E)}\phi(a)\, D_{\beta(E)}\psi(a)\, d\beta(E)$$

is always finite. I will denote this number by (ϕ, ψ) and call it the *inner product* of $\phi(E)$ and $\psi(E)$. Then

$$(\phi, \psi)=(\psi, \phi),$$
$$(\phi_1+\phi_2, \psi)=(\phi_1, \psi)+(\phi_2, \psi),$$
$$(\phi, \phi)=\|\phi\|^2,$$

and the Schwarzian inequality may be written in the form

$$|(\phi, \psi)| \leqq \|\phi\| \cdot \|\psi\|.$$

Since $$|(\phi_n-\phi, \psi)| \leqq \|\phi_n-\phi\| \cdot \|\psi\|,$$

if $$[\lim_{n\to\infty}]\, \phi_n(E)=\phi(E),$$

then $$\lim_{n\to\infty}(\phi_n, \psi)=(\phi, \psi).$$

In other words, if

$$\phi(E)=c_1\phi_1(E)+c_2\phi_2(E)+\ldots\ldots+c_n\phi_n(E)+\ldots\ldots,$$

the convergence being strong, then

$$(\phi, \psi)=c_1(\phi_1, \psi)+c_2(\phi_2, \psi)+\ldots\ldots+c_n(\phi_n, \psi)+\ldots\ldots$$

If $$(\phi, \psi)=0$$

then the two functions $\phi(E)$ and $\psi(E)$ ars said to be *orthogonal*.

7. Let $\{\psi_n\}$ be a sequence of set functions belonging to $L_2(\beta)$. If

$$(\psi_i, \psi_j)=0$$

for every pair of unequal values of i and j, the system $\{\psi_n\}$ is said to be an *orthogonal system* of set functions.

If $\|\psi_n\|$ has a value different from unity, that value can be made into unity by multiplying $\psi_n(E)$ by the factor $\frac{1}{\|\psi_n\|}$. A set function with norm 1 is said to be *normalized*. An orthogonal system of set functions is said to form a *normalized orthogonal system* when $\|\psi_n\|=1$, for all values of n.

The elements of normalized orthogonal system $\{\psi_n\}$ *are linearly independent.*

For, if there exists a sequence $\{c_n\}$ of real numbers, not all of them being zero, so that

$$c_1\psi_1(E)+c_2\psi_2(E)+\ldots\ldots+c_n\psi_n(E)+\ldots\ldots=0,$$

the convergence being strong, then, by the preceding section, taking the inner product with $\psi_i(E)$,

$$c_i\,||\,\psi_i\,||^2=c_i=0$$

for any value of i. But this contradicts the assumption.

8. An orthogonal system $\{\psi_n\}$ is said to be *complete* in a set S of functions, if no function $\phi(E)$ of S, except the null function, exists so that

$$(\phi,\psi_n)=0$$

for all values of n.

For any linear manifolds L, *there exists a normalized orthogonal system which is complete in* L.

Since $L_2(\beta)$ is separable, L is likewise so. Then there is a denumerable subset $\{\phi_n\}$ of L, which is dense in L. Take ξ_1 as the first ϕ_n, which is not zero; ξ_2 as the first ϕ_n, which is not a set function of the form $a_1\xi_1$; ξ_3 as the first ϕ_n, which is not a set function of the form $a_1\xi_1+a_2\xi_2$; and so on.

From $\{\xi_n\}$ by Schmidt's process of orthogonalization, we can construct a normalized orthogonal system $\{\psi_n\}$ so that

$$\gamma_1=\xi_1, \qquad \psi_1=\frac{\gamma_1}{||\gamma_1||},$$

$$\gamma_2=\xi_2-(\xi_2,\psi_1)\,\psi_1, \qquad \psi_2=\frac{\gamma_2}{||\gamma_2||},$$

$$\ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots,$$

$$\gamma_n=\xi_n-(\xi_n,\psi_1)\,\psi_1-(\xi_n,\psi_2)\,\psi_2-\ldots\ldots-(\xi_n,\psi_{n-1})\,\psi_{n-1}, \qquad \psi_n=\frac{\gamma_n}{||\gamma_n||},$$

$$\ldots\ldots\ldots\ldots\ldots, \qquad \ldots\ldots\ldots$$

Such a construction is always possible, that is, $||\gamma_n||\neq 0$. For, if $||\gamma_n||=0$, then ξ_n is a linear combination of $\psi_1,\psi_2,\ldots,\psi_{n-1}$, that is, a linear combination of $\xi_1,\xi_2,\ldots,\xi_{n-1}$; which is absurd.

Then the set M, composed by the linear combinations of the elements of $\{\psi_n\}$, is dense in L. If $\{\psi_n\}$ is not complete in L, then there exists a non-null function $\phi(E)$ in L, so that

$$(\phi, \psi_n)=0 \tag{1}$$

for any value of n. But there exists a sequence $\{\psi_\nu^*\}$ of set functions belonging to M so that

$$[\lim_{\nu\to\infty}]\, \psi_\nu^*(E)=\phi(E). \tag{2}$$

Since $\psi_\nu^*(E)$ is a linear combination of the elements of $\{\psi_n\}$, by (1) we have

$$(\phi, \psi_\nu^*)=0$$

for all values of ν. Hence, from (2)

$$(\phi, \phi)=||\phi||^2=0.$$

This is absurd. Therefore, $\{\psi_n\}$ is the required normalized orthogonal system, complete in L.

If we take $L_2(\beta)$ instead of L, then there exists a normalized orthogonal system which is complete in $L_2(\beta)$.

9. *If a normalized orthogonal system $\{\psi_n\}$ is complete in a linear manifold L, then $\{\psi_n\}$ is a fundamental system of L.*

For, let $\phi(E)$ be any, non-null, set function in L, and put

$$c_r=(\phi, \psi_r), \qquad (r=1, 2, \dots, n, \dots) \tag{1}$$

then some of the c's are not equal to zero. Denoting by $\sigma_n(E)$ the partial sum $\sum_{r=1}^{n} c_r\psi_r(E)$, we have

$$||\phi-\sigma_n||^2=||\phi||^2-\sum_{r=1}^{n} c_r^2.$$

It follows that, for all values of n,

$$\sum_{r=1}^{n} c_r^2 \leqq ||\phi||^2;$$

and that the series $\sum_{r=1}^{\infty} c_r^2$ converges to a number that is less than $||\phi||^2$; that is, we have the so-called *Bessel's inequality*

$$\sum_{r=1}^{\infty} c_r^2 \leqq ||\phi||^2.$$

But since

$$||\sigma_m-\sigma_u||^2=\sum_{r=n+1}^{m} c_r^2,$$

where $m>n$,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} ||\sigma_m-\sigma_n||=0.$$

Therefore, by sec. 3 the sequence $\{\sigma_n(E)\}$ converges strongly to a set function $\phi^*(E)$ of the class $L_2(\beta)$. Then

$$\phi^*(E)=c_1\psi_1(E)+c_2\psi_2(E)+\ldots\ldots+c_n\psi_n(E)+\ldots\ldots, \qquad (2)$$

$$(\phi^*, \psi_r)=c_r. \qquad (r=1, 2, \ldots\ldots, n, \ldots\ldots) \qquad (3)$$

Therefore, from (1) and (3)

$$(\phi-\phi^*, \psi_r)=0. \qquad (r=1, 2, \ldots\ldots, n, \ldots\ldots)$$

Then, since $\{\psi_n\}$ is complete in L, it must be that

$$\phi(E)-\phi^*(E)=0.$$

From (2) we have

$$\phi(E)=c_1\psi_1(E)+c_2\psi_2(E)+\ldots\ldots+c_n\psi_n(E)+\ldots\ldots \qquad (4)$$

where $\quad c_r=(\phi, \psi_r), \qquad (r=1, 2, \ldots\ldots, n, \ldots\ldots)$

and this series converges strongly. Thefore, $\{\psi_n\}$ is a fundamental system of L.

Conversely, *if a normalized orthogonal system $\{\psi_n\}$ is a fundamental system of a linear manifold L, then $\{\psi_n\}$ is complete in L.*

For, let $\phi(E)$ be any, non-null, set function of L; then it may be expressed as

$$\phi(E)=c_1\psi_1(E)+c_2\psi_2(E)+\ldots\ldots+c_n\psi_n(E)+\ldots\ldots,$$

the convergence being strong. Then, by sec. 6

$$(\phi, \psi_r)=c_r||\psi_r||^2=c_r. \qquad (r=1, 2, \ldots\ldots, n, \ldots\ldots)$$

Hence, there exists at least one set function $\psi_i(E)$ so that

$$(\phi, \psi_i)\neq 0.$$

10. Let a normalized orthogonal system $\{\psi_n\}$ be complete in a linear manifold L, and $\phi(E), \phi'(E)$ be two set functions of L. Then by the preceding section their expansions with respect to $\{\psi_n\}$ are

$$\phi(E)=c_1\psi_1(E)+c_2\psi_2(E)+\ldots\ldots+c_n\psi_n(E)+\ldots\ldots,$$

$$\phi'(E)=c_1'\psi_1(E)+c_2'\psi_2(E)+\ldots\ldots+c_n'\psi_n(E)+\ldots\ldots,$$

where

$$c_\nu=(\phi, \psi_\nu), \quad c_\nu'=(\phi', \psi_\nu). \qquad (\nu=1, 2, \ldots\ldots, n, \ldots\ldots)$$

Then

$$(\sigma_n, \phi') = c_1 c_1' + c_2 c_2' + \dots + c_n c_n',$$

where $$\sigma_n(E) = c_1 \psi_1(E) + c_2 \psi_2(E) + \dots + c_n \psi_n(E).$$

But, since

$$|(\phi - \sigma_n, \phi')| \leqq \|\phi - \sigma_n\| \cdot \|\phi'\|,$$

we have $$\lim_{n\to\infty} (\phi - \sigma_n, \phi') = 0,$$

that is $$(\phi, \phi') = \lim_{n\to\infty} (\sigma_n, \phi').$$

Therefore,

$$(\phi, \phi') = c_1 c_1' + c_2 c_2' + \dots + c_n c_n' + \dots.$$

We have now established the following theorem which is a generalization of *Parseval's theorem* in the theory of Fourier's series:

If a normalized orthogonal system $\{\psi_n\}$ is complete in a linear manifold L, and $\{c_n\}, \{c_n'\}$ are the sets of coefficients corresponding to two set functions ϕ(E), $\phi'(E)$ in L, then

$$(\phi, \phi') = c_1 c_1' + c_2 c_2' + \dots + c_n c_n' + \dots,$$

and $$\|\phi\|^2 = c_1^2 + c_2^2 + \dots + c_n^2 + \dots.$$

11. Let $\{\psi_n\}$ be a complete normalized orthogonal system in L, and

$$c_1 \psi_1(E) + c_2 \psi_2(E) + \dots c_n \psi_n(E) + \dots$$

be a series with coefficients c_n such that $\sum_{r=1}^{\infty} c_r^2$ is convergent. Denoting the partial sum of this series by $\sigma_n(E)$, we have

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|\sigma_m - \sigma_n\|^2 = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{r=n+1}^{m} c_r^2 = 0,$$

where $m > n$. Then, by sec. 3, there exists a unique set function $\phi(E)$ belonging to L, such that

$$\phi(E) = [\lim_{n\to\infty}] \sigma_n(E),$$

or $$\phi(E) = c_1 \psi_1(E) + c_2 \psi_2(E) + \dots + c_n \psi_n(E) + \dots.$$

We have thus obtained the following theorem which corresponds to the *Riesz-Fischer theorem* in the theory of Fourier's series:

If $\{\psi_n\}$ is a complete normalized orthogonal system in L, and $\{c_n\}$ is a sequence of constants such that $\sum_{r=1}^{\infty} c_r^2$ is convergent, there exists a unique set function $\phi(E)$ in L, so that

$$\phi(E) = c_1 \psi_1(E) + c_2 \psi_2(E) \dots + c_n \psi_n(E) + \dots,$$

where the series converges strongly.

