On the Location of the Roots of Linear Combinations of some Polynomials.

By

Kiyosi TODA.

(Received Jan. 8, 1937.)

I.

Let α be any complex number and ξ be any root of the equation of order n

$$A(\xi) \equiv f(\xi)+k_1(\xi-\alpha)f'(\xi)+ \dots\dots +k_n(\xi-\alpha)^n f^{(n)}(\xi) = 0\,,$$

where $f(z)$ is a given polynomial of order n, then the following equations

$$g(x) \equiv f(\xi)+f'(\xi)x+ \dots\dots +\frac{f^{(n)}(\xi)}{n\,!}x^n = 0$$

and

$$h(x) \equiv x^n+nk_1(\alpha-\xi)x^{n-1}+n(n-1)k_2(\alpha-\xi)^2x^{n-2}+ \dots\dots +n\,!\;k_n(\alpha-\xi)^n = 0$$

are apolar with each other. Hence by Grace's theorem,(1) $h(x) = 0$ has at least one root in the circle which comprises all the roots of $g(x) = 0$. If we put $x = z-\xi$ in $g(x) = 0$, we have $g(z-\xi) \equiv f(z)$. Thus if the circle C contains all the roots of $f(z) = 0$, then $h(z-\xi) = 0$ has at least one root within C. Rewriting $h(z-\xi) = 0$ in the form

$$K(y) \equiv y^n+nk_1y^{n-1}+n(n-1)k_2y^{n-2}+ \dots\dots +n\,!\;k_n = 0 \qquad \left(y = \frac{z-\xi}{\alpha-\xi}\right),$$

we obtain

Theorem I. Let z be a suitable point in the circle C containing

---

(1) After the idea used by Prof. T. Takagi in his "Note on the algebraic equations." Proc. Phys.-Math. soc. of Japan, (3), **3** (1921), 176, we start from the theorem due to J. H. Grace.

(2) In this paper, we use the word "circle" to mean the "Kreisbereich" in G. Pólya und G. Szegö, *Aufgaben und Lehrsatze aus der Analysis.* II. 55.

all the roots of the equation $f(z)=0$, and y_i is one of the roots of the equation $K(y)=0$, then any root w of the equation $A(z)=0$ is expressed in the form $w=\dfrac{z-ay_i}{1-y_i}$.

Similarly we have

Theorem I′. Let y be a suitable point in the circle which contains all of the roots of the equation $K(y)=0$, and z_j is one of the roots of the equation $f(z)=0$, then any root w of the equation $A(z)=0$ can be put in the form

$$w=\frac{z_j-ay}{1-y}=a+\frac{z_j-a}{1-y}.$$

If C_i is the circle described by w when z, in the linear transformation $L_i\left(w=\dfrac{z-ay_i}{1-y_i}\right)$, describes the circle C, we can enunciate the more precise

Theorem II. If C_i's are classified into sets of circles

$$\Gamma_1(C_1, C_2, \ldots, C_{\nu_1-1}), \qquad \Gamma_2(C_{\nu_1}, \ldots, C_{\nu_2-1}), \ldots, \Gamma_k(C_{\nu_{k-1}}, \ldots, C_n)$$

so that the sum of the circles in each set Γ_i forms a simply connected domain Γ_i and the Γ's have no point in common with each other, then the domain Γ_i has as many roots of the equation $A(z)=0$ as the number of circles C_i contained in Γ_i.

Proof: Particularly if $f(z)\equiv(z-\beta)^n$ where $\beta(\neq a)$ is a point in C, $A(z)=0$ becomes $(z-a)^n K(y)=0$[1] where $y=\dfrac{z-\beta}{z-a}$, hence the equation $A(z)=0$ has n roots β_i in C_i which correspond to β in C by L_i. Now let all of the roots of $f(z)=0$ converge to β continuously remaining within C, then as the roots of a equation are continuous with respect to its coefficients, n roots of $A(z)=0$ converge continuously to β_i respectively. But Γ's having no part in common, our assertion is valid.

