Indices of the Orthogonal Systems in the Non-Separable Hilbert Space.

By

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Let $\mathfrak{H}$ be an abstract Hilbert space; that is, $\mathfrak{H}$ is a linear vector space where the inner product is defined, and it is complete. When $\mathfrak{H}$ is separable, we usually use the natural number for the index of the orthogonal system of elements in $\mathfrak{H}$, i. e. $\{\mathfrak{g}_n\}$, n being natural numbers, and

$$(\mathfrak{g}_m, \mathfrak{g}_n) = \delta_{mn} ,$$

where $\delta_{mn} = 0$ when $m \neq n$, and $= 1$ when $m = n$. When $\{\mathfrak{g}_n\}$ is complete in $\mathfrak{H}$, $\{\mathfrak{g}_n\}$ is used as a basic of the representation of $\mathfrak{H}$. Put

$$a_n = (\mathfrak{f}, \mathfrak{g}_n) ,$$

then the sequence $\{a_n\}$ of complex numbers is the representative of $\mathfrak{f}$.

It is convenient in the theory of quantum mechanics to use a representative in which the elements of the basic system $\{\mathfrak{g}_n\}$ are eigenvectors of a self-adjoint operator. But the orthogonal system $\{\mathfrak{g}_n\}$ whose index is the natural number is applicable only in the case when the chosen self-adjoint operator has the discrete spectrum. When the chosen self-adjoint operator has the continuous spectrum, its eigenvectors have as index the real number, so that it is expressed by $\{\mathfrak{g}_r\}$ and it satisfies the following condition

$$(\mathfrak{g}_r, \mathfrak{g}_s) = \delta(r-s) ,$$

where $\delta(x)$ is the Dirac improper δ function, defined by

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\delta(x) = 0 \qquad \text{for } x \neq 0 .$$

In this case the representative of any element $\mathfrak{f}$ is a point function

$$\mathfrak{f}(r) = (\mathfrak{f}, \mathfrak{g}_r) .$$

Since the use of the improper δ function is not desirable from the standpoint of pure mathematics, I introduced in my previous paper,(1) the orthogonal system of elements $\{\mathfrak{q}(U)\}$ whose index is the set U, which I have called the vector valued differential set function. $\mathfrak{q}(U)$ satisfies the following conditions,

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = 0 \qquad \text{when } UU' = 0$$

and

$$\mathfrak{q}(U)\,[=]\sum_i \mathfrak{q}(U_i)^{(2)} \qquad \text{when } U = \sum_i U_i .$$

When $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, the representative of any element $\mathfrak{f}$ is the differential set function

$$\xi(U) = \big(\mathfrak{f}, \mathfrak{q}(U)\big) .$$

Thus in the case of the separable Hilbert space, we have used two kinds of indices, i. e. natural numbers and system of sets. But the first kind is the special case of the second kind. In the present paper I discuss the system of sets as the indices of orthogonal systems in the non-separable Hilbert space. First, introducing the non-enumerable differential set system, I extend the conception of the differential set function. Next I consider two kinds of orthogonal systems, i. e. that of elements and that of closed linear manifolds, and I discuss the relation between them. The resolution of identity is newly defined as the projections on the closed linear manifolds which compose an orthogonal system.

Differential Set Systems in the Extended Sense.

1. Let $\mathfrak{M}$ be a multiplicative system of sets in an abstract space Ω. That is, the product of any two sets U and U' belongs to $\mathfrak{M}$ with U and U'. Now, assume that $\mathfrak{M}$ contains Ω itself. Let V be a set in $\mathfrak{M}$. The decomposition of V in a sum of *finite* or *enumerably infinite* disjoint sets $\{U_n\}$ belonging to $\mathfrak{M}$:

$$V = \sum_n U_n ,$$

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(1) F. Maeda, "Representations of Linear Operators by Differential Set Functions," this journal, **6** (1936), 117.