12. Combining the last two theorems, we have the following theorem :

If $\{\psi_n\}$ *is a complete normalized orthogonal system in* L, *a necessary and sufficient condition that a series*

$$c_1\psi_1(E)+c_2\psi_2(E)\ldots\ldots+c_n\psi_n(E)+\ldots\ldots$$

expresses a set function in L, *is that* $\sum_r c_r^2$ *is convergent.*

Thus, in L, there is a one-to-one correspondence between the set function $\phi(E)$ and the sequence $\{c_n\}$ of the coefficients of its expansion. Therefore, if we consider $\{c_n\}$ as the coordinates of the set function $\phi(E)$, the distance between two set functions $\phi(E)$ and $\phi'(E)$ whose coordinates are $\{c_n\}$ and $\{c_n'\}$ is

$$\|\phi-\phi'\|=\{\sum_r (c_r-c_r')^2\}^{\frac{1}{2}}.$$

Therefore, if the fundamental system $\{\psi_n\}$ of L has only finite, say p, set functions, then L is an *Euclidian space of p dimensions*, and if the system $\{\psi_n\}$ has denumerably infinite set functions, then L is an Euclidian space of denumerably infinte dimension or *Hilbert space*.

13. If the linear manifold L is of finite dimension, then by the property of Euclidian space, every bounded sequence of set functions of L is compact in L, that is, it contains a subsequence which converges strongly to a set function of L. But, if the linear manifold L is of denumerably infinite dimension, the bounded sequence of set functions of L is not necessarily compact in L. For example, the normalized orthogonal system $\{\psi_n\}$ is bounded but is not compact. For,

$$\|\psi_m-\psi_n\|^2=\|\psi_m\|^2+\|\psi_n\|^2=2$$

for any values of m and n.

Thus, we have the following theorem :

A necessary and sufficient condition that a linear manifold L *be of finite dimension is that every bounded sequence of set functions of* L *is compact in* L.[1]

In the following sections I will consider the sequence of linear manifolds in order to investigate the properties of a bounded sequence of set functions in the linear manifolds of denumerably infinite dimension.

(1) F. Riesz proved this theorem in the case of linear manifolds of continuous point functions $f(x)$, defining $\|f\|$ as the upper bound of $|f(x)|$. (*Acta Math.*, 41 (1918), 77-78).

Sequence of Linear manifolds.

14. Let L be a linear manifold of finite or infinite dimension, and $\{\psi_n\}$ be a complete normalized orthogonal system in L. A set function $\phi(E)$ in $L_2(\beta)$ is said to be *orthogonal* to L, when

$$(\phi, \psi_i)=0$$

for all values of i. In this case, of course

$$(\phi, \psi)=0$$

for any set function $\psi(E)$ in L.

A set function $\phi(E)$ in $L_2(\beta)$ can be decomposed in one and only one way into two components, one belonging to L and the other orthogonal to L. For, let

$$\psi(E)=\sum_i (\phi, \psi_i)\, \psi_i$$

and

$$\xi(E)=\phi(E)-\psi(E).$$

Then

$$\phi(E)=\psi(E)+\xi(E), \tag{1}$$

and $\psi(E)$ is a set function in L, and $\xi(E)$ is orthogonal to L, for from (1)

$$(\phi, \psi_i)=(\phi, \psi_i)\,||\psi_i||^2+(\xi, \psi_i),$$

therefore,

$$(\xi, \psi_i)=0$$

for all values of i.

The *distance* between $\phi(E)$ and L I will call $||\xi||$ and denote it by $r(\phi, L)$; it is zero when and only when $\phi(E)$ belongs to L. Of course $r(\phi, L)$ is the least value of $||\phi-\psi||$ for all set functions $\psi(E)$ of L.

If the component of $\phi(E)$, orthogonal to L, is $\xi(E)$, then the component of $c\phi(E)$, orthogonal to L, is $c\xi(E)$; therefore,

$$r(c\phi, L)=|c|\, r(\phi, L),$$

where c is any real number.

15. Let

$$\phi_1(E),\ \phi_2(E), \ldots\ldots, \phi_p(E) \tag{1}$$

be p set functions of $L_2(\beta)$. The least value α, which always exists, of

$$||c_1\phi_1+c_2\phi_2+\ldots\ldots+c_p\phi_p||$$

for all values of c's, satisfying

$$\sum_{i=1}^{p} c_i^2=1,$$

is called the *measure of independency*[1] of the set functions (1). If a null function exists in (1), then $\alpha=0$. If there is no null function in (1), then $\alpha=0$ when and only when the set functions (1) are linearly dependent.

If $\alpha>0$, we can consider a linear manifold L of p dimensions, whose fundamental system is (1). Let $\phi(E)$ be any set function in L, then $\phi(E)$ can be expressed as

$$\phi(E)=d_1\phi_1(E)+d_2\phi_2(E)+\dots+d_p\phi_p(E).$$

If we put

$$c_i=\frac{d_i}{\sqrt{\sum_i d_i^2}},$$

then, since $\sum_i c_i^2=1$,

$$\|c_1\phi_1+c_2\phi_2+\dots+c_p\phi_p\|\geqq\alpha.$$

Therefore, we have

$$\frac{\|\phi\|}{\sqrt{\sum_i d_i^2}}\geqq\alpha,$$

that is

$$\sum_i d_i^2\leqq\frac{\|\phi\|^2}{\alpha^2}.$$

16. Let L and L' be two linear manifolds of p dimensions, and $\{\phi_i\}$ be a fundamental system of L, having the measure of independency α. *If*

$$r(\phi_i, L')<k, \qquad (i=1, 2, \dots, p) \tag{1}$$

then for any normalized set function $\eta(E)$ in L,

$$r(\eta, L')<\frac{pk}{\alpha}.$$

For, if we express $\eta(E)$ as

$$\eta=d_1\phi_1+d_2\phi_2+\dots+d_p\phi_p,$$

then by the preceding section

$$\sum_i d_i^2\leqq\frac{1}{\alpha^2}.$$

(1) For point functions, cf. Courant-Hilbert, *Methoden der Mathematischen Physik* I. zweite Aufl. (1931), 52.

Hence $$|d_i| \leqq \frac{1}{\alpha} \qquad (i=1, 2, \dots, p). \qquad (2)$$

Let $\phi_i'(E)$ be the component of $\phi_i(E)$ contained in L', then by (1)

$$||\phi_i-\phi_i'||=r(\phi_i, L')<k. \qquad (i=1, 2, \dots, p) \qquad (3)$$

Then

$$\phi'=d_1\phi_1'+d_2\phi_2'+\dots+d_p\phi_p'$$

is a set function in L', and

$$||\eta-\phi'|| \leqq |d_1|\cdot||\phi_1-\phi_1'||+|d_2|\cdot||\phi_2-\phi_2'||+\dots+|d_p|\cdot||\phi_p-\phi_p'||,$$

hence, by (2) and (3)

$$||\eta-\phi'||<\frac{pk}{\alpha}.$$

But $r(\eta, L')$ is the least values of $||\eta-\phi'||$ for all $\phi'(E)$ in L', therefore

$$r(\eta, L')<\frac{pk}{\alpha}. \qquad (4)$$

Let $\eta(E)$ be any normalized set function in L, then by (4) $r(\eta, L')$ is always less than $\frac{pk}{\alpha}$. I will denote the upper bound of $r(\eta, L')$ for all normalized set functions in L, by $\delta(L, L')$. We can define $\delta(L', L)$ similarly. Whichever of $\delta(L', L)$ and $\delta(L, L')$ is the greater, I call the *distance* between L and L', and denote it by $r(L, L')$.

$r(L, L')=0$, when and only when L and L' are identical.

17. Let L, L', L'' be 3 linear manifolds of p dimensions. Decompose a normalized set function $\eta(E)$ in L, into two components:

$$\eta(E)=\psi'(E)+\xi'(E),$$

where $\psi'(E)$ is the component which is contained in L', and $\xi'(E)$ is the component orthogonal to L'. Then

$$||\psi'|| \leqq 1,$$

and $$r(\eta, L')=||\xi'||.$$

Next, decompose $\psi'(E)$ into two components:

$$\psi'(E)=\psi''(E)+\xi''(E),$$

where $\psi''(E)$ is the component which is contained in L'', and $\xi''(E)$ is the component orthogonal to L''. Then

$$r(\psi', L'')=||\xi''||.$$

Now, since

$$\eta(E)=\psi''(E)+\xi'(E)+\xi''(E),$$

$$r(\eta,L'')\leqq\|\xi'+\xi''\|\leqq r(\eta,L')+r(\psi',L'').$$

But

$$r(\eta,L')\leqq\delta(L,L')\leqq r(L,L'),$$

and since $\|\psi'\|\leqq 1$,

$$r(\psi',L'')=\|\psi'\|\cdot r\left(\frac{\psi'}{\|\psi'\|},L''\right)\leqq\delta(L',L'')\leqq r(L',L'').$$

Therefore

$$r(\eta,L'')\leqq r(L,L')+r(L',L'').$$

But since this inequality holds for any normalized set function in L, we have

$$\delta(L,L'')\leqq r(L,L')+r(L',L'').$$

Similarly

$$\delta(L'',L)\leqq r(L'',L')+r(L',L).$$

Therefore, we have

$$r(L,L'')\leqq r(L,L')+r(L',L'').$$

That is, *the set of linear manifolds of p dimensions is a metric space.*

In the metric space of linear manifolds of p dimensions, if

$$\lim_{n\to\infty} r(L_n,L)=o,$$

then it is said that the sequence $\{L_n\}$ converges to L. In this case, for any positive number ε, an integer $N(\varepsilon)$ exists, so that

$$r(L_n,L)\leqq\varepsilon \tag{1}$$

for $n\geqq N(\varepsilon)$, and when $\varepsilon>\varepsilon'$,

$$N(\varepsilon)\leqq N(\varepsilon'),$$

and

$$\lim_{\varepsilon\to o} N(\varepsilon)=\infty.$$

Let $\phi(E)$ be a set function in L, then since

$$r(\phi,L_n)=\|\phi\|\cdot r\left(\frac{\phi}{\|\phi\|},L_n\right),$$

from (1) there exists a set function $\phi_n(E)$ in L_n, so that

$$\|\phi-\phi_n\|\leqq\frac{d}{i}\|\phi\|\quad(i=1,2,\ldots.)$$

for all values of n, satisfying $N\left(\frac{d}{i}\right)\leqq n<N\left(\frac{d}{i+1}\right)$, where d is the

upper bound of $r(L_n, L)$. Then the sequence $\{\phi_n\}$ converges strongly to $\phi(E)$.