Cor. I. If C's are discrete with each other, then the equation $A(z)=0$ has one root in every C_i respectively.

Cor. II. The equation

---

(1) a is not a root of $A(z)=0$ in this case.

$$\bar{A}(z) \equiv f(z)+k_1(z-a)f'(z)+k_2(z-a)^2f''(z)+\ldots . k_p(z-a)^p f^{(p)}(z) = 0$$

has $(n-p)$ roots in C and one root in every C_i respectively, when C's are discrete with each other, where C's are obtained by the linear transformation $w = \dfrac{z-ay_i}{1-y_i}$, y_i being the roots of the equation

$$\bar{K}(y) \equiv y^p+nk_1y^{p-1}+n(n-1)k_2y^{p-2}+\ldots .+n(n-1)\ldots .(n-p+1)k_p = 0$$

Proof: Since $k_p \neq 0$, and $k_{p+1} = \ldots . = k_n = 0$, we have $K(y) \equiv y^{n-p}\cdot\bar{K}(y)$.

Theorem III. If the roots y_i of the equation $K(y) = 0$ are such that $\mathfrak{A}(1-y_i)$ is constant, in other words, y's are distributed on a half line ending at 1, then the equation $A(z) = 0$ has all of the roots in the smallest convex polygon K^* which encloses n^2 points

$$\xi_{ij} = \frac{z_j-ay_i}{1-y_i} \qquad (i j = 1, 2, \ldots ., n),$$

where z_j is the roots of $f(z) = 0$.

Proof: Let the smallest convex polygon enclosing z_j be K and the convex polygon into which the linear transformation $L_i\left(w = \dfrac{z-ay_i}{1-y_i}\right)$ transforms K be K_i respectively,(1) then K^* must be the convex cover of $K_1, K_2, \ldots ., K_n$. Now the tangential half plane(2) π^* of K^* is also a tangential half plane π_i of some K_i, and the corresponding half plane π by the above linear transformation L_i of π_i is also tangential to K. Let π_j be the corresponding tangential half plane of K_j by L_j of π, then π_i is the convex cover of half planes π_j $(j = 1, 2, \ldots ., n)$, since K's are similar and similarly situated with each other ($\because$ $\mathfrak{A}(1-y_i) =$ const. !).

---

(1) Thus K_i is also the smallest convex polygon which encloses the points

$$\frac{z_1-ay_i}{1-y_i}, \quad \frac{z_2-ay_i}{1-y_i}, \ldots ., \frac{z_n-ay_i}{1-y_i}.$$

(2) Berwald, Über die Lage der Nullstellen von Linearkombinationen eines Polynoms und seiner Ableitungen in Bezug auf einen Punkt. Tôhoku Math. Journ. **37** (1933), 59.

Now suppose π is the circle C in the theorem I, we see that all of the root of $A(z)=0$ must be in π_i and hence within K^*. q.e.d.

Remark: The fact that K^* is the smallest convex domain which encloses all of the roots of the equation $A(z)=0$ is shown as follows;

If $f(z)\equiv(z-\beta)^n=0$, then K becomes a point β and we have $A(z)\equiv(z-\alpha)^n K\left(\frac{z-\beta}{z-\alpha}\right)$, $\therefore\ \xi_{ij}=\frac{\beta-\alpha y_i}{1-y_i}$, where $y_i=\frac{z_i-\beta}{z_i-\alpha}$ and z_i being the roots of $A(z)=0$. Thus we have $\xi_{ij}=z_i$, that is, K^* is identical with the convex cover of all the roots of $A(z)=0$.

Cor. I. If the roots y_i of the equation $\bar{K}(y)=0$ are less than unity, then the equation $\bar{A}(z)=0$ has all its roots in the smallest convex polygon which encloses z_j and np points

$$\xi_{ij}=\frac{z_i-\alpha y_i}{1-y_i}\qquad\begin{pmatrix}i=1,2,\ldots\ldots,p\\ j=1,2,\ldots\ldots,n\end{pmatrix}$$

where z_j is the roots of the equation $f(z)=0$.

