Differential Set Functions.

By

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Let $f(x)$ and $g(x)$ be continuous functions in (a, b), moreover $g(x)$ be monotone increasing. If $\lim\sum\frac{|\Delta f|^2}{\Delta g}$ exists, we denoted it by $\int_a^b\frac{|df|^2}{dg}$. For $\int_a^b\frac{|df|^2}{dg}$ to exist, it is necessary and sufficient that $f(x)$ shall be the indefinite Lebesgue-Stieltjes integral of a function $f_g'(x)$ which is integrable with respect to $g(x)$ in (a, b) with its absolute square. And we have

$$(1)\qquad \int_a^b\frac{|df|^2}{dg}=\int_a^b|f_g'(x)|^2dg(x)\,.^{(1)}$$

Put $\beta(E)=g(d)-g(c)$, when E is an open interval (c, d), and $\beta(E)=0$, when E is a point. In like wise we define $\xi(E)$ for $f(x)$. Then $\beta(E)$ and $\xi(E)$ are differential set functions in (a, b). But in the above definition of the Hellinger integral, there are considered only finite decompositions of (a, b), but not arbitrary ones. That $\xi(E)$ is completely additive is at first sight not so obvious, but follows from the inequality $|\Delta f|^2\leqq\Delta g\Delta h$, where $h(x)=\int_a^x\frac{|df|^2}{dg}$. From this we see that $\xi(E)$ is extended to the completely additive set function. These circumstances lead us to specialize the decomposition system. We define the decomposition systems M and M^* in the abstract theory of differential set functions which correspond in the above example to finite decompositions and finite or infinite decompositions of (a, b) respectively. We introduce the concept of complete additivity in $M(M^*)$ in the obvious manner. It is natural, then, to ask whether any differential set function which is completely additive in M may be extended to the ordinary set function.

(1) H. Hahn, Monatshefte für Math. und Physik, **23** (1912), 172.
J. Radon, Sitzungsberichte Wien, **122**, 2a (1913), 120.

This is treated in Part I. The answer is affirmative in general. In Part II we investigate the space of the differential set functions studied by F. Maeda, in connection with Quantum Mechanics. Let $\beta(E)$ be a non negative differential set function in the abstract set Ω which is completely additive in M. Let $\mathfrak{L}_2(\beta)$ be the aggregate of differential set functions $\xi(E)$ which are completely additive in M and for which $\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)}$ exists and is finite. $\mathfrak{L}_2(\beta)$ may be considered as a Hilbert space. If $\beta(E)$ is completely additive in M^*, then so is any $\xi(E)\in\mathfrak{L}_2(\beta)$. In this case any $\xi(E)\in\mathfrak{L}_2(\beta)$ and $\beta(E)$ are extended to ordinary set functions, and bear a relation similar to (1). Thus $\mathfrak{L}_2(\beta)$ is nothing but the space of ordinary set functions developed also by him. In the general case put $\beta^*(E)=(M^*)\int_E \beta(dE)$, and $\beta^{(s)}(E)=\beta(E)-\beta^*(E)$; then $\beta^*(E)$ is completely additive in M^*. Any $\xi(E)\in\mathfrak{L}_2(\beta)$ which is completely additive in M^* belongs to $\mathfrak{L}_2(\beta^*)$, and conversely. Among $\mathfrak{L}_2(\beta)$, $\mathfrak{L}_2(\beta^*)$ and $\mathfrak{L}_2(\beta^{(s)})$ there holds good the relation $\mathfrak{L}_2(\beta)=\mathfrak{L}_2(\beta^*)\oplus\mathfrak{L}_2(\beta^{(s)})$. Any $\xi(E)\in\mathfrak{L}_2(\beta)$ is expressed uniquely in the form:

$$\xi(E)=\xi^{(r)}(E)+\xi^{(s)}(E), \qquad \text{where} \quad \xi^{(r)}(E)\in\mathfrak{L}_2(\beta^*) \text{ and } \xi^{(s)}(E)\in\mathfrak{L}_2(\beta^{(s)}).$$

And $\xi^{(r)}(E)$ is characterized by the equation $\xi^{(r)}(E)=(M^*)\int_E \xi(dE)$. We call $\xi^{(r)}(E)$, $\xi^{(s)}(E)$ the regular and singular parts of $\xi(E)$.

We take this opportunity of thanking Prof. F. Maeda for his kind guidance.

Definitions and Notations.

1. Let us recall briefly the definitions and notations used in the multiplicative system and differential set functions introduced by A. Kolmogoroff[1] and F. Maeda.[2]

Let Ω be an abstract set. A system of subsets of Ω, to which belongs EE' with subsets E and E', is called a multiplicative system and denoted by $\mathfrak{M}$.

Consider a decomposition $\mathfrak{D}$ of Ω into distinct sets E belonging to $\mathfrak{M}$, that is, $\sum_n E_n=\Omega$. Then it is expressed by

(1) A. Kolmogoroff, Math. Ann. **103** (1930), 654–696.
(2) F. Maeda, this journal, **6** (1936), 19–45.

$$\mathfrak{D}\Omega \equiv \sum_n E_n$$

E_n is called the element of the decomposition $\mathfrak{D}$. $\mathfrak{D}$ induces the decomposition of any set E belonging to $\mathfrak{M}$, that is, $\mathfrak{D}E \equiv \sum_n EE_n$.

Let $\mathfrak{D}'\Omega \equiv \sum_m E'_m$ be another decomposition of Ω. Then $\sum_{m,n} E_n E'_m$ is also a decomposition, which is called the product of $\mathfrak{D}$ and $\mathfrak{D}'$, and is denoted by $[\mathfrak{D}\mathfrak{D}']$.

If $[\mathfrak{D}\mathfrak{D}'] = \mathfrak{D}'$, then we say $\mathfrak{D}'$ is an extension of $\mathfrak{D}$, and write

$$\mathfrak{D}' > \mathfrak{D} .$$

2. Consider an aggregate of decompositions of Ω, to which belongs $[\mathfrak{D}\mathfrak{D}']$ with $\mathfrak{D}$ and $\mathfrak{D}'$. Denote it by M. If any decomposition of Ω into elements of decompositions belonging to M belongs to M, then we say M is complete. We can always make M complete by adding such decompositions to it, and denote such a complete M by M^*. Unless otherwise said, we regard a decomposition as belonging to M.

The system of elements of all extensions of $\mathfrak{D}$ is called a differential set system and denoted by $\mathfrak{M}\mathfrak{D}\Omega$. If $\mathfrak{D}$ is identical, that is, $\mathfrak{D}\Omega \equiv \Omega$, then we write $\mathfrak{M}\Omega$ instead of $\mathfrak{M}\mathfrak{D}\Omega$. We have the following relation:

$$(\mathfrak{M}\mathfrak{D}\Omega)(\mathfrak{M}\mathfrak{D}'\Omega) = \mathfrak{M}[\mathfrak{D}\mathfrak{D}']\Omega .^{(1)}$$

3. A one or many valued set function $\xi(E)$, defined for all sets E of a differential set system $\mathfrak{M}\mathfrak{D}_0\Omega$, is called a differential set function in Ω.

If $\xi(E)$ is one valued, and

(1) $\quad \xi(E) = \sum_n \xi(E_n) \quad$ for $\quad \mathfrak{D}E \equiv \sum E_n , \quad \mathfrak{D} \varepsilon M ,$

then we say $\xi(E)$ is completely additive in M.

If the above definition (1) holds good for any $\mathfrak{D} \varepsilon M^*$, then we say $\xi(E)$ is completely additive in M^*.

Let $\mathfrak{K}$ be a closed family of subsets of Ω, and E be its element. We denote by $E\mathfrak{K}$ the subsystem of $\mathfrak{K}$ which is composed of subsets of E belonging to $\mathfrak{K}$. The one valued set function $\eta(E)$ is said to be completely additive *in* $\mathfrak{K}$, if, when $\eta(E)$ is defined for E, then so it is for $E\mathfrak{K}$, and

(1) F. Maeda, loc. cit., 21,

$$\eta(E)=\sum_n \eta(E_n)\,,$$

where E_n are distinct from each other and $E=\sum_n E_n$.

If $\eta(E)$ is defined for Ω, we say $\eta(E)$ is completely additive *on* $\mathfrak{R}$.

4. Let $\mathfrak{H}$ be a Hilbert space, that is, a complete linear space with an inner product. To avoid confusion we write $\mathfrak{N}$, V, U instead of $\mathfrak{M}$, Ω, E respectively. If for all sets U in a differential set system $\mathfrak{N}\mathfrak{D}_0 V$, a vector $\mathfrak{q}(U)$ in $\mathfrak{H}$ is determined, then we say $\mathfrak{q}(U)$ is a vector valued differential set function. And $\mathfrak{q}(U)$ is said to be completely additive in $M(M^*)$, if

$$\big(\mathfrak{q}(U),\mathfrak{q}(U')\big)=0\,, \qquad \text{when} \quad UU'=0$$

and

$$\mathfrak{q}(U)[=]\sum_n \mathfrak{q}(U_n)\,,^{(1)} \qquad \text{when} \quad \mathfrak{D}U\equiv\sum_n U_n, \qquad \mathfrak{D}\,\varepsilon\, M(M^*)\,.$$

Put $\sigma(U)=\|\mathfrak{q}(U)\|^2$, then $\sigma(U)$ is completely additive in $M(M^*)$.

In the same way as in **3** we define a completely additive vector valued set function *in* or *on* $\mathfrak{R}$.

Let $E(U)$ be a projective operator defined for all sets U in $\mathfrak{N}V$, then we say that $E(U)$ is the resolution of the identity and completely additive in $M(M^*)$ if it satisfies the following conditions:

(α) $E(U)E(U')=E(UU')\,,$

(β) $E(U)=\sum_n E(U_n)\,,$ when $\mathfrak{D}U\equiv\sum U_n\,,$ $\mathfrak{D}\,\varepsilon\, M(M^*)\,,$

(γ) $E(V)=1\,,$ that is, the identical operator.

5. Let $\xi(E)$ be a one or many valued differential set function. Following A. Kolmogoroff we say: I is the integral of $\xi(E)$ on E, if for any $\varepsilon>0$, there exists a decomposition $\mathfrak{D}_\varepsilon$ such that for any $\mathfrak{D}\supset\mathfrak{D}_\varepsilon$, $\mathfrak{D}E\equiv\sum E_n$, we have

$$\Big|\sum_n \xi(E_n)-I\Big|<\varepsilon\,.$$

And we write

$$\int_E \xi(dE)=I\,,$$

or

$$(M^*)\int_E \xi(dE)=I\,,$$

according as $\mathfrak{D}_\varepsilon$, $\mathfrak{D}\,\varepsilon\, M$ or $\varepsilon\, M^*$.

(1) [=] means the strong convergence of the series.

If we consider $\mathfrak{D}$, $\mathfrak{D}_\varepsilon$ as decompositions of E into sets of $\mathfrak{K}$, then we write

$$(\mathfrak{K})\int_E \xi(dE)=I.$$

These definitions are extended to vector valued set functions if we understand that I is a vector in $\mathfrak{H}$.