(2) [=] means the strong convergence of the series.

is expressed by $\mathfrak{D}V \equiv \sum_n U_n,$

and the sets U_n are called the elements of the decomposition $\mathfrak{D}$. Let

$$\mathfrak{D}'V \equiv \sum_m U'_m$$

be another decomposition of V, such that U'_m is a subset of any one of the elements U_n of $\mathfrak{D}$; then we say that $\mathfrak{D}'$ is an extension of $\mathfrak{D}$. Denote by $\mathfrak{M}\mathfrak{D}V$ the aggregate of the elements of all extensions of a decomposition $\mathfrak{D}V$. I said in a previous paper(1) that $\mathfrak{M}\mathfrak{D}V$ is a *differential set system* in V. Denote by $\mathfrak{M}V$ the aggregate of the elements of all decompositions of V. Then $\mathfrak{M}V$ is also a differential set system in V, and $\mathfrak{M}V$ contains V itself.

Now consider the decomposition of Ω in a sum of *non-enumerably infinite* disjoint sets $\{V_\alpha\}$ ($\alpha \in \mathfrak{A}$, $\mathfrak{A}$ being the aggregate of indices α) belonging to $\mathfrak{M}$

$$\Omega = \sum_{\alpha\in\mathfrak{A}} V_\alpha,$$

and denote it by $\mathfrak{Z}\Omega \equiv \sum_{\alpha\in\mathfrak{A}} V_\alpha.$

And construct differential set systems $\mathfrak{M}V_\alpha$ for all V_α. Denote by $\mathfrak{M}\mathfrak{Z}\Omega$ the aggregate of all the elements of $\mathfrak{M}V_\alpha$ where α runs over $\mathfrak{A}$. We say also that $\mathfrak{M}\mathfrak{Z}\Omega$ is a *differential set system* in Ω. If necessary, we say that $\mathfrak{M}\mathfrak{Z}\Omega$ is a *non-enumerable* differential set system in Ω, and $\mathfrak{M}\mathfrak{D}\Omega$ is an *enumerable* differential set system.

Space of Differential Set Functions in the Extended Sense.

2. In order to investigate the properties of the orthogonal system whose indices are sets in $\mathfrak{M}\mathfrak{Z}\Omega$, we must first consider the space of differential set function in the extended sense.(2)

If, for any set U of $\mathfrak{M}\mathfrak{Z}\Omega$, a complex valued function $\xi(U)$ is defined, and

$$\xi(U) = \sum_n \xi(U_n)$$

for any decomposition of U in $\mathfrak{M}V_\alpha$ (for any $\alpha \in \mathfrak{A}$)

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(1) F. Maeda, "Space of Differential Set Functions," this journal, **6** (1936), 21.

(2) For the space of differential set functions in the ordinary sense, cf. F. Maeda, loc. cit., this journal, **6** (1936), 23–31.

$$U = \sum_n U_n ,$$

then we say that $\xi(U)$ is a *completely additive differential set function* defined in $\mathfrak{M}\mathfrak{Z}\Omega$.

Let $\sigma(U)$ be another completely additive, non-negative differential set function defined in $\mathfrak{M}\mathfrak{Z}\Omega$. If $\xi(U) = 0$ for all sets in $\mathfrak{M}\mathfrak{Z}\Omega$, where $\sigma(U) = 0$; then we say that $\xi(U)$ is *absolutely continuous* with respect to $\sigma(U)$.

Let $\xi(U)$ be absolutely continuous with respect to $\sigma(U)$. We can not define the integral $\int_\Omega \frac{|\xi(dU)|^2}{\sigma(dU)}$ by Kolmogoroff's method,[1] since the decomposition of Ω is always non-enumerable. Hence we give to $\xi(U)$ the following restriction:

$$\xi(U) = 0 \qquad \text{for all } U \text{ in } \mathfrak{M}V_\alpha,$$

except an enumerable system of α, i.e. $\{\alpha_i\}$. And we define $\int_\Omega \frac{|\xi(dU)|^2}{\sigma(dU)}$ by $\sum_{i=1}^\infty \int_{V\alpha_i} \frac{|\xi(dU)|^2}{\sigma(dU)}$, when all $\int_{V\alpha_i} \frac{|\xi(dU)|^2}{\sigma(dU)}$ exist and $\sum_{i=1}^\infty \int_{V\alpha_i} \frac{|\xi(dU)|^2}{\sigma(dU)}$ converges to a finite value. And we denote by $\dot{\mathfrak{L}}_2(\sigma)$, the aggregate of all $\xi(U)$ such that $\int_\Omega \frac{|\xi(dU)|^2}{\sigma(dU)}$ exist, and write

$$\|\xi\| = \left[\int_\Omega \frac{|\xi(dU)|^2}{\sigma(dU)}\right]^{\frac{1}{2}}.$$