That is, *if the sequence* $\{L_n\}$ *converges to* L, *there exists a sequence* $\{\phi_n\}$, $\phi_n(E)$ *belonging to* L_n *for each value of* n, *so that*

$$[\lim_{n\to\infty}]\,\phi_n(E)=\phi(E)$$

for any set function $\phi(E)$ *in* L.

18. *The space of linear manifolds of* p *dimensions is complete.* That is, if a sequence $\{L_n\}$ of linear manifolds of p dimensions be such that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} r(L_m, L_n)=0,$$

there exists a linear manifold L of p dimensions so that

$$\lim_{n\to\infty} r(L_n, L)=0.$$

For, by the assumption, for any positive number ε, a positive number $N(\varepsilon)$ exists so that

$$r(L_m, L_n)<\varepsilon \qquad (1)$$

for any integer m and n greater than $N(\varepsilon)$, and when $\varepsilon>\varepsilon'$,

$$N(\varepsilon)\leqq N(\varepsilon').$$

Take an integer n_1 which is greater than $N\left(\frac{\varepsilon}{2}\right)$; and let a normalized fundamental system of L_{n_1} be

$$\psi_{1,1},\ \psi_{1,2},\dots,\ \psi_{1,p},$$

and its measure of independency be α.

Next take an integer n_2 which is greater than $N\left(\frac{\varepsilon}{2^2}\right)$, then by (1) there exist p set functions of L_{n_2}

$$\psi_{2,1},\ \psi_{2,2},\ \dots,\ \psi_{2,p},$$

so that

$$\|\psi_{2,i}\|\leqq 1$$

and

$$\|\psi_{1,i}-\psi_{2,i}\|<\frac{\varepsilon}{2}. \qquad (i=1,2\dots,p)$$

Next take an integer n_3 which is greater than $N\left(\frac{\varepsilon}{2^3}\right)$. Then there exist p set functions of L_{n_3}

$$\psi_{3,1},\ \psi_{3,2}, \dots,\ \psi_{3,p},$$

so that

$$\|\psi_{3,i}\| \leqq \|\psi_{2,i}\| \leqq 1$$

and
$$\|\psi_{2,i}-\psi_{3,i}\| < \frac{\varepsilon}{2^2}. \qquad (i=1, 2, \dots, p)$$

Generally, let n_ν be an integer which is greater than $N\left(\frac{\varepsilon}{2^\nu}\right)$; then there exist p set functions of L_{n_ν}

$$\psi_{\nu,1},\ \psi_{\nu,2}, \dots,\ \psi_{\nu,p},$$

so that

$$\|\psi_{\nu,i}\| \leqq 1$$

and
$$\|\psi_{\nu-1,i}-\psi_{\nu,i}\| < \frac{\varepsilon}{2^{\nu-1}}. \qquad (i=1, 2, \dots, p)$$

Thus we have p sequences of set functions

$$\{\psi_{\nu,1}\},\ \{\psi_{\nu,2}\}, \dots,\ \{\psi_{\nu,p}\}$$

and

$$\|\psi_{\nu,i}-\psi_{\mu,i}\| \leqq \sum_{t=\nu}^{\mu-1} \frac{\varepsilon}{2^t} < \frac{\varepsilon}{2^{\nu-1}}$$

when $\mu > \nu$.

Therefore, by sec. 3 there exist p set functions of $L_2(\beta)$

$$\xi_1,\ \xi_2, \dots,\ \xi_p, \tag{2}$$

so that

$$[\lim_{\nu\to\infty}]\, \psi_{\nu,i} = \xi_i,$$

and
$$\|\psi_{1,i}-\xi_i\| \leqq \varepsilon \tag{3}$$

for $i=1, 2, \dots, p$.

Next, I will shew that the meaure of independency of (2) is not zero. Let $c_1, c_2, \dots, c_p$ be any p numbers satisfying $\sum_{i=1}^{p} c_i^2 = 1$. Put

$$\psi_1 = c_1\psi_{1,1} + c_2\psi_{1,2} + \dots + c_p\psi_{1,p},$$
$$\xi = c_1\xi_1 + c_2\xi_2 + \dots + c_p\xi_p,$$

then, since the measure of independency of $\psi_{1,1}, \psi_{1,2}, \dots, \psi_{1,p}$ is α,

$$\|\psi_1\| \geqq \alpha. \tag{4}$$

Therefore, by (3)

$$\|\psi_1 - \xi\| \leqq |c_1| \cdot \|\psi_{1,1}-\xi_1\| + |c_2| \cdot \|\psi_{1,2}-\xi_2\| + \dots + |c_p| \cdot \|\psi_{1,p}-\xi_p\|$$
$$\leqq p\varepsilon.$$

Then, by (4) we have

$$||\xi|| \geqq \alpha - p\varepsilon.$$

But, since this inequality holds for any values of $c_1, c_2, \ldots, c_p$ satisfying $\sum_{i=1}^{p} c_i^2 = 1$, the measure of independency of (2) is not less than $\alpha - p\varepsilon$.

But we can take ε so small that

$$\varepsilon < \frac{\alpha}{p},$$

therefore, $\alpha - p\varepsilon > 0$, and the p set functions of (2) are linearly independent. Then, there exists a linear manifold L of p dimensions, which has (2) for its fundamental system.

Lastly, I will shew that

$$\lim_{n\to\infty} r(L_n, L) = 0.$$

Let η_1 be any normalized set function in L_{n_1}, and

$$\eta_1 = d_1 \psi_{1,1} + d_2 \psi_{1,2} + \ldots + d_p \psi_{1,p},$$

then by sec. 15,

$$\sum_i d_i^2 \leqq \frac{1}{\alpha^2}. \tag{5}$$

Now

$$\xi_1 = d_1 \xi_1 + d_2 \xi_2 + \ldots + d_p \xi_p$$

is a set function in L, and

$$||\eta_1 - \xi_1|| \leqq |d_1| \cdot ||\psi_{1,1} - \xi_1|| + |d_2| \cdot ||\psi_{1,2} - \xi_2|| + \ldots + |d_p| \cdot ||\psi_{1,p} - \xi_p||.$$

Then, by (3) and (5)

$$||\eta_1 - \xi_1|| \leqq \frac{p\varepsilon}{\alpha}.$$

Therefore,

$$r(\eta_1, L) \leqq \frac{p\varepsilon}{\alpha}.$$

But this inequality holds for any normalized set function η_1 in L_{n_1}; hence

$$\delta(L_{n_1}, L) \leqq \frac{p\varepsilon}{\alpha}. \tag{6}$$

Similarly, since the measure of independency of (2) is not less than $\alpha - p\varepsilon$, we have

$$\delta(L, L_{n_1}) \leqq \frac{p\varepsilon}{\alpha - p\varepsilon}. \tag{7}$$

By (6) and (7), we have

$$r(L, L_{n_1}) \leqq \frac{p\varepsilon}{\alpha - p\varepsilon},$$

when $n_1 > N\left(\frac{\varepsilon}{2}\right)$. Therefore, we have

$$\lim_{n\to\infty} r(L_n, L) = 0.$$

19. Let $\{\phi_\nu\}$ be a sequence of set functions of $L_2(\mathfrak{B})$. If, for any given positive number ε however small, there exists a natural number N, so that for any values of $n_1, n_2, \ldots, n_{p+1}$ greater than N, the measure of independency of $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_{p+1}}$ is less than ε; and for any positive number G however great, there exist p integers $n_1, n_2, \ldots, n_p$ greater than G, so that the measure of independency of $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_p}$ is greater than a fixed positive number α, then the sequence $\{\phi_\nu\}$ is said to have p *asymptotic dimensions.*[1]

In this case, there exists a sequence of p set functions

$$\begin{array}{l}\phi_{1_1}, \phi_{1_2}, \cdots\cdots, \phi_{1_p}, \\ \phi_{2_1}, \phi_{2_2}, \cdots\cdots, \phi_{2_p}, \\ \cdots\cdots\cdots\cdots \\ \phi_{n_1}, \phi_{n_2}, \cdots\cdots, \phi_{n_p}, \\ \cdots\cdots\cdots\cdots\end{array}$$

the measure of independency of $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_p}$ being greater than α for any value of n, and the values of $n_1, n_2, \ldots, n_p$ increasing indefinitely with n. I will call the sequence of linear manifolds $\{L_n\}$ of p dimensions, the fundamental system of L_n being $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_p}$, an *associated sequence* of $\{\phi_\nu\}$.

If we take a different sequence of p set functions, satisfying the above conditions, we have another associated sequence of $\{\phi_\nu\}$.

20. *If a bounded sequence $\{\phi_\nu\}$ of set functions is of p asymptotic dimensions, then all associated sequences $\{L_n\}$ of $\{\phi_\nu\}$ converge to a p dimensional linear manifold L, and*

$$\lim_{\nu\to\infty} r(\phi_\nu, L) = 0.$$

I will first shew that

(1) Cf. Courant-Hilbert, *loc. cit.*, 53. When $p=0$, then $\lim_{n\to\infty} \|\phi_n\| = 0$, and $[\lim_{n\to\infty}]\ \phi_n = 0$.

$$\lim_{\substack{\nu\to\infty \\ n\to\infty}} r(\phi_\nu, L_n)=0.$$

For any values of ν and n, there exists a set of values of $c_0, c_1, c_2, \ldots, c_p$ so that

$$c_0^2+c_1^2+c_2^2+\ldots+c_p^2=1,$$

and
$$\| c_0\phi_\nu+c_1\phi_{n_1}+c_2\phi_{n_2}+\ldots+c_p\phi_{n_p}\|$$
is equal to the measure of independency of $\phi_\nu, \phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_p}$.

Now

$$\|c_1\phi_{n_1}+c_2\phi_{n_2}+\ldots+c_p\phi_{n_p}\|$$
$$\leqq|c_0|\cdot\|\phi_\nu\|+\|c_0\phi_\nu+c_1\phi_{n_1}+c_2\phi_{n_2}+\ldots+c_p\phi_{n_p}\|,$$

and

$$\frac{1}{\sqrt{1-c_0^2}}\| c_1\phi_{n_1}+c_2\phi_{n_2}\ldots+c_p\phi_{n_p}\|\geqq\alpha$$

for any values of ν and n; since

$$\lim_{\substack{\nu\to\infty \\ n\to\infty}}\| c_0\phi_\nu+c_1\phi_{n_1}+c_2\phi_{n_2}+\ldots+c_p\phi_{n_p}\|=0, \tag{1}$$

and $\|\phi_\nu\|$ is always less than a fixed finite number, it must be that

$$|c_0|>k,$$

where k is a constant number independent of ν and n.

Put

$$d_i=-\frac{c_i}{c_0} \qquad (i=1, 2, \ldots, p),$$

and we have

$$\|\phi_\nu-(d_1\phi_{n_1}+d_2\phi_{n_2}+\ldots+d_p\phi_{n_p})\|$$
$$<\frac{1}{k}\| c_0\phi_\nu+c_1\phi_{n_1}+c_2\phi_{n_2}+\ldots+c_p\phi_{n_p}\|, \tag{2}$$

then by (1)

$$\lim_{\substack{\nu\to\infty \\ n\to\infty}} r(\phi_\nu, L_n)=0.$$

Substituting $\phi_{m_1}, \phi_{m_2}, \ldots, \phi_{m_p}$ for ϕ_ν, we have by sec. 16

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \delta(L_m, L_n)=0.$$

If we interchange m and n,

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \delta(L_n, L_m)=0.$$

Therefore, we have

$$\lim_{\substack{m\to\infty \\ n\to\infty}} r(L_m, L_n)=0.$$

Then by sec. 18, $\{L_n\}$ converges to a p dimensional linear manifold L.