Part I.

Extension of Completely Additive Differential Set Functions to Completely Additive Ordinary Set Functions.

§1. β-Measurable Sets and β-Measure.

6. Let $\beta(E)$ be a non negative differential set function which is completely additive in M. Let $\mathfrak{M}\mathfrak{D}_0\Omega$ be its corresponding differential set system.

Let A be any set in Ω. Consider a sequence of elements $\{A_n\}$, distinct or not, which cover A, that is,

$$A\subset \mathfrak{S}_n A_n\,,^{(1)} \qquad A_n \,\varepsilon\, \mathfrak{M}\mathfrak{D}_0\Omega\,.$$

Construct the sum $\sum_n \beta(A_n)$. Then the greatest lower bound of such sums for all covering sequences is called the *β-measure* of A and is denoted by $\beta^*(A)$. $\beta^*(A)$ has the properties of the measure function of Carathéodory ;[2]

1° $0\leqq\beta^*(A)\leqq+\infty$

2° $B\subset A$ implies $\beta^*(A)\geqq\beta^*(B)$

3° if $V=\sum_n V_n$, then $\beta^*(V)\leqq\sum_n\beta^*(V_n)$.

Now we define β-measurable sets. Let W be any set for which $\beta^*(W)<+\infty$. If

$$\beta^*(W)=\beta^*(WA)+\beta^*(WCA),$$

then A is said to be *measurable* with respect to β.

We show in the next section **7** that any set in $\mathfrak{M}\mathfrak{D}_0\Omega$ is measur-

(1) When A_n may not be distinct from each other, we write $\mathfrak{S}_n A_n$ instead of $\sum_n A_n$.

(2) C. Carathéodory, *Vorlesungen über reelle Funktionen*, (1927), 237–258.

able. From this and 1°, 2°, 3° the considerations in Carathéodory's book[1] lead us to the results;

(α) β-measurable sets form a closed family $\mathfrak{R}$ over the differential set system $\mathfrak{M}\mathfrak{D}_0\Omega$.

(β) β-measure is completely additive on $\mathfrak{R}$, if permitted the value $+\infty$.

7. We shall show that any set E in $\mathfrak{M}\mathfrak{D}_0\Omega$ is measurable. Let E be an element of $\mathfrak{D}$, $\mathfrak{D}\supset\mathfrak{D}_0$. Put

$$\mathfrak{D}\Omega\equiv E+\sum_m E_m .$$

Let W be any set in Ω for which $\beta^*(W)<+\infty$. Let $\{A_n\}$ be any covering sequence of elements of W. Then $\{EA_n\}$, $\{E_mA_n\}$ are covering sequences of WE and WCE respectively. Since

$$\sum_n \beta(A_n)=\sum_n \beta(EA_n)+\sum_{m,n}\beta(E_mA_n) ,$$

so we have

$$\sum_n \beta(A_n)\geqq\beta^*(WE)+\beta^*(WCE) .$$

This gives

$$\beta^*(W)\geqq\beta^*(WE)+\beta^*(WCE) .$$

By **6**, 3°, we have

$$\beta^*(W)=\beta^*(WE)+\beta^*(WCE) .$$

Hence E is measurable with respect to β.

8. For any set E in $\mathfrak{M}\mathfrak{D}_0\Omega$, there holds good the relation

$$(1)\qquad \beta^*(E)=\inf_{\mathfrak{D}^*\epsilon M^*}\sum_n\beta(E_n)=(M^*)\int_E\beta(dE) ,\qquad \mathfrak{D}^*E\equiv\sum_n E_n .$$

For, in fact, let $\mathfrak{D}^*$ be any decomposition belonging to M^*, and $\mathfrak{D}^*E\equiv\sum E_n$. Let E_m be an element of $\mathfrak{D}_m$. For integers $m\leqq n$, we put

$$[\mathfrak{D}_1\mathfrak{D}_2\ldots\mathfrak{D}_n]E_m\equiv\sum_p E_{mp} ,$$

then $[\mathfrak{D}_1\mathfrak{D}_2\ldots\mathfrak{D}_n]E$ is written in the form

$$[\mathfrak{D}_1\mathfrak{D}_2\ldots\mathfrak{D}_n]E\equiv\sum_{m=1}^{n}\sum_p E_{mp}+\sum_q E'_q .$$

This gives

$$\beta(E)\geqq\sum_{m=1}^{n}\sum_p\beta(E_{mp})=\sum_{m=1}^{n}\beta(E_m) .$$

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(1) C. Carathéodory, loc. cit.

Passing to the limit, we have

$$(2)\qquad \beta(E)\geqq\sum_n\beta(E_n).$$

Therefore $\sum_n\beta(E_n)$, $\mathfrak{D}^*E\equiv\sum E_n$, is a monotone decreasing function of $\mathfrak{D}^*\varepsilon M^*$, so we have

$$\inf_{\mathfrak{D}^*\varepsilon M^*}\sum_n\beta(E_n)=(M^*)\int_E\beta(dE)\,,\qquad \text{where } \mathfrak{D}^*E\equiv\sum E_n\,.$$

Next let $\{A_n\}$ be a covering sequence of E. Suppose A_n is an element of $\mathfrak{D}_n$, and E an element of $\mathfrak{D}$. Put

$$[\mathfrak{D}\mathfrak{D}_1\ldots\mathfrak{D}_n]A_n\equiv\sum_m A'_{nm}+\sum_m A''_{nm}$$

where A'_{nm} are all elements of $[\mathfrak{D}\mathfrak{D}_1\ldots\mathfrak{D}_n]A_n$ such that $A'_{nm}\subset E$ but $\not\subset\mathfrak{S}_{p=1}^{n-1}A_p$. Then

$$E=\sum_{mn}A'_{mn}\,.$$

From this it follows that $\{A'_{mn}\}$ is a covering sequence of E and

$$\sum_n\beta_n(A_n)\geqq\sum\beta(A'_{mn})\,.$$

This gives

$$\beta^*(E)=\inf_{\mathfrak{D}^*\varepsilon M^*}\sum_n\beta(E_n)\,.\qquad \text{where}\quad \mathfrak{D}^*E\equiv\sum E_n\,.$$

Therefore we have (1).

We remark that from (2)

$$\beta^*(E)\leqq\beta(E)\,.$$

The equal sign holds good when, and only when, $\beta(E)$ is completely additive in M^*. From this results the following theorem:

The condition that a non negative differential set function $\beta(E)$ which is completely additive in M should be extended to a completely additive ordinary set function is that $\beta(E)$ shall be completely additive in M^*.[(1)]

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(1) This condition is not trivial, as a simple example shows: Let Ω be an open interval (0, 1) and $\mathfrak{M}$ be a aggregate of open intervals and points in (0, 1). M consists of finite decompositions of (0, 1). Put

$$\beta(E)=\begin{cases}b-a,\ \text{or}\ b-a+1,\ \text{when } E \text{ is an open interval } (a,b) \text{ according as } b<1 \text{ or } b=1.\\ 0,\ \text{when } E \text{ is a point.}\end{cases}$$

then $\beta(E)$ is not completely additive in M^* and $\beta(\Omega)>\beta^*(\Omega)$.

9. Let A be any set in Ω. Then for any $\varepsilon > 0$ we can find a sequence of elements $\{A_n\}$ covering A such that

$$\beta^*(A) \leqq \sum_n \beta(A_n) \leqq \beta^*(A) + \varepsilon .$$

From this we see that there exists a set in the Borel family over $\mathfrak{M}\mathfrak{D}_0\Omega$ which has the same β-measure as A, and A is β-measurable when, and only when, A differs from a set in the Borel family over $\mathfrak{M}\mathfrak{D}_0\Omega$ a β-null set.

Moreover we can prove without much difficulty that any extension of $\beta(E)$, if possible, is uniquely determined for β-measurable sets.

Let $\gamma(E)$ be another non negative differential set function which is completely additive in M. If the β-null set is always a γ-null set, that is, $\gamma^*(E)$ is absolutely continuous with respect to $\beta^*(E)$, then β-measurable sets are γ-measurable.

§2. Completely Additive Differential Set Functions.

10. Let $\phi(E)$ be a differential set function which is completely additive in M, and $\mathfrak{M}\mathfrak{D}_0\Omega$ be its corresponding differential set system. We define the *total variation* of $\phi(E)$ on E as follows:

$$\sup_{\mathfrak{D} \epsilon M} \sum_n |\phi(E_n)| , \qquad \text{where} \quad \mathfrak{D}E \equiv \sum E_n .$$

And we denote it by $v_\phi(E)$.

Since $\sum_n |\phi(E_n)|$, $\mathfrak{D}E \equiv \sum_n E_n$, is a monotone increasing function of $\mathfrak{D}$, we can write

$$v_\phi(E) = \int_E |\phi(dE)| ,$$

if $v_\phi(E)$ is finite.

First we shall show that $v_\phi(E)$ is completely additive in M. For $\mathfrak{D}' \supset \mathfrak{D}$, put $\mathfrak{D}'E_n \equiv \sum_m E_{nm}$. Then we have

$$\sum_{nm} |\phi(E_{nm})| \leqq \sum_n v_\phi(E_n) .$$

This gives

$$v_\phi(E) \leqq \sum_n v_\phi(E_n) .$$

If $v_\phi(E) = +\infty$, then the equal sign obviously holds good. Let $v_\phi(E)$ be finite, then so is $v_\phi(E_n)$ from the definition of v_ϕ. For any $\varepsilon_n > 0$, we can find a $\mathfrak{D}_n$, $\mathfrak{D}_n E_n \equiv \sum_m E_{nm}$ such that

$$\sum_m |\phi(E_{nm})| + \varepsilon_n \geqq v_\phi(E_n).$$

Then if we put

$$[\mathfrak{D}_1\mathfrak{D}_2 \dots \mathfrak{D}_n]E \equiv \sum_i E'_i$$

we have

$$\sum_{p=1}^{n} v_\phi(E_p) \leqq \sum_i |\phi(E'_i)| + \sum_{p=1}^{n} \varepsilon_p$$

$$\leqq v_\phi(E) + \sum_{p=1}^{n} \varepsilon_p .$$

This gives

$$\sum_n v_\phi(E_n) \leqq v_\phi(E).$$

Therefore we have

$$\sum_n v_\phi(E_n) = v_\phi(E), \qquad \text{where} \quad \mathfrak{D}E \equiv \sum E_n, \mathfrak{D}\varepsilon M.$$

Next we shall prove that

$$v_\phi(E) = \sup_{\mathfrak{D}^* \varepsilon M^*} \sum_n |\phi(E_n)|, \qquad \text{where} \quad \mathfrak{D}^* E \equiv \sum_n E_n .$$

For, in fact, from the definition of $v_\phi(E)$, we have

$$v_\phi(E) \leqq \sup_{\mathfrak{D}^* \varepsilon M^*} \sum_n |\phi(E_n)|.$$

On the other hand, by **8** (2) we have

$$\sum_n |\phi(E_n)| \leqq \sum_n v_\phi(E_n), \qquad \mathfrak{D}^* E \equiv \sum E_n, \mathfrak{D}^* \varepsilon M^*,$$

$$\leqq v_\phi(E),$$

hence

$$v_\phi(E) \geqq \sup_{\mathfrak{D}^* \varepsilon M} \sum_n |\phi(E_n)|.$$

Therefore we have

$$v_\phi(E) = \sup_{\mathfrak{D}^* \varepsilon M} \sum_n |\phi(E_n)|, \qquad \mathfrak{D}^* E \equiv \sum_n E_n .$$

From this it follows that if $\phi(E)$ is completely additive in M^*, then so is $v_\phi(E)$, provided that the value $+\infty$ is permitted. Conversely, if $v_\phi(E)$ is finite and completely additive in M^*, then also $\phi(E)$ is completely additive in M^*. The proof is easy, so we omit it.