Cor. II. If, α and all the roots of the equation $f(z)=0$ being real, the equation $K(y)=0$ $\left(\text{or } \bar{K}(y)=0\right)$ has all the roots greater (less) than unity, then $A(z)=0$ $\left(\bar{A}(z)=0\right)$ has real roots only (for $p\leqq n$).

Cor. III. i. If the equation $K(y)=0$ has all its roots greater than unity and the equation $f(z)=0$ has negative (positive) roots only, α being positive (negative), $A(z)=0$ has also negative roots only.

ii. If the equation $\bar{K}(y)=0$ has all its roots between zero and unity and the equation $f(z)=0$ has negative (positive) roots only, α being positive (negative), $\bar{A}(z)=0$ has also negative (positive) roots only ($p\leqq n$).

iii. If the equation $\bar{K}(y)=0$ has all its roots less than zero and the equation $f(z)=0$ has all its roots of the same sign with α, then also the equation $\bar{A}(z)=0$ has the roots of the same sign with α only.

Since $\xi_{ij}=\alpha+\frac{z_j-\alpha}{1-y_i}$, we have

Cor. IV. If the equation $K(y)=0$ $\left(\text{or } \bar{K}(y)=0\right)$ has roots greater (less) than unity only, then the roots of the equation $A(z)=0$ $\left(\bar{A}(z)=0\right)$ are greater or less than α according as the roots of $f(z)=0$

are less or greater (greater or less) than α (for $p \leqq n$).

Since $\xi_{ij} = z_j + \dfrac{y_i(z_j-\alpha)}{1-y_i}$, we have

Cor. V. i. If the equation $K(y) = 0$ has all its roots greater than unity and all the roots of $f(z) = 0$ are greater (or less) than α, then the maximum (minimum) root of the equation $A(z) = 0$ is less (greater) than the maximum (minimum) root of the equation $f(z) = 0$.

ii. If the equation $\bar{K}(y) = 0$ has all its roots between zero and unity and all the roots of $f(z) = 0$ are greater (or less) than α, then the minimum (maximum) root of the equation $\bar{A}(z) = 0$ is greater (less) than the minimum root of the equation $f(z) = 0$. $(p \leqq n)$

iii. If the equation $\bar{K}(y) = 0$ has all its roots less than zero and all the roots of $f(z) = 0$ are greater (less) than α, then the maximum (minimum) root of the equation $\bar{A}(z) = 0$ is less (greater) than the maximum (minimum) root of the equation $f(z) = 0$. $(p \leqq n)$

Cor. VI. If the equation $K(y) = 0$ $\big($or $\bar{K}(y) = 0\big)$ has all its roots greater than unity (less than zero) and, then the equation $A(z) = 0$ $\big(\bar{A}(z) = 0\big)$ has negative or positive roots only according as the roots of the equation $f(z) = 0$ are negative and greater than α or positive and less than α (for $p \leqq n$).

If all the roots of the equation $\bar{K}(y) = 0$ are between zero and unity and further-more all the roots of the equation $f(z) = 0$ are greater than Max. $(\alpha, 0)$ $\big($or Min. $(\alpha, 0)\big)$, then the equation $\bar{A}(z) = 0$ has positive (negative) roots only.

II.

Lemma. $$y^{n-p} f_p\left(\alpha + \frac{1}{y}, \alpha\right) = \frac{d^p}{dy^p}\left\{y^n f\left(\alpha + \frac{1}{y}\right)\right\}$$

where $f_p(z, \alpha)$ is the p-th derivative of $f(z)$ with respect to a point α.[1]

Proof: For $p = 1$, we can easily verify that this relation is true, hence we prove by mathematical induction as follows:—

(1) G. Pólya und G. Szegö, loc. cit., 61.