If $\{L_n'\}$ and $\{L_n''\}$ be two associated sequences of $\{\phi_\nu\}$, then

$$L_1', L_1'', L_2', L_2'', \dots, L_n', L_n'', \dots$$

is also an associated sequence, say $\{L_n\}$, of $\{\phi_\nu\}$. If L be the linear manifold to which $\{L_n\}$ converges, then $\{L_n'\}$ and $\{L_n''\}$ must converge to L. Therefore, all associated sequences of $\{\phi_\nu\}$ converge to a unique linear manifold L.

Let ε be any given positive number, then by (2) there exists a number $N(\varepsilon)$ so that

$$\| \phi_\nu-(d_1\phi_{n_1}+d_2\phi_{n_2}+\dots+d_p\phi_{n_p}) \| < \varepsilon \tag{3}$$

for any values of ν and n greater than $N(\varepsilon)$. But since

$$\lim_{n\to\infty} r(L_n, L)=0,$$

there exist p set functions $\xi_1, \xi_2, \dots, \xi_p$ in L, depending on n, so that

$$\| \phi_{n_i}-\xi_i \| \leqq \varepsilon \qquad (i=1, 2, \dots, p)$$

for any value of n greater than $N'(\varepsilon)$. Now let n be greater than both $N(\varepsilon)$ and $N'(\varepsilon)$. Then

$$\begin{aligned} &\| (d_1\phi_{n_1}+d_2\phi_{n_2}+\dots+d_p\phi_{n_p})-(d_1\xi_1+d_2\xi_2+\dots+d_p\xi_p) \| \\ &\qquad \leqq \{ |d_1|+|d_2|+\dots+|d_p| \}\varepsilon, \end{aligned} \tag{4}$$

but since

$$d_i=-\frac{c_i}{c_0},$$

and

$$|c_0|>k, \qquad |c_i|\leqq 1,$$

we have

$$|d_i|<\frac{1}{k},$$

therefore, by (3) and (4)

$$\| \phi_\nu-(d_1\xi_1+d_2\xi_2+\dots+d_p\xi_p) \| \leqq \left(1+\frac{p}{k}\right)\varepsilon,$$

that is

$$\lim_{\nu\to\infty} r(\phi_\nu, L)=0.$$

Chapter II. Applications to the Theory of Integral Equations.

Space of Real Functions ot Two Sets.

21. Let a and a' be two points in the Euclidian space[1] R_n of n dimensions; then this pair of points (a, a') may be considered as a single point a^* in the Euclidian space R_{2n} of $2n$ dimensions. Then we consider the pair of sets (A, A'), A and A' being two sets in R_n, as a set of pairs of points (a, a') for all points a in A, and a' in A'. Since the interval $\overline{\mathrm{W}}^*$ in R_{2n} is a pair of intervals $(\overline{\mathrm{W}}, \overline{\mathrm{W}}')$, we can define an interval function $\beta^*(\overline{\mathrm{W}}^*)$ so that it is equal to $\beta(\overline{\mathrm{W}})\,\beta(\overline{\mathrm{W}}')$. Then from this interval function we can construct a completely additive function of normal sets $\beta^*(E^*)$ in R_{2n}. If E and E' are β-normal, then it is obvious that (E, E') is β^*-normal.[2]

Let $\mathfrak{S}(E^*)$ be a completely additive set function defined in R_{2n}, and absolutely continuous with respect to $\beta^*(E^*)$. Then by the so-called fundamental theorem which I have proved in a previous paper,[3] $\mathfrak{S}(E^*)$ has a unique finite general derivative $D_{\beta^*(E^*)}\mathfrak{S}(a^*)$ almost everywhere (β^*),[4] and

$$\mathfrak{S}(E^*)=\int_{E^*} D_{\beta^*(E^*)}\mathfrak{S}(a^*)\,d\beta^*(E^*). \tag{1}$$

When $E^*=(E, E')$, where E and E' are β-normal, as in the previous paper,[5] I write (1) as follows:

$$\mathfrak{S}(E, E')=\int_{E}\int_{E'} D_{\beta^*(E^*)}\mathfrak{S}(a, a')\,d\beta(E)\,d\beta(E')$$

and I say that $D_{\beta^*(E^*)}\mathfrak{S}(a, a')$ is *doubly integrable* with respect to $\{\beta(E), \beta(E')\}$ over (E, E'). Then

$$\int_{E'} D_{\beta^*(E^*)}\mathfrak{S}(a, a')\,d\beta(E') \quad \text{and} \quad \int_{E} D_{\beta^*(E^*)}\mathfrak{S}(a, a')\,d\beta(E)$$

exist almost everywhere (β)[4] in E and E' respectively; and

(1) In the Euclidian space, the monotone uniformity of $\beta(E)$ at A has no bearing on the problem. Cf. F. Maeda, this journal, 2 (1932), 33.

(2) Cf. F. Maeda, this journal, 2 (1932), 160.

(3) This journal, 2 (1932), 37.

(4) In what follows, I omit the words "almost everywhere (β^*) or (β)."

(5) This journal, 2 (1932), 161.

$$\int_E\int_{E'} D_{\beta^*(E^*)}\mathfrak{S}(a,a')\,d\beta(E)\,d\beta(E')=\int_E\left\{\int_{E'} D_{\beta^*(E^*)}\mathfrak{S}(a,a')\,d\beta(E')\right\}d\beta(E)$$

$$=\int_{E'}\left\{\int_E D_{\beta^*(E^*)}\mathfrak{S}(a,a')\,d\beta(E)\right\}d\beta(E');$$

that is, we can *change the order of the integration*.

Then by the fundamental relation

$$D_{\beta(E)}\mathfrak{S}(a,E')=\int_{E'} D_{\beta^*(E^*)}\mathfrak{S}(a,a')\,d\beta(E'),^{(1)}$$

where $D_{\beta(E)}\mathfrak{S}(a,E')$ is the general derivative, at a, of $\mathfrak{S}(E,E')$ which is a function of set E, with respect to $\beta(E)$, and

$$D_{\beta(E')}D_{\beta(E)}\mathfrak{S}(a,a')=D_{\beta^*(E^*)}\mathfrak{S}(a,a').$$

Similarly we have

$$D_{\beta(E)}D_{\beta(E')}\mathfrak{S}(a,a')=D_{\beta^*(E^*)}\mathfrak{S}(a,a').$$

Therefore, if $D_{\beta^*(E^*)}\mathfrak{S}(a,a')$ is doubly integrable with respect to $\{\beta(E),\beta(E')\}$, then we can *change the order of the differentiation*.

I have proved in a previous paper,(2) that on the same assumption we can *change the order of differentiation and integration*; i.e.

$$D_{\beta(E)}\int_{E'} D_{\beta(E')}\mathfrak{S}(a,a')\,d\beta(E')=\int_{E'} D_{\beta(E)}D_{\beta(E')}\mathfrak{S}(a,a')\,d\beta(E').$$

If $\mathfrak{S}(E,E')$ belongs to the class $L_2(\beta^*)$, that is,

$$\int_A\int_A [D_{\beta(E)}D_{\beta(E')}\mathfrak{S}(a,a')]^2 d\beta(E)\,d\beta(E')$$

exists, then since

$$[D_{\beta(E')}\mathfrak{S}(E,a')]^2=\left[\int_E D_{\beta(E)}D_{\beta(E')}\mathfrak{S}(a,a')\,d\beta(E)\right]^2$$

$$\leqq\beta(E)\cdot\int_E [D_{\beta(E)}D_{\beta(E')}\mathfrak{S}(a,a')]^2 d\beta(E),$$

$\int_A [D_{\beta(E')}\mathfrak{S}(E,a')]^2 d\beta(E')$ exists. Therefore, $\mathfrak{S}(E,E')$ belongs to $L_2(\beta)$, as a function of E', E being constant. Similarly $\mathfrak{S}(E,E')$ belongs to $L_2(\beta)$, as a function of E, E' being constant.

22. I will define the *norm* $\|\mathfrak{S}\|$ of $\mathfrak{S}(E,E')$ and the *inner pro-*

(1) In this peper, I use the symbol (=), as signifying "equal almost everywhere (β^*) or (β)."

(2) This journal, 2 (1932), 163.

duct $(\mathfrak{S}_1, \mathfrak{S}_2)$ of $\mathfrak{S}(E, E')$ and $\mathfrak{S}_2(E, E')$ as follows:

$$\|\mathfrak{S}\| = \left[\int_A\int_A [D_{\beta(E)} D_{\beta(E')} \mathfrak{S}(a, a')]^2 d\beta(E)\, d\beta(E')\right]^{\frac{1}{2}},$$

$$(\mathfrak{S}_1, \mathfrak{S}_2) = \int_A\int_A D_{\beta(E)} D_{\beta(E')} \mathfrak{S}_1(a, a')\, D_{\beta(E)} D_{\beta(E')} \mathfrak{S}_2(a, a')\, d\beta(E)\, d\beta(E'),$$

and I will use the same terminology as in chapter I.

Let $\{\eta_\nu(E)\}$ and $\{\eta_\nu'(E')\}$ be two normalized orthogonal systems, belonging to $L_2(\beta)$, then $\{\eta_\nu(E)\,\eta_\nu'(E')\}$ is a normalized orthogonal system of set functions, belonging to $L_2(\beta^*)$. For, since

$$D_{\beta(E)} D_{\beta(E')} [\eta_\nu(a)\, \eta_\nu'(a')] = D_{\beta(E)}\, \eta(a)\, D_{\beta(E')}\, \eta_\nu'(a'),$$

we have

$$\|\eta_\nu \eta_\nu'\| = \|\eta_\nu\| \cdot \|\eta_\nu'\| = 1, \qquad (\nu = 1, 2, \ldots)$$

and

$$(\eta_\mu \eta_\mu', \eta_\nu \eta_\nu') = (\eta_\mu, \eta_\nu)\,(\eta_\mu', \eta_\nu') = 0,$$

for every pair of unequal values of μ and ν.

Therefore, if the normalized orthogonal system $\{\eta_\nu(E)\,\eta_\nu'(E')\}$ is complete in a linear manifold L^* of functions of two sets, then any function $\mathfrak{S}(E, E')$ in L^* can be expanded uniquely as follows:

$$\mathfrak{S}(E, E') = c_1\eta_1(E)\,\eta_1(E') + c_2\eta_2(E)\,\eta_2'(E') + \ldots\ldots$$
$$+ c_\nu\eta_\nu(E)\,\eta_\nu'(E') + \ldots\ldots,$$

where $c_\nu = (\mathfrak{S}, \eta_\nu\eta_\nu'), \qquad (\nu = 1, 2, \ldots\ldots)$

and all the considerations and theorems of chapter I for the normalized orthogonal system hold also in this case.

Linear Set Functional Transformations.