Denote by $\xi_\alpha(U)$, the completely additive differential set function defined only for the sets U in $\mathfrak{M}V_\alpha$, and

$$\xi_\alpha(U) = \xi(U) \qquad \text{for all } U \in \mathfrak{M}V_\alpha ;$$

and call $\xi_\alpha(U)$ the *component* of $\xi(U)$. Similarly we define the component $\sigma_\alpha(U)$ of $\sigma(U)$. Then, since

$$\int_{V\alpha} \frac{|\xi_\alpha(dU)|^2}{\sigma_\alpha(dU)} = \int_{V\alpha} \frac{|\xi(dU)|^2}{\sigma(dU)} ,$$

we have the following theorem:

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(1) A. Kolmogoroff, Math. Ann. **103** (1930), 676.

In order that $\xi(U) \in \mathfrak{L}_2(\sigma)$, *it is necessary and sufficient that*

(i) $\xi_a(U) \in \mathfrak{L}_2(\sigma_a)$ *for all* $a \in \mathfrak{A}$,

(ii) $\| \xi_a \| = 0$ *for all* $a \in \mathfrak{A}$, *except an enumerable system of* a,

(iii) $\sum_{a \in \mathfrak{A}} \| \xi_a \|^2$ (1) *is finite.*

And in this case

$$\sum_{a \in \mathfrak{A}} \| \xi_a \|^2 = \| \xi \|^2 .$$

From this theorem, we can easily prove that $\mathfrak{L}_2(\sigma)$ is linear.

Let $\xi(U)$ and $\eta(U)$ be two set functions in $\mathfrak{L}_2(\sigma)$. Then, since

$$\sum_{a \in \mathfrak{A}} \int_{V_a} \frac{\xi_a(dU)\overline{\eta_a(dU)}}{\sigma_a(dU)} = \sum_{a \in \mathfrak{A}} (\xi_a, \eta_a)$$

is finite, we denote this value by $\int_{\Omega} \frac{\xi(dU)\overline{\eta(dU)}}{\sigma(dU)}$, and define the inner product (ξ, η) by

$$(\xi, \eta) = \int_{\Omega} \frac{\xi(dU)\overline{\eta(dU)}}{\sigma(dU)} .$$

The completeness of $\mathfrak{L}_2(\sigma)$ can also be proved by modifying the proof of the completeness of the class of all sequences of complex numbers $\{a_n\}$ such that $\sum_n | a_n |^2$ converges.(2)

Thus $\mathfrak{L}_2(\sigma)$ is a Hilbert space.

Orthogonal System of Elements.

3. Let $\{\mathfrak{q}(U)\}$ be a system of elements in $\mathfrak{H}$, whose index is the set U in $\mathfrak{M}\mathfrak{Z}\Omega$. We may assume that $\mathfrak{q}(U)$ is defined for all U in $\mathfrak{M}\mathfrak{Z}\Omega$. For, if not, let $\mathfrak{q}(U) = 0$ for any undefined U. When $\{\mathfrak{q}(U)\}$ satisfies the following conditions, we say that $\{\mathfrak{q}(U)\}$ is an *orthogonal system of elements.*(3)

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(1) $\sum_{a \in \mathfrak{A}}$ is the summation of the terms which are not zero.