If $\phi(E)$ is completely additive in M^* and $v_\phi(E)$ is finite on E, then we can write

$$v_\phi(E) = (M^*)\int_E |\phi(dE)|.$$

11. Let $\phi(E)$ be a differential set function which is completely additive in M. We shall solve the question: *When can $\phi(E)$ be extended to a completely additive ordinary set function?* In order that $\phi(E)$ may be extended to such an ordinary set function it is necessary that $\phi(E)$ shall be completely additive in M^* and $v_\phi(E)$ be finite for any set in $\mathfrak{M}\mathfrak{D}_0\Omega$. In the following sections we shall show that this condition is sufficient.[1]

12. Suppose that $\phi(E)$ is completely additive in M^* and $v_\phi(E)$ is finite in any set in $\mathfrak{M}\mathfrak{D}_0\Omega$.

Let A be any set in Ω. If A is v_ϕ-measurable, then we say that A is *ϕ-measurable*. We assume that A is ϕ-measurable and $v_\phi^*(A) < +\infty$.

For any $\varepsilon > 0$, we take any covering sequence of elements $\{A_n\}$ of A such that

$$(1) \qquad v_\phi^*(A) \leqq \sum_n v_\phi(A_n) \leqq v_\phi^*(A) + \varepsilon .$$

Let $\{\bar{A}_m\}$ be another such sequence of elements corresponding to $\bar{\varepsilon}$:

$$(2) \qquad v_\phi^*(A) \leqq \sum_m v_\phi(\bar{A}_m) \leqq v_\phi^*(A) + \bar{\varepsilon} .$$

Let A_n be an element of $\mathfrak{D}_n$, and put

$$[\mathfrak{D}_1\mathfrak{D}_2 \dots \mathfrak{D}_n] A_n \equiv \sum_m A'_{nm} + \sum_m A''_{nm} ,$$

where A'_{nm} are all elements of $[\mathfrak{D}_1\mathfrak{D}_2 \dots \mathfrak{D}_n] A_n$ such that

$$A'_{nm} \not< \mathop{\mathfrak{S}}_{p=1}^{n-1} A_p .$$

Since

$$\sum_{nm} A'_{nm} > A ,$$

so we have

$$\sum_{nm} v_\phi(A'_{nm}) = v_\phi^*(\sum_{nm} A'_{nm}) \geqq v_\phi^*(A) .$$

This gives

$$\sum_{nm} v_\phi(A''_{nm}) \leqq \varepsilon .$$

Therefore we have

$$(3) \qquad |\sum_n \phi(A_n) - \sum_{nm} \phi(A'_{nm})| \leqq \varepsilon .$$

First we assume that A_n are distinct from each other and that the same is true of $\bar{A}_m$. Then we have

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(1) That these conditions are not trivial follows from an example given by A. Kolmogoroff. Cf. A. Kolmogoroff, loc. cit., 673.

$$\begin{aligned}|\sum_n \phi(A_n)-\sum_m \phi(\bar{A}_m)| &\leqq |\sum_n \phi(A_n)-\sum_{n,m}\phi(A_n\bar{A}_m)| \\ &\quad +|\sum_m \phi(\bar{A}_m)-\sum_{n,m}\phi(A_n\bar{A}_m)| \\ &\leqq v_\phi^*(\sum_n A_n-\sum_m \bar{A}_m)+v_\phi^*(\sum_m \bar{A}_m-\sum_n A_n) \\ &\leqq \varepsilon+\bar{\varepsilon}.\end{aligned}$$

Hence for general A_n and $\bar{A}_m$ we have

$$(4)\qquad |\sum_n \phi(A_n)-\sum_m \phi(\bar{A}_m)| \leqq 2(\varepsilon+\bar{\varepsilon}).$$

If we tend ε to zero, then $\sum_n \phi(A_n)$ converges to some fixed number which we denote by $\phi^*(A)$. Then (4) gives

$$(5)\qquad |\sum_n \phi(A_n)-\phi^*(A)\| \leqq 2\varepsilon.$$

For any set E in $\mathfrak{M}\mathfrak{D}_0\Omega$, take $A_1=E$, and $A_n=0$ for $n\geqq 2$, then we have

$$\phi^*(E)=\phi(E).$$

Let $\mathfrak{K}$ be the system of ϕ-measurable sets. Then $\mathfrak{K}\supset\mathfrak{M}\mathfrak{D}_0\Omega$, and if $\phi(E)$ is defined for E, then so it is for $E\mathfrak{K}$. Hence, if we succeed in proving the complete additivity of $\phi^*(A)$, then $\phi^*(A)$ is such an extension of $\phi(E)$ as we desire. We shall prove this in the next section.

13. Consider a sequence of distinct measurable sets $\{V_n\}$ such that $V=\sum_n V_n$ and $v_\phi^*(V)<+\infty$. We shall prove

$$(1)\qquad \phi^*(V)=\sum_n \phi^*(V_n).$$

Let $\{A_{nm}\}$ be a sequence of elements covering V_n such that for a given $\varepsilon_n>0$

$$(2)\qquad v_\phi^*(V_n)\leqq\sum_m v_\phi(A_{nm})\leqq v_\phi^*(V_n)+\varepsilon_n.$$

From (2) we have

$$(3)\qquad v_\phi^*(V)\leqq\sum_{nm} v_\phi(A_{nm})\leqq v_\phi^*(V)+\sum_n \varepsilon_n.$$

From **12** (1), (5) and **13** (2), (3) it follows that

$$|\sum_m \phi(A_{nm})-\phi^*(V_n)|\leqq 2\varepsilon_n$$

and

$$|\sum_{mn} \phi(A_{nm})-\phi^*(V)|\leqq 2\sum_n \varepsilon_n.$$

These two inequalities give

$$|\phi^*(V)-\sum_n \phi^*(V_n)| \leqq 4\sum_n \varepsilon_n .$$

Since $\sum_n \varepsilon_n$ may be taken as small as we please, we thus have (1). Hence we have the following theorem:

A differential set function $\phi(E)$ which is completely additive in M can be extended to a completely additive ordinary set function when, and only when, $\phi(E)$ is completely additive in M^* and has finite total variation.

14. Now consider the total variation of $\phi^*(A)$, which is expressed by

$$(\mathfrak{R})\int_A |\phi^*(dE)|,$$

where A is ϕ-measurable and $v_\phi^*(A) < +\infty$.

From the definition of $\phi^*(A)$ we have

$$|\phi^*(A)| \leqq v_\phi^*(A) .$$

Hence

$$(\mathfrak{R})\int_A |\phi^*(dE)| \leqq v_\phi^*(A) .$$

But on any set E in $\mathfrak{M}\mathfrak{D}_0\Omega$ we have

$$(\mathfrak{R})\int_E |\phi^*(dE)| \geqq v_\phi(E) .$$

From this we have

$$(\mathfrak{R})\int_E |\phi^*(dE)| = v_\phi(E) .$$

Since by **9** any extension of $v_\phi(E)$ is uniquely determined on ϕ-measurable sets, so we have

$$(\mathfrak{R})\int_A |\phi^*(dE)| = v^*(A) .$$

§3. Vector Valued Differential Set Functions and the Resolution of the Identity.

15. Let $\mathfrak{q}(U)$ be a vector valued differential set function defined for all sets U in $\mathfrak{N}\mathfrak{D}_0 V$, which is completely additive in M. In order that $\mathfrak{q}(U)$ may be extended to an ordinary vector valued set function it is necessary that $\mathfrak{q}(U)$ shall be completely additive in M^*. We shall show that this condition is also sufficient.

Let $\mathfrak{q}(U)$ be completely additive in M^*. Then $\sigma(U)=\|\mathfrak{q}(U)\|^2$ is non negative and completely additive in M^*. By the theorem in **8** we can extend $\sigma(U)$ to the ordinary set function $\sigma^*(U)$ which is completely additive in a closed family $\mathfrak{K}$ over $\mathfrak{N}\mathfrak{D}_0V$.

Let A be a σ-measurable set and $\sigma^*(A)<+\infty$. Then for any $\varepsilon>0$ take any sequence of elements $\{A_n\}$ covering A such that

$$(1) \qquad \sigma^*(A)\leqq\sum_n\sigma(A_n)\leqq\sigma^*(A)+\varepsilon\,.$$

Let $\{\bar{A}_m\}$ be another such sequence of elements covering A corresponding to $\bar{\varepsilon}$, that is,

$$(2) \qquad \sigma^*(A)\leqq\sum_m\sigma(\bar{A}_m)\leqq\sigma^*(A)+\bar{\varepsilon}\,.$$

Then we have

$$\begin{aligned}(3) \qquad &\|\sum_n\mathfrak{q}(A_n)-\sum_m\mathfrak{q}(\bar{A}_m)\|^2\\ &=\sum_n\sigma(A_n)+\sum_m\sigma(\bar{A}_m)-2\sum_{mn}\sigma(A_nA_m)\\ &\leqq\{\sum_n\sigma(A_n)-\sigma^*(A)\}+\{\sum_m\sigma(\bar{A}_m)-\sigma^*(A)\}\\ &\leqq\varepsilon+\bar{\varepsilon}\,.\end{aligned}$$

If we tend ε to zero, then $\sum_n\mathfrak{q}(A_n)$ converges to a certain vector in the Hilbert space $\mathfrak{H}$, which we denote by $\mathfrak{q}^*(A)$. From this follows

$$\mathfrak{q}^*(U)=\mathfrak{q}(U) \qquad \text{for} \quad U \text{ in } \mathfrak{N}\mathfrak{D}_0V\,.$$

Passing to the limit in (3), we have

$$(4) \qquad \|\sum_n\mathfrak{q}(A_n)-\mathfrak{q}^*(A)\|^2\leqq\varepsilon\,.$$

Let B be another σ-measurable set and $\sigma^*(B)<+\infty$. Let $\{B_m\}$ be a sequence of elements such that

$$\sigma^*(B)\leqq\sum_m\sigma(B_m)\leqq\sigma^*(B)+\varepsilon$$

holds good, ε being a positive number, as small as desired. And we suppose that A_n are district from each other and so are B_m. Then we have

$$(5)\qquad \Big|\Big(\sum_n \mathfrak{q}(A_n),\ \sum_m \mathfrak{q}(B_m)\Big)-\sigma^*(AB)\Big|$$
$$=\sigma^*(\sum_{nm} A_nB_m)-\sigma^*(AB)$$
$$\leqq \sigma^*(\sum_n A_n-A)+\sigma^*(\sum_m B_m-B)\leqq 2\varepsilon\,.$$