23. Let $\mathfrak{K}(E, E')$ be a real, non-null, function of two sets belonging to $L_2(\beta^*)$, then, since by sec. 21 $\mathfrak{K}(E, E')$ belongs also to $L_2(\beta)$ as a function of set E, and as a function of set E', the following two integrals

$$\int_A D_{\beta(E')} \mathfrak{K}(E, a')\, D_{\beta(E')} \phi(a')\, d\beta(E') \qquad (1)$$

$$\int_A D_{\beta(E)} \mathfrak{K}(a, E')\, D_{\beta(E)}\, \phi(a)\, d\beta(E) \qquad (1')$$

exist, $\phi(E)$ being any set function of $L_2(\beta)$. As I have found in the preceding paper,[1] (1) and (1′) represent two set functions of $L_2(\beta)$, say $\Phi(E)$ and $\Phi'(E')$. (1) and (1′) are *linear transformations* in the space $L_2(\beta)$, and $\mathfrak{K}(E, E')$ are their *kernels*. I denote these transformations by

$$\Phi(E) = T_{\mathfrak{K}}\phi(E),$$
$$\Phi'(E) = T_{\mathfrak{K}}'\phi(E).$$

Since $D_{\beta(E)}D_{\beta(E')}\mathfrak{K}(a, a')\, D_{\beta(E')}\phi(a')$ is doubly integrable with respect to $\{\beta(E), \beta(E')\}$ over (A, A), we have

$$D_{\beta(E)}T_{\mathfrak{K}}\phi(a) = \int_A D_{\beta(E)}D_{\beta(E')}\mathfrak{K}(a, a')\, D_{\beta(E')}\phi(a')\, d\beta(E'),$$

then by the Schwarzian inequality, we have

$$[D_{\beta(E)}T_{\mathfrak{K}}\phi(a)]^2 \leqq \int_A [D_{\beta(E)}D_{\beta(E')}\mathfrak{K}(a, a')]^2 d\beta(E') \cdot \|\phi\|^2,$$

then $$\|T_{\mathfrak{K}}\phi\| \leqq \|\mathfrak{K}\| \cdot \|\phi\|. \qquad (2)$$

Similarly, we have

$$\|T_{\mathfrak{K}}'\phi\| \leqq \|\mathfrak{K}\| \cdot \|\phi\|.$$

From (2), *if*

$$[\lim_{n\to\infty}]\, \phi_n(E) = \phi(E),$$

then $$[\lim_{n\to\infty}]\, T_{\mathfrak{K}}\phi_n(E) = T_{\mathfrak{K}}\phi(E). \qquad (3)$$

And, *if*

$$[\lim_{n\to\infty}]\, \mathfrak{K}_n(E, E') = \mathfrak{K}(E, E')$$

then $$[\lim_{n\to\infty}]\, T_{\mathfrak{K}_n}\phi(E) = T_{\mathfrak{K}}\phi(E). \qquad (4)$$

Similarly for the transformation $T_{\mathfrak{K}}'$.

24. We will now apply theorems (3) and (4) of the preceding section to Volterra's solution of the integral equation:

$$\phi(E) = \psi(E) + T_{\mathfrak{K}}\phi(E),$$

which I have considered in a previous paper.[2]

Let us construct the iterated kernels:

$$\mathfrak{K}_1(E, E') = \mathfrak{K}(E, E'),$$

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(1) This journal, 2 (1932), 164.

(2) This journal, 2 (1932), 168-174.

$$\mathfrak{K}_2(E, E') = T_{\mathfrak{K}}\mathfrak{K}_1(E, E')^{(1)} = \int_A D_{\beta(E'')}\mathfrak{K}(E, a'')\, D_{\beta(E'')}\mathfrak{K}(a'', E')\, d\beta(E''),$$

$$\begin{aligned}\mathfrak{K}_3(E, E') &= T_{\mathfrak{K}}\mathfrak{K}_2(E, E')\\ &= \int_A D_{\beta(E''')}\mathfrak{K}(E, a''')\Big[\int_A D_{\beta(E'')} D_{\beta(E''')}\mathfrak{K}(a''', a'')\\ &\qquad\qquad \times D_{\beta(E'')}\mathfrak{K}(a'', E')\, d\beta(E'')\Big]\, d\beta(E''')\\ &= \int_A D_{\beta(E'')}\mathfrak{K}_2(E, a'')\, D_{\beta(E'')}\mathfrak{K}(a'', E')\, d\beta(E'')\\ &= T_{\mathfrak{K}_2}\mathfrak{K}(E, E').\end{aligned}$$

Generally,

$$\mathfrak{K}_n(E, E') = T_{\mathfrak{K}}\mathfrak{K}_{n-1}(E, E') = T_{\mathfrak{K}_{n-1}}\mathfrak{K}(E, E').$$

Then, since

$$\|\mathfrak{K}_n\| \leqq \|\mathfrak{K}\| \cdot \|\mathfrak{K}_{n-1}\|, \qquad (n = 2.\ 3, \ldots.)$$

we have

$$\|\mathfrak{K}_n\| \leqq \|\mathfrak{K}\|^n.$$

Let

$$\mathfrak{S}_n(E, E') = \mathfrak{K}_1(E, E') + \mathfrak{K}_2(E, E') + \ldots\ldots + \mathfrak{K}_n(E, E').$$

If $\|\mathfrak{K}\| < 1$, since

$$\begin{aligned}\|\mathfrak{S}_m - \mathfrak{S}_n\| &\leqq \|\mathfrak{K}_{n+1}\| + \|\mathfrak{K}_{n+2}\| + \ldots\ldots + \|\mathfrak{K}_m\|\\ &< \frac{\|\mathfrak{K}\|^{n+1}}{1 - \|\mathfrak{K}\|},\end{aligned}$$

where $m > n$, we have

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \|\mathfrak{S}_m - \mathfrak{S}_n\| = 0.$$

Therefore, since $L_2^*(\beta)$ is complete as proved in sec. 3, $\{\mathfrak{S}_n(E, E')\}$ converges strongly to a function of $L_2^*(\beta)$, say $-\mathfrak{k}(E, E')$; and

$$-\mathfrak{k}(E, E') = \mathfrak{K}_1(E, E') + \mathfrak{K}_2(E, E') + \ldots\ldots + \mathfrak{K}_n(E, E') + \ldots\ldots^{(2)}$$

By (3) of the preceding section, we have

$$-T_{\mathfrak{K}}\mathfrak{k}(E, E') = [\lim_{n\to\infty}]\, T_{\mathfrak{K}}\mathfrak{S}_n(E, E')$$

(1) This denotes the transformation of $\mathfrak{K}_1(E, E')$ as a function of E, E' being constant.

(2) This proof of the convergence of the series is simpler than that of the preceding paper. But the present method can not be applied to the case where the kernel is of the form $\mathfrak{K}(E, a')$ or $\mathfrak{K}(a, a')$.

$$= \mathfrak{K}_2(E, E') + \mathfrak{K}_3(E, E') + \dots + \mathfrak{K}_{n+1}(E, E') + \dots.$$

Similarly, by (4) of the preceding section, we have

$$- T_{\mathfrak{K}} \mathfrak{K}(E, E') = [\lim_{n \to \infty}] T_{\mathfrak{S}_n} \mathfrak{K}(E, E')$$
$$= \mathfrak{K}_2(E, E') + \mathfrak{K}_3(E, E') + \dots + \mathfrak{K}_{n+1}(E, E') + \dots.$$

Therefore, we have the reciprocal property between $\mathfrak{K}(E, E')$ and $\mathfrak{k}(E, E')$, i.e.

$$\mathfrak{K}(E, E') + \mathfrak{k}(E, E') = T_{\mathfrak{K}} \mathfrak{k}(E, E') = T_{\mathfrak{k}} \mathfrak{K}(E, E').$$

Using this property, as in the preceding paper, we have the followng theorem:

In the integral equation

$$\phi(E) = \psi(E) + \int_A D_{\beta(E')} \mathfrak{K}(E, a') \, D_{\beta(E')} \phi(a') \, d\beta(E'),$$

if $\mathfrak{K}(E, E')$ *and* $\psi(E)$ *belong to* $L_2(\beta^*)$ *and* $L_2(\beta)$ *respectively, and*

$$\|\mathfrak{K}\| < 1,$$

then this equation has one and only one solution belonging to $L_2(\beta)$ *and this solution is given by*

$$\phi(E) = \psi(E) - \int_A D_{\beta(E')} \mathfrak{k}(E, a') \, D_{\beta(E')} \psi(a') \, d\beta(E').$$

Linear Manifold of Set Functions, determined by a Real Symmetric Kernel.

25. Let the kernel $\mathfrak{K}(E, E')$ be symmetric, i.e.

$$\mathfrak{K}(E, E') = \mathfrak{K}(E', E),$$

then

$$T_{\mathfrak{K}} \phi(E) = T_{\mathfrak{K}}' \phi(E).$$

Since

$$(\mathfrak{K}, \phi\psi) = \int_A \int_A D_{\beta(E)} D_{\beta(E')} \mathfrak{K}(a, a') \, D_{\beta(E)} \phi(a) \, D_{\beta(E')} \psi(a') \, d\beta(E) \, d\beta(E'),$$

we have the fundamental relations

$$(\mathfrak{K}, \phi\psi) = (T_{\mathfrak{K}} \phi, \psi) = (\phi, T_{\mathfrak{K}} \psi). \qquad (1)$$

Let λ be a real number. Then if, in $L_2(\beta)$, the null function being excepted, there exists a set function $\eta(E)$, which satisfies

$$\eta(E) = \lambda T_{\mathfrak{K}} \eta(E), \qquad (2)$$

then λ is called a *characteristic constant* of the kernel $\mathfrak{K}(E, E')$ and

$\eta(E)$ a *characteristic function* of $\mathfrak{K}(E, E')$ with respect to λ.

For any symmetric kernel, there exists at least one characteristic constant.

By sec. 23 for any set function $\phi(E)$ in $L_2(\beta)$,

$$\| T_{\mathfrak{K}}\phi \| \leqq \| \mathfrak{K} \| \cdot \| \phi \|,$$

therefore, there exists a finite upper bound, say l, of $\dfrac{\| T_{\mathfrak{K}}\phi \|}{\| \phi \|}$ for all set functions $\phi(E)$ in $L_2(\beta)$. By (2), we have

$$\frac{1}{|\lambda|} = \frac{\| T_{\mathfrak{K}}\eta \|}{\| \eta \|};$$

therefore, if $\mathfrak{K}(E, E')$ has characteristic constants, their absolute values can not be less than $\dfrac{1}{l}$. In what follows, I will show that at least one value of $\dfrac{1}{l}$ or $-\dfrac{1}{l}$ is a characteristic constant of $\mathfrak{K}(E, E')$.

To prove this, I will first show that the lower bound of

$$\| \psi - \frac{1}{l} T_{\mathfrak{K}}\psi \| \quad \text{or} \quad \| \psi + \frac{1}{l} T_{\mathfrak{K}}\psi \|$$

for any normalized set function $\psi(E)$ in $L_2(\beta)$, is zero.[1] For, if it is not, then a positive number ε exists so that

$$\| \phi - \frac{1}{l} T_{\mathfrak{K}}\phi \| > \varepsilon \| \phi \|, \tag{3}$$

$$\| \phi + \frac{1}{l} T_{\mathfrak{K}}\phi \| > \varepsilon \| \phi \| \tag{4}$$

for all set functions $\phi(E)$ in $L_2(\beta)$, except the null function. Then

$$\| \phi - \frac{1}{l^2} T_{\mathfrak{K}} T_{\mathfrak{K}}\phi \| = \| (\phi - \frac{1}{l} T_{\mathfrak{K}}\phi) + \frac{1}{l} T_{\mathfrak{K}}(\phi - \frac{1}{l} T_{\mathfrak{K}}\phi) \|$$

by (4) $$> \varepsilon \| \phi - \frac{1}{l} T_{\mathfrak{K}}\phi \|,$$

by (3) $$> \varepsilon^2 \| \phi \|, \tag{5}$$

for all set functions $\phi(E)$ in $L_2(\beta)$, except the null function.