(2) Cf. M. H. Stone, *Linear Transformations in Hilbert Space,* (1932), 15. In the proof of the completeness of $\mathfrak{L}_2(\beta)$ in my previous paper, loc. cit., this journal, **6** (1936), 30, (2) is incorrect. (3) is obtained as follows: As sec. 3 (3) we have $| \sum_{i=1}^{k} \xi_m(E_i) - \sum_{i=1}^{k} \xi_n(E_i) |^2 \leqq \beta(E)\, \varepsilon^2$ when $m, n > N$ for any k. Let $m \to \infty$ then we have (3). The right hands of the inequalities (3), (4) and (5) must be written $\varepsilon \sqrt{\beta(E)}$.

(3) The simplest example of this is the normalized orthogonal system $\{\mathfrak{g}_a\}$ whose index is the transfinite ordinal number a. In this case V_a is the set which has only one element $\mathfrak{a}$.

(i) $\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = 0$ when $UU' = 0$,

(ii) $\mathfrak{q}(U) [=] \sum_n \mathfrak{q}(U_n)$

for any decomposition of U in $\mathfrak{M}V_a$ (for any $a \in \mathfrak{A}$)

$$U = \sum_n U_n .$$

Thus $\mathfrak{q}(U)$ is nothing but the completely additive vector valued differential set function which was investigated in a previous paper,[1] the domain of the definition being extended. If we put $\sigma(U) = \| \mathfrak{q}(U) \|^2$, then it is evident that $\sigma(U)$ is a completely additive, non-negative differential set function defined in $\mathfrak{M}\mathfrak{Z}\Omega$.

Now *we have the relation*

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = \sigma(UU') .$$

To prove this, when $U, U' \in \mathfrak{M}V_a$, consider two decompositions of V_a such that

$$V_a = U + \sum_n U_n , \qquad V_a = U' + \sum_m U'_m .$$

Then $\qquad U' = UU' + \sum_n U_n U' , \qquad U = UU' + \sum_m UU'_m ,$

Hence by (ii)

$$\mathfrak{q}(U') [=] \mathfrak{q}(UU') + \sum_n \mathfrak{q}(U_n U') , \qquad \mathfrak{q}(U) [=] \mathfrak{q}(UU') + \sum_m \mathfrak{q}(UU'_m) .$$

Therefore by (i), we have

$$\big(\mathfrak{q}(U), \mathfrak{q}(U')\big) = \big(\mathfrak{q}(UU'), \mathfrak{q}(UU')\big) = \sigma(UU') .$$

Denote by $\mathfrak{q}_a(U)$, the completely additive vector valued differential set function defined only for the sets U in $\mathfrak{M}V_a$, and

$$\mathfrak{q}_a(U) = \mathfrak{q}(U) \qquad \text{for all } U \in \mathfrak{M}V_a ,$$

and call $\mathfrak{q}_a(U)$ the *component* of $\mathfrak{q}(U)$.

Let $\xi(U)$ be a completely additive differential set function, which is absolutely continuous with respect to $\sigma(U)$. If $\xi_a(U) = 0$ for all $U \in \mathfrak{M}V_a$, except an enumerable system $\{a_i\}$, and $\sum_i \int_{V_a} \frac{\xi_{a_i}(dU)\mathfrak{q}_{a_i}(dU)}{\sigma_{a_i}(dU)}$ converges strongly to an element $\mathfrak{f}$ of $\mathfrak{H}$, then we say that the integral

(1) F. Maeda, loc. cit., this journal, **6** (1936), 33.

$\int_{\Omega} \frac{\mathfrak{x}(dU)\mathfrak{q}(dU)}{\sigma(dU)}$ exists, and its value is $\mathfrak{f}$. Since $\int_{V\alpha} \frac{\mathfrak{x}_\alpha(dU)\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}$ exists when and only when $\mathfrak{x}_\alpha(U)$ belongs to $\mathfrak{L}_2(\sigma_\alpha)$,[1] from sec. 3, $\int_{\Omega} \frac{\mathfrak{x}(dU)\mathfrak{q}(dU)}{\sigma(dU)}$ exists when and only when $\mathfrak{x}(U)$ belongs to $\mathfrak{L}_2(\sigma)$. And we can easily prove that when