Hence we have

$$\big(\mathfrak{q}^*(A),\ \mathfrak{q}^*(B)\big)=\sigma^*(AB)\,.$$

In the same way as in **13** we can derive the complete additivity of $\mathfrak{q}^*(A)$. Thus $\mathfrak{q}(U)$ is extended to the ordinary vector valued set function.[1]

16. Next we shall show that any extension of $\mathfrak{q}(U)$ is uniquely determined on σ-measurable sets.

Let $\bar{\mathfrak{q}}(A)$ be another extension of $\mathfrak{q}(U)$, and A be σ-measurable and $\sigma^*(A)<+\infty$. Then $\|\bar{\mathfrak{q}}(A)\|^2$ is an extension of $\sigma(U)$, hence, by **9**, we have

$$\|\bar{\mathfrak{q}}(A)\|^2=\sigma^*(A)\,.$$

Then for any distinct sequence $\{A_n\}$ in **15** (1) we have

$$\|\bar{\mathfrak{q}}(A)-\sum_n \mathfrak{q}(A_n)\|^2=\|\bar{\mathfrak{q}}(A)-\sum_n \bar{\mathfrak{q}}(A_n)\|^2$$
$$=\sigma^*(\sum_n A_n-A)\leqq\varepsilon\,.$$

This gives

$$\bar{\mathfrak{q}}(A)=\mathfrak{q}^*(A)\,.$$

17. Let $E(U)$ be the resolution of the identity which is completely additive in M^*.[2]

In a similar manner to that used in my previous paper,[3] we can extend $E(U)$ to the ordinary resolution of the identity defined for all sets in the closed family over $\mathfrak{R}V$, since $E(U)\mathfrak{f}$, $\mathfrak{f}$ being a vector in $\mathfrak{H}$, is a vector valued differential set function which is completely additive in M^*. I shall give later another method of extension.

§4. Integrals of Stieltjes Type.

18. Let $\beta(E)$ be a non negative differential set function which is completely additive in M.

(1) We shall give another method of extension $\mathfrak{q}(U)$ by the aid of the space of differential set functions. Cf. Part II §3.

(2) Cf. **4.**

(3) T. Ogasawara, this journal, **5** (1935), 117–130.

Let $f(\lambda)$ be any finite complex valued point function defined in Ω. Put

$$\xi(E)=f(\lambda)\beta(E), \qquad \lambda \varepsilon E.$$

Then $\xi(E)$ is a many valued differential set function. If $\xi(E)$ is integrable on E,[1] then we say $f(\lambda)$ *is integrable* with respect to $\beta(E)$ and denote the integral by $\int_E f(\lambda)\beta(dE)$ or $(M^*)\int_E f(\lambda)\beta(dE)$, according as we confine ourselves to M or M^*. For $\xi(E)$ to be integrable on E, it is necessary and sufficient that for any $\varepsilon>0$ there shall exist a $\mathfrak{D}_\varepsilon$ such that for any $\mathfrak{D}.\ \mathfrak{D}'\supset\mathfrak{D}_\varepsilon$, we have

(1) $\sum_n \xi(E_n)$ absolutely converges,

and (2) $|\sum_n \xi(E_n)-\sum_n \xi(E'_n)|<\varepsilon$, where $\mathfrak{D}E\equiv\sum_n E_n$, $\mathfrak{D}'E\equiv\sum_n E'_n$.

These conditions are equivalent to the following:

(3) there shall exist a $\mathfrak{D}$, $\mathfrak{D}E\equiv\sum E_n$ such that

$$\sum_n \sup_{\lambda\varepsilon E_n}|f(\lambda)|\beta(E_n)<+\infty \text{ }^{(2)}$$

(4) for any $\varepsilon>0$ there shall exist a $\mathfrak{D}$, $\mathfrak{D}E\equiv\sum E_n$ such that

$$\sum_n \operatorname{Osc} f(E_n)\beta(E_n)<\varepsilon^{(2)}$$

where Osc $f(E_n)$ means the oscillation of $f(\lambda)$ on E_n.

The proof is easy, so we omit it.

If $f(\lambda)$ is integrable on Ω, then since $\beta(E)\geqq\beta^*(E)$, it follows from (3), (4) that $f(\lambda)$ is integrable with respect to $\beta^*(E)$ and $(\mathfrak{K})\int_\Omega f(\lambda)\beta^*(dE)$ exists,[3] where $\mathfrak{K}$ is the system of β-measurable sets. Hence if $f(\lambda)$ is integrable, then $f(\lambda)$ is β-measurable.[3]

19. The integrals have the following properties:

(1) If $f(\lambda)$ integrable, then so is $|f(\lambda)|$, but not conversely.

(2) If $\int_E f(\lambda)\beta(dE)$ exists, then so does $\int_{E'} f(\lambda)\beta(dE)$ for $E'\subset E$, and

$$\int_E f(\lambda)\beta(dE)=\sum_n\int_{E_n} f(\lambda)\beta(dE), \qquad \mathfrak{D}E\equiv\sum E_n,\ \mathfrak{D}\varepsilon M.$$

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(1) Cf. **5**. E is allowable to be Ω.

(2) Here we regard $\infty\times\beta(E_n)=0$ when $\beta(E_n)=0$.

(3) T. Ogasawara, this journal, **6** (1936), 47-54.

(3) Put $\phi(E)=\int_E f(\lambda)\beta(dE)$, if $\int_E f(\lambda)\beta(dE)$ exists. Then we have

$$v_\phi(E)=\int_E |f(\lambda)|\beta(dE),$$

$$v_\phi^*(E)=\int_E |f(\lambda)|\beta^*(dE),^{(1)}$$

and

$$(M^*)\int_E \phi(dE)=\int_E f(\lambda)\beta^*(dE),$$

Thus, if $\phi(E)$ is completely additive in M^*, then we have

$$\phi(E)=\int_E f(\lambda)\beta^*(dE).$$

(4) If $\int_E f(\lambda)\beta(dE)$ exists, then so does $(M^*)\int_E f(\lambda)\beta(E)$, but not conversely.

(5) If $(M^*)\int_E f(\lambda)\beta(dE)$ exists, then so does $(M^*)\int_{E'} f(\lambda)\beta(dE)$ for $E'\subset E$, and

$$(M^*)\int_E f(\lambda)\beta(dE)=\sum_n (M^*)\int_{E_n} f(\lambda)\beta(dE).$$

(6) Put $\Psi(E)=(M^*)\int_E f(\lambda)\beta(dE)$, if $(M^*)\int_E f(\lambda)\beta(dE)$ exists. Then we have

$$v_\psi(E)=(M^*)\int_E |f(\lambda)|\beta(dE)=(M^*)\int_E |f(\lambda)|\beta^*(dE),$$

therefore

$$\Psi(E)=(M^*)\int_E f(\lambda)\beta^*(dE).$$

(7) If $\beta(E)$ is completely additive in M^* and $\int_E f(\lambda)\beta(dE)$ exists, then

$$\int_E f(\lambda)\beta(dE)=(M^*)\int_E f(\lambda)\beta(dE)=(\mathfrak{R})\int_E f(\lambda)\beta(dE).$$

The proof is not very difficult, so we pass on.

20. Let $\mathfrak{M}_1$ be a multiplicative system of open intervals and points in one-dimensional Euclidean space $\mathfrak{R}_1$. M consists of all decompositions of $\mathfrak{R}_1$ into open intervals and points. Then from **18** (4) it follows that

(1) Here $v_\phi^*(E)$ means the v_ϕ-measure of E or $(M^*)\int_E v_\phi(dE)$.

(1) the set of discontinuous points of $f(\lambda)$, except possibly on the discontinuous points of $\beta(E)$, is a β-null set.(1)

Then this condition and **18** (3) are necessary and sufficient for $f(\lambda)$ to be integrable. The proof is as in the ordinary Riemann integral.(2)

21. Lec $\phi(E)$ be a differential set function defined for all sets E in $\mathfrak{M}\mathfrak{D}_0\Omega$, which is completely additive in M and for which $v_\phi(E) < +\infty$. We say: $f(\lambda)$ is absolutely integrable with respect to $\phi(E)$, if it is integrable with respect to v_ϕ.

If this is the case, then $\xi(E) = f(\lambda)\phi(E)$, $\lambda \varepsilon E$, will be integrable, as easily seen. And the integral is denoted by $\int_E f(\lambda)\phi(dE)$ or $(M^*)\int_E f(\lambda)\phi(dE)$, according as we confine ourselves to M or M^*.

Then the integrals have analogous properties as in **19** and **20.**

22. Here we introduce the concept of the absolute continuity of differential set functions.

Let $\beta(E)$ be a non negative differential set function which is completely additive in M^*. We say: A differential set function $\phi(E)$, which is completely additive in M, is *absolutely continuous* with respect to $\beta(E)$ if, for any $\varepsilon > 0$, there corresponds to each element E, for which $\beta(E)$ and $\phi(E)$ are defined, a positive number η such that for any sequence of distinct elements $\{A_n\}$ in E the condition

$$\sum_n \beta(A_n) < \eta$$

implies

$$\sum_n |\phi(A_n)| < \varepsilon .$$

Let $\phi(E)$ be absolutely continuous with respect to $\beta(E)$. Then we shall show that $\phi(E)$ is completely additive in M^* and has finite total variation, hence there corresponds to $\phi(E)$ a point function $\phi(\lambda)$ such that

$$\phi(E) = (\mathfrak{R})\int_E \varphi(\lambda)\beta^*(dE) .$$

(1) This corresponds to the fundamental theorem in the Rieman integral: A bounded function $f(\lambda)$ is integrable on a finite interval when, and only when, the set of the discontinuous points of $f(\lambda)$ in this interval is Lebesgue measure zero.

(2) Cf. H. Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, 21–26.

(3) In this definition we may take $|\sum_n \phi(A_n)| < \varepsilon$ instead of $\sum_n |\phi(A_n)| < \varepsilon$ and A_n finite in numher,

For, in fact, let $\mathfrak{D}^*E\equiv\sum_n E_n$, $\mathfrak{D}^*\varepsilon M$ and E_n be an element of $\mathfrak{D}_n\varepsilon M$. In the same way as in **8**, we can write

$$[\mathfrak{D}_1\ldots.\mathfrak{D}_n]E_m\equiv\sum_p E_{mp},\ m\leqq n.$$

$$[\mathfrak{D}_1\ldots.\mathfrak{D}_n]E\equiv\sum_{m=1}^{n}\sum_p E_{mp}+\sum_q E'_q.$$

Hence

$$\phi(E)-\sum_{m=1}^{n}\phi(E_m)=\sum_q\phi(E'_q),$$

But

$$\sum_q\beta(E'_q)=\sum_{m>n}\beta(E_m).$$

Take n so large that

$$\sum_{m>n}\beta(E_m)<\eta$$

then

$$|\phi(E)-\sum_{m=1}^{n}\phi(E)|=|\sum_q\phi(E'_q)|<\varepsilon.$$

This gives

$$\phi(E)=\sum_n\phi(E_n).$$

Therefore $\phi(E)$ is completely additive in M^*.