On the other hand, in $L_2(\beta)$ there exists a set function $\phi(E)$ which satisfies the following inequality

$$\| T_{\mathfrak{K}}\phi \|^2 > l^2\left(1 - \frac{\varepsilon^4}{2}\right) \| \phi \|^2. \tag{6}$$

(1) I owe this property to F. Riesz, *Math. Ann.* 69 (1910), 484-487.

With respect to this set function $\phi(E)$

$$\|\phi-\frac{1}{l^2}T_{\mathfrak{R}}T_{\mathfrak{R}}\phi\|^2=\|\phi\|^2-\frac{2}{l^2}(\phi,\ T_{\mathfrak{R}}T_{\mathfrak{R}}\phi)+\frac{1}{l^4}\|T_{\mathfrak{R}}T_{\mathfrak{R}}\phi\|^2.$$

Then, since, by (1) and (6)

$$(\phi,\ T_{\mathfrak{R}}T_{\mathfrak{R}}\phi)=(T_{\mathfrak{R}}\phi,\ T_{\mathfrak{R}}\phi)=\|T_{\mathfrak{R}}\phi\|^2>l^2\left(1-\frac{\varepsilon^4}{2}\right)\|\phi\|^2,$$

and
$$\|T_{\mathfrak{R}}T_{\mathfrak{R}}\phi\|^2\leqq l^2\|T_{\mathfrak{R}}\phi\|^2\leqq l^4\|\phi\|^2,$$

we have

$$\|\phi-\frac{1}{l^2}T_{\mathfrak{R}}T_{\mathfrak{R}}\phi\|^2<\|\phi\|^2\left\{1-2\left(1-\frac{\varepsilon^4}{2}\right)+1\right\}$$
$$=\varepsilon^4\|\phi\|^2,$$

which contradicts (5).

Let the value, $\frac{1}{l}$ or $-\frac{1}{l}$, be denoted by λ, so that the lower bound of

$$\|\psi-\lambda T_{\mathfrak{R}}\psi\|$$

for any normalized set function $\psi(E)$ is zero. Then there exists a sequence $\{\psi_\nu\}$ which satisfies

$$\lim_{\nu\to\infty}\|\psi_\nu-\lambda T_{\mathfrak{R}}\psi_\nu\|=0.$$

I will call such a sequence $\{\psi_\nu\}$ the *minimal sequence* of $\|\psi-\lambda T_{\mathfrak{R}}\psi\|$.

Let $\{\psi_\nu\}$ be of p asymptotic dimensions; then there exists an associated sequence $\{L_n\}$ of linear manifolds of p dimensions, and the measure of independency of the fundamental system of L_n

$$\psi_{n_1},\ \psi_{n_2},\ \ldots,\ \psi_{n_p}$$

is greater than a fixed postive number α, where $n_1, n_2, \ldots, n_p$ increase indefinitely with n.

Let $\phi_n(E)$ be a set function of L_n, and let $\|\phi_n\|\leqq k$, k being a constant number indepedent of n. If we express $\phi_n(E)$ as

$$\phi_n=d_{n_1}\psi_{n_1}+d_{n_2}\psi_{n_2}+\ldots+d_{n_p}\psi_{n_p},$$

then by sec. 15

$$\sum_i d_{n_i}^2\leqq\frac{k^2}{\alpha^2},$$

and
$$|d_{n_i}|\leqq\frac{k}{\alpha}\qquad(i=1, 2, \ldots, p)$$

for any value of n. Then, since

$$\|\phi_n-\lambda T_{\mathfrak{K}}\phi_n\| \leqq |d_{n_1}|\cdot\|\psi_{n_1}-\lambda T_{\mathfrak{K}}\psi_{n_1}\|+|d_{n_2}|\cdot\|\psi_{n_2}-\lambda T_{\mathfrak{K}}\psi_{n_2}\| + \cdots + |d_{n_p}|\cdot\|\psi_{n_p}-\lambda T_{\mathfrak{K}}\psi_{n_p}\|,$$

we have

$$\lim_{n\to\infty}\|\phi_n-\lambda T_{\mathfrak{K}}\phi_n\|=0. \qquad (7)$$

If we take the normalized orthogonal system in L_n

$$\eta_{n_1},\ \eta_{n_2},\ \ldots\ldots,\ \eta_{n_p}$$

instead of $\phi_n(E)$, we have

$$\lim_{n\to\infty}\|\eta_{n_i}-\lambda T_{\mathfrak{K}}\eta_{n_i}\|=0. \quad (i=1, 2, \ldots., p) \quad (8)$$

Then, since $\|\eta_{n_i}\|=1$ for any value of n,

$$\lim_{n\to\infty}|\lambda|\cdot\|T_{\mathfrak{K}}\eta_{n_i}\|=1. \qquad (9)$$

Therefore, since

$$\|\eta_{n_i}-\lambda T_{\mathfrak{K}}\eta_{n_i}\|^2=\|\eta_{n_i}\|^2+\lambda^2\|T_{\mathfrak{K}}\eta_{n_i}\|^2-2\lambda(\eta_{n_i}, T_{\mathfrak{K}}\eta_{n_i}),$$

by (8) and (9), we have

$$\lim_{n\to\infty}(\eta_{n_i}, T_{\mathfrak{K}}\eta_{n_i})=\frac{1}{\lambda}. \qquad (i=1, 2,\ldots., p)$$

Now by Bessel's inequality

$$\|\mathfrak{K}\|^2\geqq\sum_{i=1}^{p}(\mathfrak{K}, \eta_{n_i}\eta_{n_i})^2=\sum_{i=1}^{p}(\eta_{n_i}, T_{\mathfrak{K}}\eta_{n_i})^2$$

for any value of n, therefore we have

$$\|\mathfrak{K}\|^2\geqq\frac{p}{\lambda^2}.$$

That is, the asymptotic dimensions of minimal sequences of $\|\psi-\lambda T_{\mathfrak{K}}\psi\|$ can not be greater than $\lambda^2\|\mathfrak{K}\|^2$. But, for any minimal sequence $\{\psi_\nu\}$ the asymptotic dimension is $\geqq 1$, for, $\|\psi_\nu\|=1$ for all values of ν.

Let $\{\psi_\nu\}$ be the minimal sequence of $\|\psi-\lambda T_{\mathfrak{K}}\psi\|$, which has the greatest asymptotic dimension, say $p\,(\geqq 1)$, and $\{L_n\}$ be the associated sequence of $\{\psi_\nu\}$. Then by sections 20 and 17, there exist a linear manifold L of p dimensions, and p sequences of set functions

$$\{\phi_{n_1}\},\ \{\phi_{n_2}\},\ \ldots,\ \{\phi_{n_p}\} \qquad (10)$$

where $\phi_{n_1}, \phi_{n_2}, \ldots, \phi_{n_p}$ belong to L_n, for each value of n, so that the sequences of (10) converge strongly to p linearly independent set functions

$$\xi_1,\ \xi_2,\ \ldots\ldots,\ \xi_p$$

of L, respectively. That is,

$$\lim_{n\to\infty} \| \phi_{n_i} - \xi_i \| = 0. \quad (i=1, 2, \ldots, p) \quad (11)$$

Then, $\{\phi_{n_i}\}$ being bounded, from (7) we have

$$\lim_{n\to\infty} \| \phi_{n_i} - \lambda T_{\mathfrak{K}} \phi_{n_i} \| = 0. \quad (i=1, 2, \ldots, p) \quad (12)$$

But

$$\begin{aligned} & \| (\phi_{n_i} - \lambda T_{\mathfrak{K}} \phi_{n_i}) - (\xi_i - \lambda T_{\mathfrak{K}} \xi_i) \| \\ \leqq & \| \phi_{n_i} - \xi_i \| + |\lambda| \cdot \| T_{\mathfrak{K}} (\phi_{n_i} - \xi_i) \| \\ \leqq & (1 + |\lambda| \cdot \| \mathfrak{K} \|) \| \phi_{n_i} - \xi_i \|. \end{aligned}$$

Then, by (11) and (12), we have

$$\xi_i - \lambda T_{\mathfrak{K}} \xi_i = 0. \qquad (i=1, 2, \ldots, p)$$

Therefore, λ is a characteristic constant of $\mathfrak{K}(E, E')$ and

$$\xi_1, \ \xi_2, \ \ldots\ldots, \ \xi_p$$

are characteristic functions of $\mathfrak{K}(E, E')$ with respect to λ. Hence *all set functions in L are characteristic functions of $\mathfrak{K}(E, E')$ with respect to λ.*

Conversely, *all chararacteristic functions of $\mathfrak{K}(E, E')$ with respect to λ belong to L.* For, let ξ be a characteristic function with respect to λ; then the sequence $\{\psi_\nu'\}$ where

$$\psi'_{2\nu-1} = \xi, \qquad\qquad (\nu = 1, 2, \ldots)$$

and

$$\psi'_{2\nu} = \psi_\nu$$

is also a minimal sequence of $\| \psi - \lambda T_{\mathfrak{K}} \psi \|$, $\{\psi_\nu\}$ being the minimal sequence above defined. Then, $\{\psi_\nu'\}$ being of p asymptotic dimensions, the associated sequence of $\{L_n\}$ of $\{\psi_\nu\}$ is also an associated sequence of $\{\psi_\nu'\}$. Then, by sec. 20

$$\lim_{\nu\to\infty} r(\psi_\nu', L) = 0,$$

hence ξ must be a set function of L.

I will call this linear manifold L the *characteristic manifold* of $\mathfrak{K}(E, E')$ with respect to λ.

26. In the preceding section, we found a characteristic constant of $\mathfrak{K}(E, E')$ which has the least absolute value. We will denote this characteristic constant by $\lambda^{(1)}$ and the characteristic manifold of $\mathfrak{K}(E, E')$ with respect to $\lambda^{(1)}$ by $L^{(1)}$,

$$\eta_1^{(1)}, \ \eta_2^{(1)}, \ \ldots\ldots, \ \eta_{p_1}^{(1)}$$

being a complete normalized orthogonal system in $L^{(1)}$.

I will now find other characteristic constants and characteristic manifolds.

Let it be assumed that two different characteristic constants λ and λ' exist, and let η and η' be the characteristic functions with respect to λ and λ' respectively. That is,

$$\eta=\lambda T_{\mathfrak{K}}\eta, \tag{1}$$

$$\eta'=\lambda' T_{\mathfrak{K}}\eta', \tag{2}$$

From (1), we have

$$(\eta, \eta')=\lambda\,(T_{\mathfrak{K}}\eta, \eta')=\lambda\,(\mathfrak{K}, \eta\eta').$$

And from (2), we have

$$(\eta, \eta')=\lambda'\,(\eta, T_{\mathfrak{K}}\eta')=\lambda'\,(\mathfrak{K}, \eta\eta').$$

Since $\lambda\neq\lambda'$, it must be that

$$(\eta, \eta')=0.$$

Therefore, to find characteristic functions of $\mathfrak{K}(E, E')$ with respect to λ, different from $\lambda^{(1)}$, it is sufficient to consider the set functions which are orthogonal to $L^{(1)}$.