$$y^{n-(p+1)}f_{p+1}\left(a+\frac{1}{y}, a\right) = y^{n-(p+1)}\left\{(n-p)f_p\left(a+\frac{1}{y}, a\right) - \frac{1}{y}\frac{d}{d\left(a+\frac{1}{y}\right)}f_p\left(a+\frac{1}{y}, a\right)\right\}$$

$$= f_p\left(a+\frac{1}{y}, a\right)\frac{dy^{n-p}}{dy} - y^{n-p-2}(-y^2)\frac{df_p\left(a+\frac{1}{y}, a\right)}{dy}$$

$$= \frac{d}{dy}\left\{y^{n-p}f_p\left(a+\frac{1}{y}, a\right)\right\}$$

$$= \frac{d^{p+1}}{dy^{p+1}}\left\{y^n f\left(a+\frac{1}{y}, a\right)\right\}. \qquad \text{q.e.d.}$$

Now put $z = a+\frac{1}{y}$ in the equation

$$B(z) \equiv f(z)+k_1 f_1(z, a)+ \dots +k_n f_n(z, a) = 0 ,$$

then we have

$$y^n\left\{f\left(a+\frac{1}{y}\right)+k_1 f_1\left(a+\frac{1}{y}, a\right)+ \dots +k_n f_n\left(a+\frac{1}{y}, a\right)\right\} = 0$$

that is $$\varphi(y)+k_1 y\varphi(y)+ \dots +k_n y^n \varphi^{(n)}(y) = 0 \quad \dots\dots \quad (1)$$

where $$\varphi(y) \equiv y^n f\left(a+\frac{1}{y}\right).$$

Thus corresponding to those theorems especially for $a = 0$ in I, we can obtain several theorems about the equation $B(z) = 0$. Corresponding to the theorem I, we have the

Theorem I. Let z be a suitable point in the circle which comprises all of the roots of the equation $f(z) = 0$ and y_i is one of the roots of the equation $K(y) = 0$, then the roots of the equation $B(z) = 0$ is expressed as $w = a+(1-y_i)(z-a)$

Proof: From the relation

$$w = \frac{1}{\eta}+a \qquad \text{and} \qquad \eta = \frac{y}{1-y_i},$$

where η is the corresponding root of the equation (1) and y is a suitable point in the circle comprising all the roots of $\varphi(y) = 0$, which can be expressed as $\frac{1}{y}+a = z$, z being a point as in the assumption, we can easily deduce the result.

Theorem I′. Let y be a suitable point in the circle which contains all of the roots of the equation $K(y) = 0$, and z_j is one of the roots of the equation $f(z) = 0$, then any root w of the equation $B(z) = 0$ can be put in the form

$$w = a+(z_j-a)(1-y) = z_j+(a-z_j)y .$$

Theorem II. Let y_i be the roots of the equation $K(y) = 0$ and the roots of $f(z) = 0$ be all in the circle C; if C_i be the circle into which C is transformed by the linear transformation $w = a+(1-y_i)(z-a)$ and C's are classified into

$$\Gamma_1(C_1, \dots, C_{\nu_1-1}), \qquad \Gamma_2(C_{\nu_1}, \dots, C_{\nu_2-1}), \dots, \Gamma_k(C_{\nu_{k-1}}, \dots, C_n)$$

so that the C's in each Γ_i form a simply connected domain Γ_i and Γ's are discrete with each other, then the equation $B(z) = 0$ has as many roots in the domain Γ_i as the number of circles C_i contained in Γ_i.

Theorem III. If the roots y_i of the equation $K(y) = 0$ are such that $\mathfrak{A}(1-y_i)$ are constant, then the equation $B(z) = 0$ has all of its roots in the smallest convex polygon H^* which covers the n^2 points $\xi_{ij} = a+(1-y_i)(z_j-a)$, where z_j is the roots of $f(z) = 0$.

Theorem III′. If the roots of the equation $f(z) = 0$ are distributed on a half line ending at a, then all the roots of the equation $B(z) = 0$ are in H^*.