Next we shall show that $v_\phi(E)<+\infty$.

Consider any decomposition $\mathfrak{D}\varepsilon M^*$, $\mathfrak{D}E\equiv E_n$, then the number of elements for which $\beta(E_n)\geqq\eta$ is less than $\frac{\beta(E)}{\eta}$. Hence there exists a decomposition for which the number of such elements is maximum. Let $\mathfrak{D}$, $\mathfrak{D}E\equiv\sum_n E_n$ be such a decomposition and $E_1, E_2,\ldots.E_m$, be such elements. Take p so large that $\sum_{n>p}\beta(E_n)<\eta$. Then for any $\mathfrak{D}'>\mathfrak{D}$ we put $\mathfrak{D}'E_n\equiv\sum_q E_{mq}$. We have

$$\sum_{n>m}\sum_q|\phi(E_{nq})|\leqq(p-m+1)\varepsilon.$$

In $\mathfrak{D}'E_n$, $n\leqq m$ there exists at most one element for which $\beta(E_{nq})\geqq\eta$, so we have for $n\leqq m$

$$\sum_q|\phi(E_{nq})|\leqq\phi(E_n)|+\frac{4\varepsilon}{\eta}\beta(E_n).$$

Hence

$$\sum_{nq}|\phi(E_{nq})|\leqq\frac{4\varepsilon}{\eta}\beta(E)+(p-m+1)\varepsilon+\sum_n|\phi(E_n)|,$$

therefore $v_\phi(E)<+\infty$.

And from these it follows that for any $\varepsilon > 0$ there corresponds to each element E an $\eta > 0$ such that for any sequence of distinct elements $\{A_n\}$ in E

$$\sum_n \beta(A_n) < \eta$$

implies

$$\sum_n v_\phi(A_n) < \varepsilon .$$

Consequently a β-null set is a ϕ-null set. Therefore there exists a $\phi(\lambda)$ such that

$$\phi(E) = (\mathfrak{R})\int_E \phi(\lambda)\beta^*(dE)^{(1)}$$

where $\mathfrak{R}$ is the Borel family over $\mathfrak{M}\mathfrak{D}_0\Omega$.

23. Consider a vector valued differential set function $\mathfrak{q}(U)$ which is completely additive in M. Let $f(\lambda)$ be a complex valued point function defined in Ω.

Put

$$\xi(U) = f(\lambda)\mathfrak{q}(U), \; \lambda \varepsilon U, \qquad U \varepsilon \mathfrak{N}\mathfrak{D}_0 V .$$

We say: $f(\lambda)$ *is integrable* with respect to $\mathfrak{q}(U)$, if $\xi(U)$ is integrable. For $f(\lambda)$ to be integrable on U,[2] it is necessary and sufficient that the following conditions shall be satisfied.

(1) There shall exist a decomposition $\mathfrak{D}$, $\mathfrak{D}U \equiv \sum_n U_n$ such that

$$\sum_n \sup_{\lambda \varepsilon E_n} |f(\lambda)|^2 \sigma(U_n) < +\infty .^{(3)}$$

(2) For any $\varepsilon > 0$, there shall exist a $\mathfrak{D}$, $\mathfrak{D}U \equiv \sum_n U_n$ such that

$$\sum_n \left(\operatorname{Osc} f(U_n)\right)^2 \sigma(U_n) < \varepsilon^{(4)}$$

where $\operatorname{Osc} f(U_n)$ denotes the oscillation of $f(\lambda)$ on U_n.

Since

$$\sum_n \Big| |f(\lambda')|^2 - |f(\lambda'')|^2 \Big| \sigma(U_n) \qquad \lambda', \lambda'' \varepsilon U_n$$
$$\leqq \left\{\sum |f(\lambda') - f(\lambda'')|^2 \sigma(U_n)\right\}^{\frac{1}{2}} \left\{\Big(\sum_n |f(\lambda')|^2 \sigma(U_n)\Big)^{\frac{1}{2}} + \Big(\sum_n |f(\lambda'')|^2 \sigma(U_n)\Big)^{\frac{1}{2}}\right\},$$

we have from (1), (2),

(1) S. Saks, *Théorie de l'intégrale,* (1933), 255.
(2) U is allowable to be V.
(3) Cf. 229, footnote (2).
(4) Cf. 229, footnote (2).

(3) for any $\varepsilon > 0$, there shall exist a $\mathfrak{D}$, $\mathfrak{D}U \equiv \sum_n U_n$ such that

$$\sum_n \operatorname{Osc} |f|^2(U_n)\,\sigma(U_n) < \varepsilon .$$

Let $f(\lambda)$ be integrable on V with respect to $\mathfrak{q}(U)$. Since $\sigma(U) \geqq \sigma^*(U)$, it follows that $f(\lambda)$ is measurable.(1) And from (1), (3) $|f(\lambda)|^2$ is integrable on V with respect to σ.

If $\mathfrak{q}(U)$ is completely additive in M^* and $\int_U f(\lambda)\,\mathfrak{q}(dU)$ exists, then $(M^*)\int_U f(\lambda)\,\mathfrak{q}(U)$ and $(\mathfrak{R})\int_U f(\lambda)\,\mathfrak{q}^*(U)$ exist and are equal.

In $\mathfrak{M}$, the conditions that $f(\lambda)$ shall be integrable on R_1 with respect to $\mathfrak{q}(U)$ are

(3) that there shall exist a $\mathfrak{D}$, $\mathfrak{D}R_1 \equiv \sum U_n$ such that

$$\sum_n \sup_{\lambda \varepsilon U_n} |f(\lambda)|^2 \sigma(U_n) < +\infty .$$

(4) that the set of discontinuous points of $f(\lambda)$, except possibly on the discontinuous points of $\sigma(U)$, shall be a σ-null set.

Part II.

Space of the Differential Set Functions.

§1. Space of Differential Set Functions which are Completely Additive in M.

24. Let $\beta(E)$, $\xi(E)$.... be differential set functions defined for all sets in $\mathfrak{M}\mathfrak{D}_0\Omega$, $\mathfrak{M}\mathfrak{D}_0'\Omega$, ... which are completely additive in M. Let $\beta(E)$ be non negative. Consider the differential set function

$$\frac{|\xi(E)|^2}{\beta(E)}, \qquad E\,\varepsilon\,\mathfrak{M}[\mathfrak{D}_0\mathfrak{D}_0']\,\Omega,$$

where we define that when $\beta(E)=0$, it takes the value 0 or $+\infty$, according as $\xi(E)=0$ or $\neq 0$.

For $\mathfrak{D} > [\mathfrak{D}_0\mathfrak{D}_0']$ we define:

$$\xi_E(\mathfrak{D}) = \sum_n \frac{|\xi(E_n)|^2}{\beta(E_n)}, \qquad \mathfrak{D}E \equiv \sum_n E_n ,$$

where E is a set in $\mathfrak{M}\mathfrak{D}_0\Omega$, or Ω.

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(1) T. Ogasawara, loc. cit., 51.

$\xi_E(\mathfrak{D})$ has the properties:

(1) $\mathfrak{D}' \supset \mathfrak{D}$ implies $\xi_E(\mathfrak{D}') \geqq \xi_E(\mathfrak{D})$[1]

(2) $\mathfrak{D}' \supset \mathfrak{D}$, $\mathfrak{D}E \equiv \sum E_n$ implies $\xi_E(\mathfrak{D}') = \sum \xi_{E_n}(\mathfrak{D}')$.

Then (1) gives

$$\sup_{\mathfrak{D} \varepsilon M} \xi_E(\mathfrak{D}) = \lim_{\mathfrak{D}} \xi_E(\mathfrak{D}),$$

which we denote by $\varkappa(E)$. When it is finite, we write

$$\varkappa(E) = \int_E \frac{|\xi(dE)|^2}{\beta(dE)}. \tag{2}$$

Then, in the same way as in **10,** $\varkappa(E)$ is completely additive in M, and

$$\varkappa(E) = \sup_{\mathfrak{D}^* \varepsilon M^*} \xi_E(\mathfrak{D}^*), \tag{4}$$

hence, if $\beta(E)$ and $\xi(E)$ are completely additive in M^*, so is $\varkappa(E)$, and if $\varkappa(E)$ is, moreover, finite, then

$$\varkappa(E) = (M^*)\int_E \frac{|\xi(dE)|^2}{\beta(dE)}. \tag{5}$$

25. Denote by $\mathfrak{L}_2(\beta)$ the aggregate of differential set functions $\xi(E)$ for which $\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)}$ exists. Let $\xi(E)$ and $\eta(E) \varepsilon \mathfrak{L}_2(\beta)$. We define the inner product as follows:

$$(\xi, \eta) = \int_\Omega \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)}.\text{[2]}$$

Then $\mathfrak{L}_2(\beta)$ is a Hilbert space.[3] The proof presents little difficulty, so wo pass on. F. Maeda has developed the theory of $\mathfrak{L}_2(\beta)$ under the less general assumption that the aggregate of decompositions M consists of all decompositions of Ω. But we shall now investigate the $\mathfrak{L}_2(\beta)$ under the general assumption on M.

From **24** (3) we have $|\xi(E)|^2 \leqq \varkappa(E)\beta(E)$.

Let $\{A_n\}$ be a sequence of elements in $\mathfrak{M}[\mathfrak{D}_0\mathfrak{D}_0']\Omega$, then, by the aid of Hölder's inequality, we have

$$|\sum_n \xi(A_n)|^2 \leqq \sum_n |\xi(A_n)|^2 \leqq \{\sum_n \varkappa(A_n)\}\{\sum_n \beta(A_n)\}. \tag{1}$$

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(1) F. Maeda, this journal, **6** (1936), 23.

(2) F. Maeda, ibid, 25.

(3) F. Maeda, ibid. 19–45. Correction, F. Maeda, this volume, 107, footnote.

From this follows: Let $E \varepsilon \mathfrak{M}\mathfrak{D}_0\Omega$. Put $[\mathfrak{D}_0\mathfrak{D}_0']E \equiv \sum E_n$. Then, from (1), $\sum_n \xi(E_n)$ is absolutely convergent. Hence we may assume that $\xi(E)$ is defined for all sets in $\mathfrak{M}\mathfrak{D}_0\Omega$. And by **21,** if $\beta(E)$ is completely additive in M^*, then $\xi(E)$ is absolutely continuous with respect to $\beta(E)$, hence it is completely additive in M^*[1] and extended to the ordinary set function.