Put

$$\mathfrak{K}^{(1)}(E,E)=\eta_1^{(1)}(E)\,\eta_1^{(1)}(E')+\eta_2^{(1)}(E)\,\eta_2^{(1)}(E')+\ldots\ldots+\eta_{p_1}^{(1)}(E)\,\eta_{p_1}^{(1)}(E').$$

Then $T_{\mathfrak{K}^{(1)}}$ transforms each of the functions $\phi(E)$ in $L_2(\beta)$ into functions of $L^{(1)}$. For, decompose $\phi(E)$ into two components

$$\phi(E)=\phi^{(1)}(E)+\psi^{(1)}(E),$$

where $\phi^{(1)}(E)$ is constained in $L^{(1)}$, and $\psi^{(1)}(E)$ is orthogonal to $L^{(1)}$. Then, since

$$T_{\mathfrak{K}^{(1)}}\eta_i^{(1)}(E)=\eta_i^{(1)}(E), \qquad (i=1, 2, \ldots, p)$$

and

$$T_{\mathfrak{K}^{(1)}}\psi^{(1)}(E)=0,$$

we have

$$T_{\mathfrak{K}^{(1)}}\phi(E)=\phi^{(1)}(E). \tag{3}$$

From this equality, we can say that $\mathfrak{K}^{(1)}(E, E')$ has only one characteristic constant, that is 1, and a set function $\phi(E)$ is a characteristic function of $\mathfrak{K}^{(1)}(E, E')$ when and only when $\phi(E)$ is a set function in $L^{(1)}$.

Now put

$$\mathfrak{L}_1(E, E')=\mathfrak{K}(E, E')-\frac{1}{\lambda^{(1)}}\mathfrak{K}^{(1)}(E, E') \tag{4}$$

Then

$$T_{\mathfrak{L}_1}\phi(E)=T_{\mathfrak{K}}\phi(E)-\frac{1}{\lambda^{(1)}}T_{\mathfrak{K}^{(1)}}\phi(E).$$

Since

$$\lambda^{(1)}T_{\mathfrak{K}}\phi^{(1)}(E)=\phi^{(1)}(E),$$

we have, by (3)

$$T_{\mathfrak{L}_1}\phi(E)=\frac{1}{\lambda^{(1)}}\phi^{(1)}(E)+T_{\mathfrak{K}}\psi^{(1)}(E)-\frac{1}{\lambda^{(1)}}\phi^{(1)}(E),$$

that is,

$$T_{\mathfrak{L}_1}\phi(E)=T_{\mathfrak{K}}\psi^{(1)}(E). \tag{5}$$

If $\phi(E)$ is a characteristic function of $\mathfrak{K}(E, E')$ with respect to λ different from $\lambda^{(1)}$, then, $\phi(E)$ must be orthogonal to $L^{(1)}$, that is,

$$\phi(E)=\psi^{(1)}(E).$$

Hence by (5), $\phi(E)$ is also a characteristic function of $\mathfrak{L}_1(E, E')$ with respect to the same characteristic constant λ.

Next let $\phi(E)$ be a characteristic function of $\mathfrak{L}_1(E, E')$ with respect to λ, then, since by (5) $T_{\mathfrak{L}_1}$ transforms all set functions in $L^{(1)}$ into the null function, $\phi(E)$ must be orthogonal to $L^{(1)}$. That is,

$$\phi(E)=\psi^{(1)}(E).$$

Then by (5), $\phi(E)$ is also a characteristic function of $\mathfrak{K}(E, E')$ with respect to the same characteristic constant λ.

Therefore, to find all the characteristic constants and corresponding characteristic manifolds of $\mathfrak{K}(E, E')$, different from $\lambda^{(1)}$ and $L^{(1)}$, it is sufficient to consider all the characteristic constants and corresponding characteristic manifolds of $\mathfrak{L}_1(E, E')$.

Since

$$T_{\mathfrak{L}_1}\eta_i^{(1)}(E)=0, \qquad (i=1, 2, \ldots, p)$$

we have

$$(\mathfrak{L}_1, \mathfrak{K}^{(1)})=\sum_{i=1}^{p_1}(\mathfrak{L}_1, \eta_i^{(1)}\eta_i^{(1)})=\sum_{i=1}^{p_1}(T_{\mathfrak{L}_1}\eta_i^{(1)}, \eta_i^{(1)})=0.$$

Therefore, by (4)

$$\|\mathfrak{K}\|^2=\|\mathfrak{L}_1\|^2+\frac{1}{\lambda^{(1)2}}\|\mathfrak{K}^{(1)}\|^2.$$

Then

$$\|\mathfrak{L}_1\|^2=\|\mathfrak{K}\|^2-\frac{p_1}{\lambda^{(1)2}}.$$

If

$$\|\mathfrak{K}\|^2-\frac{p_1}{\lambda^{(1)2}}=0,$$

then
$$\mathfrak{K}(E, E') = \frac{1}{\lambda^{(1)}} \mathfrak{K}^{(1)}(E, E').$$

But, if
$$\|\mathfrak{K}\|^2 - \frac{p_1}{\lambda^{(1)2}} > 0,$$
by the method of the preceding section, we find a characteristic constant $\lambda^{(2)}$ of $\mathfrak{L}_1(E, E')$ which has the least absolute value, and a linear manifold of $L^{(2)}$ of p_2 dimensions. Then
$$|\lambda^{(1)}| \leqq |\lambda^{(2)}|,$$
and
$$\frac{p_2}{\lambda^{(2)2}} \leqq \|\mathfrak{L}_1\|^2 = \|\mathfrak{K}\|^2 - \frac{p_1}{\lambda^{(1)2}}.$$

Now put
$$\mathfrak{L}_2(E, E') = \mathfrak{L}_1(E, E') - \frac{1}{\lambda^{(2)}} \mathfrak{K}^{(2)}(E, E'),$$
where
$$\mathfrak{K}^{(2)}(E, E') = \eta_1^{(2)}(E)\,\eta_1^{(2)}(E') + \eta_2^{(2)}(E)\,\eta_2^{(2)}(E') + \dots\dots + \eta_{p_2}^{(2)}(E)\eta_{p_2}^{(2)}(E'),$$
$\eta_1^{(2)}(E), \eta_2^{(2)}(E), \dots\dots\dots, \eta_{p_2}^{(2)}(E)$ being the complete orthogonal system in $L^{(2)}$, and apply the above methods to $\mathfrak{L}_2(E, E')$ instead of $\mathfrak{L}_1(E, E')$, and so on.

If an integer n exists so that
$$\|\mathfrak{K}\|^2 - \frac{p_1}{\lambda^{(1)2}} - \frac{p_2}{\lambda^{(2)2}} - \dots\dots - \frac{p_n}{\lambda^{(n)2}} = 0,$$
then
$$\mathfrak{K}(E, E') = \frac{1}{\lambda^{(1)}} \mathfrak{K}^{(1)}(E, E') + \frac{1}{\lambda^{(2)}} \mathfrak{K}^{(2)}(E, E') + \dots\dots + \frac{1}{\lambda^{(n)}} \mathfrak{K}^{(n)}(E, E') \tag{6}$$
is the required result.

But, if
$$\|\mathfrak{K}\|^2 - \frac{p_1}{\lambda^{(1)2}} - \frac{p_2}{\lambda^{(2)2}} - \dots\dots - \frac{p_n}{\lambda^{(n)2}} > 0$$
for any value of n, then since the infinite series
$$\frac{p_1}{\lambda^{(1)2}} + \frac{p_2}{\lambda^{(2)2}} + \dots\dots + \frac{p_n}{\lambda^{(n)2}} + \dots\dots$$
converges,
$$\frac{1}{\lambda^{(1)}} \mathfrak{K}^{(1)}(E, E') + \frac{1}{\lambda^{(2)}} \mathfrak{K}^{(2)}(E, E') + \dots\dots + \frac{1}{\lambda^{(n)}} \mathfrak{K}^{(n)}(E, E') + \dots\dots$$

converges strongly to a set function, say $\mathfrak{P}(E,E')$, of $L_2^*(\beta)$; and

$$\lim_{n\to\infty}|\lambda^{(n)}|=\infty.$$

Since we have successively taken the characteristic constants which have the least absolute values, there exist no characteristic constants except $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}, \ldots$.

I will now show that $\mathfrak{K}(E,E')$ is equal to $\mathfrak{P}(E,E')$. For, if not, put

$$\mathfrak{Q}(E,E')=\mathfrak{K}(E,E')-\mathfrak{P}(E,E');$$

then since $\mathfrak{Q}(E,E')$ is symmetric, by the preceding section, $\mathfrak{Q}(E,E')$ has at least one characteristic function, say $\eta(E)$, with respect to a characteristic constant λ, i.e.

$$\eta(E)=\lambda T_{\mathfrak{Q}}\eta(E). \tag{7}$$

Since

$$T_{\mathfrak{Q}}\eta_i^{(n)}=T_{\mathfrak{K}}\eta_i^{(n)}-T_{\mathfrak{P}}\eta_i^{(n)}=\frac{\eta_i^{(n)}}{\lambda^{(n)}}-\frac{\eta_i^{(n)}}{\lambda^{(n)}}=0. \quad \begin{pmatrix} i=1,2,\ldots,p_n \\ n=1,2,\ldots\ldots \end{pmatrix}$$

we have

$$(\eta,\eta_i^{(n)})=(\lambda T_{\mathfrak{Q}}\eta,\eta_i^{(n)})=\lambda(\eta,T_{\mathfrak{Q}}\eta_i^{(n)})=0. \quad \begin{pmatrix} i=1,2,\ldots,p_n \\ n=1,2,\ldots\ldots \end{pmatrix}$$

That is, $\eta(E)$ is orthogonal to all the characteristic functions with respect to $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}, \ldots$. Therefore,

$$T_{\mathfrak{P}}\eta(E)=0.$$

Consequently,

$$T_{\mathfrak{Q}}\eta(E)=T_{\mathfrak{K}}\eta(E)-T_{\mathfrak{P}}\eta(E)=T_{\mathfrak{K}}\eta(E);$$

then by (7)

$$\eta(E)=\lambda T_{\mathfrak{K}}\eta(E).$$

Therefore, λ, being a characteristic constant of $\mathfrak{K}(E,E')$, must be equal to one of $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}, \ldots$. But $\eta(E)$ is orthogonal to all the characteristic functions with respect to $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}, \ldots \ldots$, which is absurd. Consequently, it must be that

$$\mathfrak{Q}(E,E')=0.$$

Now, we have the required results:

$$\mathfrak{K}(E,E')=\frac{1}{\lambda^{(1)}}\mathfrak{K}^{(1)}(E,E')+\frac{1}{\lambda^{(2)}}\mathfrak{K}^{(2)}(E,E') + \ldots + \frac{1}{\lambda^{(n)}}\mathfrak{K}^{(n)}(E,E')+\ldots, \tag{8}$$

and

$$\|\mathfrak{K}\|^2=\frac{p_1}{\lambda^{(1)2}}+\frac{p_2}{\lambda^{(2)2}}+\dots+\frac{p_n}{\lambda^{(n)2}}+\dots\dots$$