Cor. I. If the roots y_i of the equation $\bar{K}(y) = 0$ are less than unity, then the equation

$$\bar{B}(z) \equiv f(z)+k_1 f_1(z, a)+ \dots +k_p f_p(z, a) = 0$$

has all of its roots in the smallest convex polygon which covers z_j and np points

$$\xi_{ij} = a+(1-y_i)(z_j-a) \qquad (i = 1, 2, \dots, p;\ j = 1, 2, \dots, n).$$

Cor. II. a and all the roots of the equation $f(z) = 0$ being real, if the equation $K(y) = 0$ $\left(\text{or } \bar{K}(y) = 0\right)$ has real roots only which are greater (less) than unity, then the equation $B(z) = 0$ $\left(\bar{B}(z) = 0\right)$ has real roots only (for $p \leqq n$).

Cor. III. If the equation $K(y) = 0$ $\left(\text{or } \bar{K}(y) = 0\right)$ has roots greater (less) than unity only, then the roots of the equation $B(z) = 0$ $\left(\bar{B}(z) = 0\right)$ are greater or less than a according as the roots of the equation $f(x) = 0$ are less or greater (greater or less) than a (for $p \leqq n$).

Cor. IV.[1] If the roots of the equation $\bar{K}(y) = 0$ are all greater than unity or all negative (or all between zero and unity) and the equation $f(z) = 0$ has all its roots of the sign different from (same with) that of a, then the roots of the equation $\bar{B}(z) = 0$ are of the same (opposite) sign with a. $(p \leqq n)$

Proof: $\xi_{ij} = ay_i + (1-y_i)z_j$

Cor. V. If the roots of the equation $\bar{K}(y) = 0$ are all positive (or negative), then the minimum root of the equation $\bar{B}(z) = 0$ is greater than that of the equation $f(z) = 0$, or the maximum root of the equation $\bar{B}(z) = 0$ is less than that of the equation $f(z) = 0$ according as the roots of the equation $f(z) = 0$ are all less or greater (greater or less) than a.

Proof. Since $\xi_{ij} = (a-z_j)y_i + z_j$, we can conclude this from Theorem III′.

Specially from the above Cor. V, we have

Cor. VI. If $\bar{K}(y) = 0$ has positive (or negative) roots only, then the equation $\bar{B}(z) = 0$ has positive or negative roots only according as the equation $a(z) = 0$ has those roots only which are positive and less (greater) than a or negative and greater (less) than a. $(p \leqq n)$

III.

Putting $k_i = \dfrac{h_i}{a^i}$ in $\bar{K}(z) = 0$, $\bar{B}(z) = 0$ respectively, we have

$$(ay)^p + nh_1(ay)^{p-1} + \ldots\ldots + n(n-1)\ldots\ldots(n-p+1)h_p = 0$$

(1) For the case where the roots of $\bar{K}(y) = 0$ are greater than unity, we can conclude, if we use Theorem III, this Cor. only for $p = n$; but if we use Theorem III′, we can obtain this result for $p \leqq n$.

and $$f(z)+h_1\frac{f_1(z, \alpha)}{\alpha}+h_2\frac{f_2(z, \alpha)}{\alpha^2}+\dots\dots+h_p\frac{f_p(z, \alpha)}{\alpha^p}=0\,.$$

If η is the root of the equation

$$y^p+nh_1y^{p-1}+n(n-1)h_2y^{p-1}+\dots\dots+n(n-1)\dots\dots(n-p+1)h_p=0$$

we have

$$\begin{aligned} w &= \alpha+(1-y_i)(z-\alpha) \\ &= \alpha+z-\alpha-\frac{\eta_i}{\alpha}z+\eta_i \\ &= z+\eta_i-\frac{z\eta_i}{\alpha}\,. \end{aligned}$$

Now let $\alpha\to\infty$, considering the relation

$$\lim_{\alpha\to\infty}\frac{(\alpha-z)^p f^{(p)}(z)}{\alpha^p}=f^{(p)}(z) \qquad \text{or} \qquad \lim_{\alpha\to\infty}\frac{f_p(z, \alpha)}{\alpha^p}=f^{(p)}(z)\,,$$

we can obtain several theorems from the results enunciated in II.