26. Let $\xi(E) \varepsilon \mathfrak{L}_2(\beta)$ be completely additive in M^*; then we shall show that $\xi(E) \varepsilon \mathfrak{L}_2(\beta^*)$ and

(1) $$\int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)} = \int_\Omega \frac{|\xi(dE)|^2}{\beta^*(dE)},^{(2)}$$

where $\beta^*(E) = (M^*)\int_E \beta(dE)$.[3]

From 25 (1) we have

(2) $|\xi(E)|^2 \leqq \varkappa^*(E)\beta^*(E)$, where $\varkappa^*(E) = (M^*)\int_E \varkappa(dE)$.[3]

This gives

$$\varkappa^*(\Omega) \geqq \int_\Omega \frac{|\xi(dE)|^2}{\beta^*(dE)},$$

but since $\beta^*(E) \leqq \beta(E)$, so we have

$$\int_\Omega \frac{|\xi(dE)|^2}{\beta^*(dE)} \geqq \int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)} = \varkappa(\Omega).$$

Since $\varkappa(\Omega) \geqq \varkappa^*(\Omega)$, so we have (1). And $\xi(E) \varepsilon \mathfrak{L}_2(\beta^*)$. Here we remark that

(3) $$\varkappa^*(E) = \varkappa(E).$$

Let $\xi(E)$, $\eta(E) \varepsilon \mathfrak{L}_2(\beta)$ be completely additive in M^*. Then

(4) $$\int_\Omega \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)} = \int_\Omega \frac{\xi(dE)\overline{\eta(dE)}}{\beta^*(dE)}$$

This follows from the relation:

(5) $$(\xi, \eta) = \left\| \frac{\xi+\eta}{2} \right\|^2 - \left\| \frac{\xi-\eta}{2} \right\|^2 + \left\| \frac{\xi+i\eta}{2} \right\|^2 - \left\| \frac{\xi-i\eta}{2} \right\|^2.$$

(1) It is also so when $\varkappa(E)$ is completely additive in M^*.
(2) This equation holds good for E.
(3) Cf. **8.**

Since $\beta(E) \geqq \beta^*(E)$, so we have $\mathfrak{L}_2(\beta) \supset \mathfrak{L}_2(\beta^*)$.

We call $\mathfrak{L}_2(\beta^*)$ the *regular part* of $\mathfrak{L}_2(\beta)$ and its orthogonal complement the *singular part* of $\mathfrak{L}_2(\beta)$, which we denote by $\mathfrak{L}_2^{(s)}(\beta)$. Then we can write

(6) $$\mathfrak{L}_2(\beta) = \mathfrak{L}_2(\beta^*) \oplus \mathfrak{L}_2^{(s)}(\beta) .$$

27. Let $\xi(E) \varepsilon \mathfrak{L}_2(\beta)$. Denote by $\xi^{(r)}(E)$, $\xi^{(s)}(E)$ its components on $\mathfrak{L}_2(\beta^*)$, $\mathfrak{L}_2^{(s)}(\beta)$ and call them the *regular and singular parts* of $\xi(E)$ respectively. Put $\beta_E^*(E') = \beta^*(EE')$; then $\beta_E^*(E') \varepsilon \mathfrak{L}_2(\beta^*)$. Now we have

$$\big(\xi(E'),\, \beta_E^*(E')\big) = \big(\xi^{(r)}(E'),\, \beta_E^*(E')\big) + \big(\xi^{(s)}(E'),\, \beta_E^*(E')\big) .$$

Since, by **26** (4), (6),

$$\big(\xi^{(r)}(E'),\, \beta_E^*(E')\big) = \xi^{(r)}(E) ,$$

$$\big(\xi^{(s)}(E'),\, \beta_E^*(E')\big) = 0 ,$$

so we have

(1) $$\xi^{(r)}(E) = \int_E \frac{\xi(dE)\,\beta^*(dE)}{\beta(dE)} .$$

If we put $\beta_E(E') = \beta(EE')$, then (1) gives $\beta_E^{(r)}(E') = \beta_E^*(E')$, hence we have

$$\beta_E^{(s)}(E') = \beta_E(E') - \beta_E^{(r)}(E') .$$

Therefore if we put

$$\beta^{(s)}(E) = \beta(E) - \beta^*(E) ,$$

then

(2) $$\xi^{(s)}(E) = \int_E \frac{\xi(dE)\,\beta^{(s)}(dE)}{\beta(dE)} .$$

From (2) it follows that

(3) $$\beta^{(s)}(E) = \int_E \frac{|\beta^{(s)}(dE)|^2}{\beta(dE)} .$$

Next we define the differential set function $\xi_{\mathfrak{D}}(E)$ by

(4) $$\xi_{\mathfrak{D}}(E) = \sum_n \frac{\xi(E_n)}{\beta(E_n)} \beta(E_n E) , \qquad \mathfrak{D}\Omega \equiv \sum_n E_n,\ E \varepsilon \mathfrak{M}\mathfrak{D}_0\Omega .$$

Then it belongs to $\mathfrak{L}_2(\beta)$, and we have

(5) $$\xi_{\mathfrak{D}}^{(r)}(E) = \sum_n \frac{\xi(E_n)}{\beta(E_n)} \beta^*(E_n E) ,$$

(6) $$\xi^{(s)}_{\mathfrak{D}}(E)=\sum_n \frac{\xi(E_n)}{\beta(E_n)}\beta^{(s)}(E_nE).$$

Since $\lim_{\mathfrak{D}\in M}\xi_{\mathfrak{D}}(E)[=]\xi(E)$, so $\lim_{\mathfrak{D}\in M}\xi^{(r)}_{\mathfrak{D}}(E)[=]\xi^{(r)}(E)$ and $\lim_{\mathfrak{D}\in M}\xi^{(s)}_{\mathfrak{D}}(E)[=]\xi^{(s)}(E)$. Denote by $\| \quad \|^{(s)}$ the norm in $\mathfrak{L}_2(\beta^{(s)})$. Then since, by (3),

$$\| \beta^{(s)}_E(E') \| = \| \beta^{(s)}_E(E') \|^{(s)}$$

so $\xi^{(s)}(E)\varepsilon\mathfrak{L}_2(\beta^{(s)})$, and $\lim_{\mathfrak{D}\in M}\xi^{(s)}_{\mathfrak{D}}(E)=\xi^{(s)}(E)$ in $\mathfrak{L}_2(\beta^{(s)})$. And we have

(7) $$\int_\Omega \frac{|\xi^{(s)}(dE)|^2}{\beta(dE)}=\int_\Omega \frac{|\xi^{(s)}(dE)|^2}{\beta^{(s)}(dE)}.$$ (1)

Thus $\mathfrak{L}^{(s)}_2(\beta)\subset\mathfrak{L}_2(\beta^{(s)})$.

But any $\xi(E)\varepsilon\mathfrak{L}_2(\beta^{(s)})$ is approximated by

$$\sum_n \frac{\xi(E_n)}{\beta^{(s)}(E_n)}\beta^{(s)}(E_nE), \qquad \mathfrak{D}\Omega\equiv\sum E_n,$$

and $\beta^{(s)}(E_nE)\,\varepsilon\,\mathfrak{L}^{(s)}_2(\beta)$, hence

$$\mathfrak{L}^{(s)}_2(\beta)=\mathfrak{L}_2(\beta^{(s)})=\mathfrak{M}(\beta^{(s)}).$$

And in like manner we have $\mathfrak{L}_2(\beta^{(r)})=\mathfrak{M}(\beta^{(r)})$.

Thus we have

$$\mathfrak{L}_2(\beta)=\mathfrak{L}_2(\beta^*)\oplus\mathfrak{L}_2(\beta^{(s)}),$$

and

$$\mathfrak{L}_2(\beta^*)=\mathfrak{M}(\beta^*), \qquad \mathfrak{L}_2(\beta^{(s)})=\mathfrak{M}(\beta^{(s)}).$$

28. From **25** (1) we have

$$\left|(M^*)\int_E \xi^{(s)}(dE)\right|^2 \leqq \varkappa^*(E)\left\{(M^*)\int_E \beta^{(s)}(dE)\right\},$$

but since $(M^*)\int_E \beta^{(s)}(dE)=0$, so we have $(M^*)\int_E \xi^{(s)}(dE)=0$. Consequently

(1) $$(M^*)\int_E \xi(dE)=\xi^{(r)}(E).$$

Next we shall show that

(2) $$\int_E \frac{|\xi^{(r)}(dE)|^2}{\beta^*(dE)}=\varkappa^*(E).$$

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(1) Similarly we have $\int_\Omega \frac{\xi^{(s)}(dE)\overline{\eta^{(s)}(dE)}}{\beta(dE)}=\int_\Omega \frac{\xi^{(s)}(dE)\overline{\eta^{(s)}(dE)}}{\beta^{(s)}(dE)}$.

For this purpose we denote $\int_E \frac{|\xi(dE)|^2}{\beta(dE)}$ by $\|\xi\|_E^2$. Since $\lim_{\mathfrak{D}\in M}\xi_{\mathfrak{D}}(E)$ $[=]\xi(E)$, so we have

$$\lim_{\mathfrak{D}\in M}\|\xi_{\mathfrak{D}}\|_E^2=\|\xi\|_E^2=\varkappa(E)\,.$$

Now

$$\|\xi_{\mathfrak{D}}\|_E^{2*}=\sum_n\left|\frac{\xi(E_n)}{\beta(E_n)}\right|^2\beta^*(E_nE)$$
$$=\|\xi_{\mathfrak{D}}^{(r)}\|_E^2\,.$$

This gives

$$\lim_{\mathfrak{D}}\|\xi_{\mathfrak{D}}\|_E^{2*}=\int_E\frac{|\xi^{(r)}(dE)|^2}{\beta^*(dE)}\,.$$

But, on the other hand,

$$\varkappa^*(E)-\|\xi_{\mathfrak{D}}^{(r)}\|_E^2=\{\|\xi\|_E^2-\|\xi_{\mathfrak{D}}\|_E^2\}^*\,,$$

which is non negative, and less than $\|\xi\|_E^2-\|\xi_{\mathfrak{D}}\|_E^2$, since $\|\xi\|_E^2\geqq\|\xi_{\mathfrak{D}}\|_E^2$. Hence, passing to the limit, we have (2).