If we put as follows:

$$\begin{aligned}&\eta_1=\eta_1^{(1)}, \quad \eta_2=\eta_2^{(1)}, \quad \dots\dots\dots, \quad \eta_{p_1}=\eta_{p_1}^{(1)},\\ &\eta_{p_1+1}=\eta_1^{(2)}, \quad \eta_{p_1+2}=\eta_2^{(2)}, \quad \dots\dots\dots, \quad \eta_{p_1+p_2}=\eta_{p_1}^{(2)},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

and

$$\begin{aligned}&\lambda_1=\lambda_2=\dots\dots\dots=\lambda_{p_1}=\lambda^{(1)},\\ &\lambda_{p_1+1}=\lambda_{p_1+2}=\dots\dots\dots=\lambda_{p_1+p_2}=\lambda^{(2)},\\ &\dots\dots\dots\dots\dots\dots\dots\\ &\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

then (6) and (8) may be written as follows:

$$\mathfrak{K}(E,E')=\sum_{\nu}\frac{\eta_\nu(E)\,\eta_\nu(E')}{\lambda_\nu}$$

and

$$\|\mathfrak{K}\|^2=\sum_{\nu}\frac{1}{\lambda_\nu^2}.$$

Thus, *we can always expand the symmetric kernel with respect to the orthogonal system of characteristic functions.*[1]

Similarly, we can expand the iterated kernels of $\mathfrak{K}(E,E')$. Thus[2]

$$\begin{aligned}\mathfrak{K}_2(E,E')&=T_{\mathfrak{K}}\mathfrak{K}(E,E')\\ &=[\lim_{n\to\infty}]\,T_{\mathfrak{K}}\sum_{\nu=1}^{n}\frac{\eta_\nu(E)\,\eta_\nu(E')}{\lambda_\nu} \qquad \text{(by sec. 23, (3))}\\ &=[\lim_{n\to\infty}]\sum_{\nu=1}^{n}\frac{\eta_\nu(E')}{\lambda_\nu}\,T_{\mathfrak{K}}\eta_\nu(E)\\ &=[\lim_{n\to\infty}]\sum_{\nu=1}^{n}\frac{\eta_\nu(E')}{\lambda_\nu}\cdot\frac{\eta_\nu(E)}{\lambda_\nu}\\ &=\sum_{\nu=1}^{\infty}\frac{\eta_\nu(E)\,\eta_\nu(E')}{\lambda_\nu^2}.\end{aligned}$$

(1) When the symmetric kernel is a continuous point function, the expansion is possible when the expanded series is uniformly convergent.

(2) When $\mathfrak{K}(E,E')$ is expanded in the form (6), it is simple.

Generally

$$\mathfrak{K}_i(E,E')=\sum_{\nu=1}^{\infty}\frac{\eta_\nu(E)\,\eta_\nu(E')}{\lambda_\nu^{\,i}},$$

the convergence being strong.

27. Let $L_{\mathfrak{K}}$ denote the linear manifold whose fundamental system is composed of the characteristic functions

$$\eta_1(E),\ \eta_2(E),\ \ldots.,\ \eta_\nu(E),\ \ldots..$$

of the symmetric kernel $\mathfrak{K}(E,E')$.

Let $\phi(E)$ be any set function in $L_2(\beta)$. If we put

$$\mathfrak{S}_n(E,E')=\sum_{\nu=1}^{n}\frac{\eta_\nu(E)\,\eta_\nu(E')}{\lambda_\nu},$$

then by sec. 23, (4)

$$T_{\mathfrak{K}}\phi(E)=[\lim_{n\to\infty}]\,T_{\mathfrak{S}_n}\phi(E)=[\lim_{n\to\infty}]\sum_{\nu=1}^{n}\frac{(\phi,\eta_\nu)}{\lambda_\nu}\eta_\nu(E)$$

$$=\frac{(\phi,\eta_1)}{\lambda_1}\eta_1(E)+\frac{(\phi,\eta_2)}{\lambda_2}\eta_2(E)+\ldots.+\frac{(\phi,\eta_\nu)}{\lambda_\nu}\eta_\nu(E)+\ldots..$$

Therefore, $T_{\mathfrak{K}}\phi(E)$ is a function of $L_{\mathfrak{K}}$; that is, $T_{\mathfrak{K}}$ *transforms all set functions in* $L_2(\beta)$ *into set functions in* $L_{\mathfrak{K}}$.

If $\phi(E)$ is orthogonal to $L_{\mathfrak{K}}$, then

$$(\phi,\eta_\nu)=0. \qquad (\nu=1,2,\ldots.)$$

therefore,

$$T_{\mathfrak{K}}\phi(E)=0.$$

Conversely, if

$$T_{\mathfrak{K}}\phi(E)=0,$$

then

$$(\phi,\eta_\nu)=(\phi,\lambda_\nu T_{\mathfrak{K}}\eta_\nu)=\lambda_\nu(T_{\mathfrak{K}}\phi,\eta_\nu)=0,\quad (\nu=1,2,\ldots.)$$

that is, $\phi(E)$ is orthogonal to $L_{\mathfrak{K}}$.

Consequently, *the necessary and sufficient condition that a set function* $\phi(E)$ *in* $L_2(\beta)$ *is orthogonal to* $L_{\mathfrak{K}}$, *is that*

$$T_{\mathfrak{K}}\phi(E)=0.$$

Solution of Integral Equations with Real Symmetric Kernels.

28. Consider the linear integral equation of the *first kind*

$$\psi(E)=T_{\mathfrak{K}}\phi(E) \qquad (1)$$

with real symmetric kernel $\mathfrak{K}(E,E')$. Then, by the preceding section, it is necessary that the given function $\psi(E)$ is a set function in $L_{\mathfrak{K}}$. Hence $\psi(E)$ may be expressed as follows

$$\psi(E)=\sum_{\nu} c_{\nu}\eta_{\nu}(E),$$

$\sum_{\nu} c_{\nu}^2$ being convergent. Let

$$\phi(E)=\sum_{\nu} a_{\nu}\eta_{\nu}(E)+\phi'(E),$$

where $\phi'(E)$ is orthogonal to $L_{\mathfrak{K}}$, and $\sum_{\nu} a_{\nu}^2$ is convergent. Since

$$T_{\mathfrak{K}}\phi'(E)=0,$$

$\phi'(E)$ is arbitrary. We can determine the unknown coefficients a_1, a_2, ..., a_{ν}, ... by the following equality, which we have from (1)

$$\sum_{\nu} c_{\nu}\eta_{\nu}(E)=T_{\mathfrak{K}}\{\sum_{\nu} a_{\nu}\eta_{\nu}(E)\}$$

$$=\sum_{\nu}\frac{a_{\nu}}{\lambda_{\nu}}\eta_{\nu}(E).$$

Therefore, we haue

$$c_{\nu}=\frac{a_{\nu}}{\lambda_{\nu}} \qquad (\nu=1,2,\ldots.)$$

Consequently, if $\sum\lambda_{\nu}^2 c_{\nu}^2$ is convergent, the solutions of (1) are given by

$$\phi(E)=\sum_{\nu}\lambda_{\nu}c_{\nu}\eta_{\nu}(E)+\phi'(E),$$

$\phi'(E)$ being an arbitrary function orthogonal to $L_{\mathfrak{K}}$.

29. Next, consider the linear integral equation of the *second kind*

$$\phi(E)=\psi(E)+\lambda T_{\mathfrak{K}}\phi(E) \qquad (1)$$

with real symmetric kernel $\mathfrak{K}(E,E')$. Let the given function $\psi(E)$ and the unknown function $\phi(E)$ be

$$\psi(E)=\sum_{\nu} c_{\nu}\eta_{\nu}(E)+\psi'(E),$$

$$\phi(E)=\sum_{\nu} a_{\nu}\eta_{\nu}(E)+\phi'(E),$$

where $\psi'(E)$, $\phi'(E)$ are orthogonal to $L_{\mathfrak{K}}$, and $\sum_\nu c_\nu^2$, $\sum_\nu a_\nu^2$ are convergent. Since

$$T_{\mathfrak{K}}\phi'(E)=0,$$

we have from (1)

$$\sum_\nu a_\nu \eta_\nu(E)+\phi'(E)=\sum_\nu c_\nu \eta_\nu(E)+\psi'(E)+\lambda T_{\mathfrak{K}}\{\sum_\nu a_\nu \eta_\nu(E)\},$$

Then

$$\phi'(E)=\psi'(E),$$

and

$$\sum_\nu a_\nu \eta(E)=\sum_\nu c_\nu \eta_\nu(E)+\lambda \sum_\nu \frac{a_\nu}{\lambda_\nu}\eta_\nu(E).$$

Hence

$$a_\nu=c_\nu+\lambda\frac{a_\nu}{\lambda_\nu}. \qquad (\nu=1, 2, \ldots .) \qquad (2)$$

If λ is not a characteristic constant of $\mathfrak{K}(E, E')$, then from (2)

$$a_\nu=\frac{\lambda_\nu c_\nu}{\lambda_\nu-\lambda}.$$

Therefore, we have a unique solution

$$\phi(E)=\sum_\nu \frac{\lambda_\nu c_\nu}{\lambda_\nu-\lambda}\eta_\nu(E)+\psi'(E),$$

or

$$\phi(E)=\psi(E)+\lambda \sum_\nu \frac{c_\nu}{\lambda_\nu-\lambda}\eta_\nu(E), \qquad (3)$$

since

$$\frac{\lambda_\nu c_\nu}{\lambda_\nu-\lambda}=c_\nu+\frac{\lambda c_\nu}{\lambda_\nu-\lambda}.$$

Of course, the series (3) converges strongly, for $\left|\frac{1}{\lambda_\nu-\lambda}\right|<1$ when ν is sufficiently large.

If λ is a characteristic constant, and

$$\lambda=\lambda_i=\lambda_{i+1}=\ldots .=\lambda_{i+j},$$

then from (2), it must be that

$$c_\nu=0, \qquad (\nu=i, i+1, \ldots, i+j)$$

that is, the given function $\psi(E)$ must be orthogonal to all characteristic functions with respect to λ, and a_i, a_{i+1}, $\ldots$, a_{i+j} can be arbitrary. The solutions of (1) are given by

$$\phi(E) = \sum_\nu{}' \frac{\lambda_\nu c_\nu}{\lambda_\nu - \lambda} \eta_\nu(E) + \psi'(E) + \sum_{\nu=i}^{i+j} a_\nu \eta_\nu(E),$$

or

$$\phi(E) = \psi(E) + \lambda \sum_\nu{}' \frac{c_\nu}{\lambda_\nu - \lambda} \eta_\nu(E) + \sum_{\nu=i}^{i+j} a_\nu' \eta_\nu(E), \tag{4}$$

where $\sum_\nu{}'$ represents the summation with respect to all values of ν, except $\nu = i,\ i+1,\ \ldots,\ i+j$, and $a'_i\ a'_{i+1},\ \ldots,\ a'_{i+j}$ are arbitrary constants.

(3) and (4) correspond to Schmidt's solutions[1] of the ordinary integral equation.

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(1) E. Schmidt, *Math. Ann.* **63** (1907), 454.