Theorem I.(1) Let z be a suitable point in a circle C which comprises all the roots of the given equation $f(z)=0$ and y_i be one of roots of the equation $K(y)=0$, then any root w of the equation

$$C(z)\equiv f(z)+k_1f'(z)+k_2f''(z)+\dots\dots+k_nf^{(n)}(z)$$

is expressed in the form

$$w=z+y_i\,.$$

Or more precisely, we have the

Theorem II. Let C_i be the circle into which C is translated by $w=z+y_i$ when z describes C, and those C's be classed into sets of circles

$$\Gamma_1(C_1,\dots\dots,C_{\nu_1-1})\,,\qquad \Gamma_2(C_{\nu_1},\dots\dots,C_{\nu_2-1}),\dots\dots,\Gamma_k(C_{\nu_{k-1}},\dots\dots,C_n)$$

---

(1) What corresponds to theorem I′ becomes identical with theorem I. (Cf. Y. Uchida, On the Roots of the algebraic equation of the form $f+k_1f'+\dots\dots+k_nf^{(n)}=0$. Tôhoku Math. Journ. **14** (1918), 325.)

S. Kakeya, On algebraic equations having the roots of limited magnitude. Proc. Phys.-Math. Soc. of Japan, **3** (1921), 99.

J. L. Walsch, On the location of the roots of certain types of polynomials. Trans. Amer. Math. Soc. **24** (1922), 163–180.

so that the C's in each Γ_i form a simply connected domain Γ_i while all the Γ's are discreted with each other, then the domain Γ_i has as many roots of the equation $C(z) = 0$ as the number of circles C_i contained in Γ_i.

Cor. I. If all of C's are discreted with each other, then the equation $C(z) = 0$ has one root in each circle C_i respectively.

Cor. II. The equation

$$\overline{C}(z) \equiv f(z) + k_1 f'(z) + \dots + k_p f^{(p)}(z) = 0 \qquad (p \leqq n)$$

has $(n-p)$ roots in C and one root in each circle C_i respectively, whenever C's, which are obtained from C corresponding to the roots of the equation $\overline{K}(y) = 0$, are discrete with each other.

Theorem III. If y_i and z_j are the roots of the equations $K(y) = 0$ and $f(z) = 0$ respectively, then the equation $C(z) = 0$ has all the roots in the smallest convex polygon which covers n^2 points $\xi_{ij} = y_i + z_j$, that is to say, in the sum of two polygon covering y's and z's respectively.

Cor. I.(1) The equation $\overline{C}(z) = 0$ has all its roots in the smallest convex polygon which contains z_j and np points $\xi_{ij} = y_i + z_j$ where y_i and z_j are the roots of the equations $\overline{K}(y) = 0$ and $f(z) = 0$ respectively, in other words, they are in the sum of two convex polygon covering $(0, y_1, \dots, y_p)$ and $(z_1, \dots, z_n)$ respectively.

Cor. II.(2) If both the equations $f(z) = 0$ and $\overline{K}(y) = 0$ have real roots only, then also the equation $\overline{C}(z) = 0$ has real roots only $(p \leqq n)$.

Cor. III. If both the equations $f(z) = 0$ and $\overline{K}(y) = 0$ have positive (negative) roots only, then the equation $\overline{C}(z) = 0$ has also positive (negative) roots only $(p \leqq n)$.

---

(1) M. Fujiwara, Einige Bemerkungen über die elementare Theorie der algebraischen Gleichungen. Tôhoku Math. Journ. **9** (1916), 104.

(2) K. Ôisi, On the roots of an algebraic equation $f + k_1 f' + k_2 f'' + \dots + k_n f^{(k)} = 0$. Tôhoku Math. Journ. **20** (1921), 1–17.