If we put

$$\varkappa^{(s)}(E)=\varkappa(E)-\varkappa^*(E)\,,$$

then, from (2), we have

(3) $$\int_E\frac{|\xi^{(s)}(dE)|^2}{\beta^{(s)}(dE)}=\varkappa^{(s)}(E)\,.$$

Let $\xi(E)$, $\eta(E)\varepsilon\mathfrak{L}_2(\beta)$, and put $\phi(E)=\int_E\frac{\xi(dE)\,\overline{\eta(dE)}}{\beta(dE)}$.

then, from (2), (3) and **26** (5), we have

$$(M^*)\int_E\phi(dE)=\int_E\frac{\xi^{(r)}(dE)\,\overline{\eta^{(r)}(dE)}}{\beta^*(dE)}$$

$$\phi(E)-(M^*)\int_E\phi(dE)=\int_E\frac{\xi^{(s)}(dE)\,\overline{\eta^{(s)}(dE)}}{\beta^{(s)}(dE)}\,.$$

29. Lastly we shall show that for $\xi(E)$, $\eta(E)\varepsilon\mathfrak{L}_2(\beta)$

(1) $$(M^*)\int_\Omega\frac{|\xi(dE)|^2}{\beta(dE)}=\int_\Omega\frac{|\xi^{(r)}(dE)|^2}{\beta^*(dE)}$$

(2) $$(M^*)\int_\Omega\frac{\xi(dE)\,\overline{\eta(dE)}}{\beta(dE)}=\int_\Omega\frac{\xi^{(r)}(dE)\,\overline{\eta^{(r)}(dE)}}{\beta^*(dE)}\,.$$

It is sufficient to prove (1), as (2) follows from (1) by **26** (5). In fact, for any $\varepsilon>0$ there exists a $\mathfrak{D}^*\varepsilon M^*$ such that for any $\mathfrak{D}\supset\mathfrak{D}_*$, $\mathfrak{D}\varepsilon M^*$, $\mathfrak{D}\Omega\equiv\sum_n E_n$ we have

$$\sum_n \varkappa(E_n)\leqq\varkappa^*(\Omega)+\varepsilon\,.$$

This gives

$$\xi_\Omega(\mathfrak{D})\leqq\varkappa^*(\Omega)+\varepsilon\,. \tag{3}$$

But, on the other hand, for any $\varepsilon>0$ there exists a $\mathfrak{D}_0^*\varepsilon M$ such that

$$\varkappa^*(\Omega)-\varepsilon\leqq\xi_\Omega^{(r)}(\mathfrak{D}_0^*)\,.$$

Since $\frac{y^2}{x}$ is a continuous function of (x, y), so we can determine a $\mathfrak{D}_n\varepsilon M^*$ such that for anp $\mathfrak{D}'_n\supset\mathfrak{D}_n$, $\mathfrak{D}'_n\varepsilon M^*$, $\mathfrak{D}_nE'_n\equiv\sum_n E'_{nm}$, we have

$$\frac{|\xi^*(E_n)|^2}{\beta^*(E_n)}-\varepsilon_n\leqq\frac{|\sum_m\xi(E'_{nm})|^2}{\sum_m\beta(E'_{nm})} \tag{4}$$

$$\leqq\sum_m\frac{|\xi(E'_{nm})|^2}{\beta(E'_{nm})}\,.$$

If we put $\mathfrak{D}_0'^*\Omega\equiv\sum_n\mathfrak{D}_nE'_n$, then, from (4), for any $\mathfrak{D}\supset\mathfrak{D}_0^*$, we have

$$\varkappa^*(\Omega)-2\varepsilon\leqq\xi_\Omega(\mathfrak{D})\,. \tag{5}$$

From (3), (5) follows (1).

§2. Space of Differential Set Functions which are Completely Additive in M^*.

30. Let $\beta(E)$ be non negative and completely additive in M^*. From **25** any $\xi(E)\varepsilon\mathfrak{L}_2(\beta)$ is completely additive in M^* and extended to the ordinary completely additive set function $\xi^*(E)$, and

$$\int_\Omega\frac{|\xi(dE)|^2}{\beta(dE)}=(M^*)\int_\Omega\frac{|\xi(dE)|^2}{\beta(dE)}\,, \tag{1}$$

$$\int_\Omega\frac{\xi(dE)\,\overline{\eta(dE)}}{\beta(dE)}=(M^*)\int_\Omega\frac{\xi(dE)\,\overline{\eta(dE)}}{\beta(dE)}\,, \tag{2}$$

where $\eta(E)$ is another differential set function belonging to $\mathfrak{L}_2(\beta)$.

It is sometimes convenient to make use of this formula in the actual calculation. For example, in the differential set system of open intervals and points, we can confine ourselves to decompositions whose point elements have no limiting point.

31. Let A be any set in the Borel family over $\mathfrak{M}\mathfrak{D}_0\Omega$, and $\beta^*(A) < +\infty$. For any we take a sequence of elements $\{A_n\}$ covering A for which we have[1]

$$v_\xi^*(A) \leqq \sum_n v_\xi(A_n) \leqq v_\xi^*(A) + \varepsilon$$

$$\varkappa^*(A) \leqq \sum_n \varkappa(A_n) \leqq \varkappa^*(A) + \varepsilon$$

$$\beta^*(A) \leqq \sum_n \beta(A_n) \leqq \beta^*(\mathrm{A}) + \varepsilon .$$

Then, from **25** (1), we have

$$(1) \qquad |\xi^*(A)|^2 \leqq \varkappa^*(A)\beta^*(A) .$$

In the following we omit *, for simplicity. Then (1) is written

$$(1') \qquad |\xi(A)|^2 \leqq \varkappa(A)\beta(A) .$$

In the same way as in **26** we have

$$(2) \qquad \int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)} = (\mathfrak{R}) \int_\Omega \frac{|\xi(dE)|^2}{\beta(dE)} ,$$

where $\mathfrak{R}$ denotes the Borel family over $\mathfrak{M}\mathfrak{D}_0\Omega$.

Accordingly

$$(3) \quad \int_\Omega \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)} = (\mathfrak{R}) \int_\Omega \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)} . \qquad \xi, \eta \,\varepsilon\, \mathfrak{L}_2(\beta) .$$ [2]

From (1′), $\xi(A)$ is absolutely continuous with respect to $\beta(A)$, hence there exists a point function $\xi(\lambda)$ which is determined except on β-null sets, such that

$$(4) \qquad \xi(A) = (\mathfrak{R}) \int_A \xi(\lambda)\beta(dE) .$$ [3]

There exists a decomposition $\overline{\mathfrak{D}}$ of Ω into sets A_n of $\mathfrak{R}$ on each of which

$$(5) \qquad \left| \frac{|\xi(A_n)|^2}{\beta(A_n)} - (\mathfrak{R}) \int_{A_n} |\xi(\lambda)|^2 \beta(dE) \right| \leqq \varepsilon_n ,$$

(1) Cf. **12.**
(2) This follows from (2) and **26** (5).
(3) Cf. **22.** S. Saks, loc. cit., 255.

where $\varepsilon_n > 0$, small at pleasure.

Then, from (5), we have

$$\int_{\Omega} \frac{|\xi(dE)|^2}{\beta(dE)} = (\mathfrak{R}) \int_{\Omega} |\xi(\lambda)|^2 \beta(dE) . \tag{6}$$

Accordingly we have

$$\int_{\Omega} \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)} = (\mathfrak{R}) \int_{\Omega} \xi(\lambda)\overline{\eta(\lambda)}\beta(dE) , \qquad \xi, \eta \varepsilon \mathfrak{L}_2(\beta) \tag{7}$$

where $\eta(\lambda)$ corresponds to $\eta(E)$ as $\xi(\lambda)$ to $\xi(E)$.

From (1), (4) any β-null set is a $\varkappa$- and ξ-null set; hence we may assume that $\mathfrak{R}$ is the system of β-measurable sets.

Since obviously

$$\varkappa(E) = (\mathfrak{R}) \int_{E} |\xi(\lambda)|^2 \beta(dE) ,$$

then for any set A in $\mathfrak{R}$ we have

$$\varkappa(A) = (\mathfrak{R}) \int_{A} |\xi(\lambda)|^2 \beta(dE) . \tag{8}$$

Similarly, if we put

$$\phi(E) = \int_{E} \frac{\xi(dE)\overline{\eta(dE)}}{\beta(dE)} , \qquad E \varepsilon \mathfrak{M}\mathfrak{D}_0 \Omega ,$$

then

$$\phi(A) = (\mathfrak{R}) \int_{A} \xi(\lambda)\overline{\eta(\lambda)}\beta(dE) .$$

Next consider any completely additive set function $\xi'(A)$ in $\mathfrak{R}$, which is absolutely continuous with respect to $\beta(E)$. Let $(\mathfrak{R})\int_{\Omega} |\xi'(\lambda)|^2 \beta(dE) < +\infty$, then $\xi'(E)$ may be considered a differential set function which is completely additive in M^*. Since $\int_{\Omega} \frac{|\xi'(dE)|^2}{\beta(dE)} \leqq (\mathfrak{R}) \int_{\Omega} \frac{|\xi'(dE)|^2}{\beta(dE)}$, hence $\xi'(E) \varepsilon \mathfrak{L}_2(\beta)$.

Thus $\mathfrak{L}_2(\beta)$ can be considered the space of ordinary set functions studied by F. Maeda.(1) in this journal.

32. If we put for any $A \varepsilon \mathfrak{R}$

$$\xi_A(E) = \xi(AE) ,$$

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(1) F. Maeda, this journal, **3** (1933) 1–42, 243–273; **5** (1935), 107–116; **6** (1936) 19, footnote (2).

then

$$\int_{\Omega} \frac{|\xi_A(dE)|^2}{\beta(dE)} = (\mathfrak{R})\int_A |\xi(\lambda)|^2 \beta(dE),$$

therefore we have

(1) If $A = \sum A_n$, then

$$\int_{\Omega} \frac{|\xi_A(dE)|^2}{\beta(dE)} = \sum_n \int_{\Omega} \frac{|\xi_{A_n}(dE)|^2}{\beta(dE)}.$$

(2) $$\int_{\Omega} \frac{\xi_A(dE)\overline{\eta_B(dE)}}{\beta(dE)} = \int_{\Omega} \frac{\xi_{AB}(dE)\overline{\eta(dE)}}{\beta(dE)}.$$

§3. Vector valued Differential Set Function and the Resolution of the Identity.

33. Let $\mathfrak{q}(U)$ be a vector valued differential set function which is completely additive in M. Let $\xi(U)$ be a differential set function which is completely additive in M. Then the vector valued differential set function

$$\frac{\xi(U)\mathfrak{q}(U)}{\sigma(U)}, \qquad U \varepsilon \mathfrak{N}\mathfrak{D}_0 V$$

is integrable on Ω when, and only when, $\xi(U)$ belongs to $\mathfrak{L}_2(\sigma)$.[1] $\mathfrak{L}_2(\sigma)$ is isomorphic with the closed linear manifold $\mathfrak{M}(\mathfrak{q})$ under the correspondence :[2]

(1) $$\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big), \qquad \text{where } \mathfrak{f} \varepsilon \mathfrak{M}(\mathfrak{q}).$$

(2) $$\mathfrak{f} = \int_V \frac{\xi(dU)\mathfrak{q}(dU)}{\sigma(dU)}.$$

From 27 we have

$$\mathfrak{L}_2(\sigma) = \mathfrak{L}_2(\sigma^*) \oplus \mathfrak{L}_2(\sigma^{(s)}).$$

Put $\sigma_A^*(U) = \sigma^*(AU)$, where A is σ-measurable and $\sigma^*(A) < +\infty$. To this there corresponds a vector in $\mathfrak{M}(\mathfrak{q})$ which we shall denote by $\mathfrak{q}^*(A)$. Then we have, obviously,

(3) $$\big(\mathfrak{q}^*(A), \mathfrak{q}^*(B)\big) = \big(\sigma_A^*(U), \sigma_B^*(U)\big) = \sigma^*(AB).$$

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(1) For the proof see F. Maeda, loc. cit., 34.

(2) Cf. F. Maeda, loc. cit., 36.

(4) $$\mathfrak{q}^*(A)[=]\sum_n \mathfrak{q}^*(A_n), \qquad \text{when} \quad A=\sum A_n.$$

Thus $\mathfrak{q}^*(U)$ is a vector valued differential set function which is completely additive in M^*. If $\mathfrak{q}(U)$ is completely additive in M^*, we have

$$\mathfrak{q}^*(U)=\mathfrak{q}(U),$$

hence $\mathfrak{q}^*(A)$ is the extension of $\mathfrak{q}(U)$, whose uniqueness has been already proved in **16.**

Let $\mathfrak{q}^{(s)}(U)$ be a vector corsesponding to $\sigma_U^{(s)}(U')$; then it is a vector-valued differential set function which is completely additive in M and orthogonal to $\{\mathfrak{q}^*(U)\}$. We see, obviously, that $\mathfrak{M}(\mathfrak{q}^*)$ corresponds to $\mathfrak{L}_2(\beta^*)$ and $\mathfrak{M}(\mathfrak{q}^{(s)})$ to $\mathfrak{L}_2(\beta^{(s)})$. Since $\|\mathfrak{q}^{(s)}(U)\|^2=\sigma^{(s)}(U)$, so we have

$$(M^*)\int_U \mathfrak{q}(dU)=\mathfrak{q}^*(U),$$

$$(M^*)\int_U \mathfrak{q}^{(s)}(dU)=0,$$

and

$$\mathfrak{q}(U)=\mathfrak{q}^*(U)+\mathfrak{q}^{(s)}(U).$$

34. Since, by **27,**

$$\xi^{(r)}(U)=\left(\xi(U'),\, \sigma_U^*(U')\right), \qquad \xi^{(s)}(U)=\left(\xi(U'),\, \sigma_U^{(s)}(U')\right),$$

so we have

$$\xi^{(r)}(U)=\left(\mathfrak{f},\, \mathfrak{q}^*(U)\right), \qquad \xi^{(s)}(U)=\left(\mathfrak{f},\, \mathfrak{q}^{(s)}(U)\right).$$

Let $\mathfrak{f}^{(r)}$, $\mathfrak{f}^{(s)}$ be the components of $\mathfrak{f}$ on $\mathfrak{M}(\mathfrak{q}^*)$, $\mathfrak{M}(\mathfrak{q}^{(s)})$. Then they are expressed by various forms:

(1) $$\int_V \frac{\xi^{(r)}(dU)\,\mathfrak{q}(dU)}{\sigma(dU)}, \qquad \int_V \frac{\xi^{(s)}(dU)\,\mathfrak{q}(dU)}{\sigma(dU)},$$

(2) $$\int_V \frac{\xi(dU)\,\mathfrak{q}^*(dU)}{\sigma(dU)}, \qquad \int_V \frac{\xi(dU)\,\mathfrak{q}^{(s)}(dU)}{\sigma(dU)},$$

(3) $$\int_V \frac{\xi^{(r)}(dU)\,\mathfrak{q}^*(dU)}{\sigma(dU)}, \qquad \int_V \frac{\xi^{(s)}(dU)\,\mathfrak{q}^{(s)}(dU)}{\sigma(dU)},$$

(4) $$\int_V \frac{\xi^{(r)}(dU)\,\mathfrak{q}^*(dU)}{\sigma^*(dU)}, \qquad \int_V \frac{\xi^{(s)}(dU)\,\mathfrak{q}^{(s)}(dU)}{\sigma^{(s)}(dU)}.$$

These follow from the correspondence formula **33** (1).

35. We shall observe $\mathfrak{M}(\mathfrak{q})$ when $\mathfrak{q}(U)$ is completely additive in M^*. Since $\xi(U)$ becomes completely additive in M^*, any $\xi(U)\,\varepsilon\,\mathfrak{L}_2(\sigma)$ is written in the form:

(1) $$\xi(U)=(\mathfrak{K})\int_U \xi(\lambda)\,\sigma(dU).^{(1)}$$

Then the vector $\mathfrak{f}$ which corresponds to $\xi(U)$ is

(2) $$\mathfrak{f}=(\mathfrak{K})\int_V \xi(\lambda)\,\mathfrak{q}^*(dU).$$

This follows from the formula **33** (1).

Conversely, any $\mathfrak{f}$ which is expressed by (2) belongs to $\mathfrak{M}(\mathfrak{q})$ and corresponds to $\xi(U)$ in (1).

36. Next consider the resolution of the identity $E(U)$. In virtue of the theorem of O. Teichmüller[2] it follows that there exists an orthogonal system $\{\mathfrak{q}_\alpha(U)\}$ which is complete in $\mathfrak{H}$, where $\mathfrak{q}_\alpha(U)=E(U)\mathfrak{b}_\alpha$, $\mathfrak{b}_\alpha\,\varepsilon\,\mathfrak{H}$, $\alpha\,\varepsilon\,\mathfrak{A}$.[3]

Let $\mathfrak{f}$ be any vector in $\mathfrak{H}$, and $\mathfrak{f}_\alpha$ be its component on $\mathfrak{M}(\mathfrak{q}_\alpha)$. Then $\mathfrak{f}_\alpha$ is expressed by

(1) $$\int_V \frac{\xi(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma(dU)}, \qquad \text{where} \quad \sigma_\alpha(U)=\|\mathfrak{q}_\alpha(U)\|^2$$

and

(2) $$\xi_\alpha(U)=\big(\mathfrak{f},\,\mathfrak{q}_\alpha(U)\big).$$

Since for only finite or enumerable α (1) does not vanish, we have

(3) $$\mathfrak{f}[=]\sum_\alpha\int_V \frac{\xi_\alpha(dU)\,\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}.$$

If we put $\xi_{\alpha U'}(U)=\xi_\alpha(UU')$, then from (1) (2) we have

(4) $$E(U)=\mathfrak{f}[=]\sum_\alpha\int_V \frac{\xi_{\alpha U'}(dU')\,\mathfrak{q}_\alpha(dU')}{\sigma_\alpha(dU')}.$$

First consider the case when $E(U)$ is completely additive in M^*.

Let $\mathfrak{K}$ be the system of all sets which are σ_α-measurable for all $\alpha\,\varepsilon\,\mathfrak{A}$. Let $A\,\varepsilon\,\mathfrak{K}$; then we shall define $E^*(A)$ by

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(1) $\mathfrak{K}$ is the Borel family over $\mathfrak{R}\mathfrak{D}_0 V$.

(2) O. Teichmüller, Journal für die reine u. angew. Math., **174** (1935), 78.

(3) $\mathfrak{A}$ is an aggregate of indices.

$$(5)\qquad E^*(A)\mathfrak{f}[=]\sum_a\int_V \frac{\xi_{aA}(dU)\mathfrak{q}_a(dU)}{\sigma_a(dU)}\,.$$

By (4) we have $E^*(U)=E(U)$. Now we shall show that $E^*(A)$ is the ordinary resolution of the identity.

$$(1^\circ)\qquad \big(E^*(A)\mathfrak{f},\,\mathfrak{g}\big)=\big(\mathfrak{f},\,E^*(A)\mathfrak{g}\big)\,.$$

For put $\eta_a(U)=\big(\mathfrak{g},\,\mathfrak{q}_a(U)\big)$; then by (1) we have

$$\mathfrak{g}_a=\int_V \frac{\eta_a(dU)\,\mathfrak{q}_a(dU)}{\sigma_a(dU)}\,,$$

hence

$$\big(E^*(A)\mathfrak{f},\,\mathfrak{g}\big)\,[=]\sum_a\int_V \frac{\xi_{aA}(dU)\,\eta_a(dU)}{\sigma_a(dU)}\,,$$

$$[=]\sum_a\int_V \frac{\xi_a(dU)\,\eta_{aA}(dU)}{\sigma_a(dU)}\,,\qquad \text{since by 32,}$$

$$=\big(\mathfrak{f},\,E^*(A)\mathfrak{g}\big)\,,$$

$$(2^\circ)\qquad E^*(A)E^*(A')=E^*(AA')\,.$$

For put $\eta_a(U)=\big(E^*(A)\mathfrak{f},\,\mathfrak{g}_a(U)\big)$; then from (3) we have

$$\eta_a(U)=\big(E^*(A)\mathfrak{f},\,\mathfrak{g}_a(U)\big)\,,$$

$$=\xi_{aA}(U)\,,$$

$$=\xi_a(AU)\,.$$

This gives $\eta_{aA'}(U)=\xi_a(AA'U)=\xi_{aAA'}(U)$; hence we have (2)

$$(3^\circ)\qquad E^*(A)=\sum E^*(A_n)\,,\qquad \text{when}\quad A=\sum A_n\,,$$

$$(4^\circ)\qquad E(V)=1$$

(3°) (4°) follow obviously from the definition of $E^*(A)$ and 32.

Thus $E^*(A)$ is the resolution of the identity defined on the closed family to which $E(U)$ is extended. Consider any extension $\overline{E}(U)$ of $E(U)$, and let $A\,\varepsilon\,\mathfrak{R}$.

Put $\mathfrak{q}(U)=E(U)\mathfrak{f}$, $\mathfrak{f}$ being fixed. Then $\overline{E}(A)\mathfrak{f}$ is an extension of

$\mathfrak{q}(U)$. Since we can easily see that A is σ-measurable, $\overline{E}(A)\mathfrak{f}$ is uniquely determined by **16.** This holds good for any $\mathfrak{f}$ in $\mathfrak{H}$. Thus $\overline{E}(A)$ is uniquely determined on sets of $\mathfrak{K}$.

37. Next consider the general case when $E(U)$ is completely additive in M. Since

$$\mathfrak{M}(\mathfrak{q}_a)=\mathfrak{M}(\mathfrak{q}_a^*)\oplus\mathfrak{M}(\mathfrak{q}_a^{(s)}),$$

therefore we have $\mathfrak{H}=\sum_a\left(\mathfrak{M}(\mathfrak{q}_a^*)\,;\,\oplus\right)\oplus\sum_a\left(\mathfrak{M}(\mathfrak{q}_a^{(s)})\,;\,\oplus\right)$.[1]

Put $\mathfrak{H}_1=\sum_a\left(\mathfrak{M}(\mathfrak{q}_a^*)\,;\,\oplus\right)$ and $\mathfrak{H}_2=\sum_a\left(\mathfrak{M}(\mathfrak{q}_a^{(s)})\,;\,\oplus\right)$. Then the resolution of the identity $E^*(U)$ and $E^{(s)}(U)$ in $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is defined by the equation analogous to **36** (3). If for any $\mathfrak{f}_2$ in $\mathfrak{H}_2$ we define

$$E^*(U)\mathfrak{f}_2=0,$$

and similarly for any $\mathfrak{f}_1$ in $\mathfrak{H}_1$ $E^{(s)}(U)\mathfrak{f}_1=0$, then

$$E(U)=E^*(U)+E^{(s)}(U),$$

where

$$(M^*)\int_U E(dU)=E^*(U), \qquad \text{and} \quad (M^*)\int_U E^{(s)}(dU)=0.$$

As $E^*(U)$ is completely additive in M^*, we can extend it.

(1) $\sum_a(\mathfrak{M}(\mathfrak{q}_a);\,\oplus)$ means $\mathfrak{M}(\mathfrak{q}_a)\perp\mathfrak{M}(\mathfrak{q}_\beta)$ for $a\neq\beta$, and denotes the least closed linear manifold over $\mathfrak{M}(\mathfrak{q}_a)$ for all a in $\mathfrak{A}$.