$$\mathfrak{f} = \int_{\Omega} \frac{\mathfrak{x}(dU)\mathfrak{q}(dU)}{\sigma(dU)}, \qquad \mathfrak{g} = \int_{\Omega} \frac{\eta(dU)\mathfrak{q}(dU)}{\sigma(dU)},$$

then

$$(\mathfrak{f}, \mathfrak{g}) = (\mathfrak{x}, \eta).$$

We say that the orthogonal system $\{\mathfrak{q}(U)\}$ is *complete* in $\mathfrak{H}$, when there exists no element, except null, which is orthogonal to all $\mathfrak{q}(U)$ $(U \in \mathfrak{M}\mathfrak{Z}\Omega)$. Let $\mathfrak{M}(\mathfrak{q}_\alpha)$ be the closed linear manifold determined by the system $\{\mathfrak{q}_\alpha(U)\}$, U being any set in $\mathfrak{M}V_\alpha$. Then

$$\mathfrak{M}(\mathfrak{q}_\alpha) \perp \mathfrak{M}(\mathfrak{q}_\beta)^{(2)} \qquad \text{for } \alpha \neq \beta,$$

and when $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$

$$\mathfrak{H} = \sum_{\alpha \in \mathfrak{A}} \left(\mathfrak{M}(\mathfrak{q}_\alpha);\ \oplus\right).^{(3)} \tag{1}$$

Now assume that $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$. Let $\mathfrak{f}$ be any element in $\mathfrak{H}$, and denote bv $\mathfrak{f}_\alpha$ the projection of $\mathfrak{f}$ on $\mathfrak{M}(\mathfrak{q}_\alpha)$. Then

$$\sum_\nu \| \mathfrak{f}_{\alpha_\nu} \|^2 \leqq \| \mathfrak{f} \|^2$$

for any enumerable sequence $\{\alpha_\nu\}$. Hence

$$\mathfrak{f}_\alpha = 0$$

except an enumerable system of α. And from (1) we have

$$\mathfrak{f} [=] \sum_{\alpha \in \mathfrak{A}} \mathfrak{f}_\alpha . \tag{2}$$

The representative of $\mathfrak{f}$ with respect to $\{\mathfrak{q}(U)\}$ is

$$\mathfrak{x}(U) = \left(\mathfrak{f}, \mathfrak{q}(U)\right),$$

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(1) F. Maeda, loc. cit., this journal, **6** (1936), 35.

(2) This means that $\mathfrak{M}(\mathfrak{q}_\alpha)$ is orthogonal to $\mathfrak{M}(\mathfrak{q}_\beta)$.

(3) This means that $\mathfrak{H}$ is the closed linear manifold determined by the elements of $\mathfrak{M}(\mathfrak{q}_\alpha)$ for all $\alpha \in \mathfrak{A}$.

and the components of $\mathfrak{x}(U)$ are

$$\mathfrak{x}_\alpha(U) = \big(\mathfrak{f}, \mathfrak{q}_\alpha(U)\big) = \big(\mathfrak{f}_\alpha, \mathfrak{q}_\alpha(U)\big).$$

Hence $\mathfrak{x}_\alpha(U) \in \mathfrak{L}_2(\sigma_\alpha)$ and $\| \mathfrak{x}_\alpha \| = \| \mathfrak{f}_\alpha \|$.[1]

Therefore $\mathfrak{x}(U)$ belongs to $\mathfrak{L}_2(\sigma)$ and since $\mathfrak{f}_\alpha = \int_{V\alpha} \frac{\mathfrak{x}_\alpha(dU)\mathfrak{q}_\alpha(dU)}{\sigma_\alpha(dU)}$, by (2) we have the relation

$$\mathfrak{f} = \int_\Omega \frac{\mathfrak{x}(dU)\mathfrak{q}(dU)}{\sigma(dU)}.$$

This is the expansion of $\mathfrak{f}$ with respect to the orthogonal system $\{\mathfrak{q}(U)\}$.

In general, the orthogonal system is not composed of one vector valued differential set function; for example, it is composed of $\{\mathfrak{q}_\beta(U)\}$ $(\beta \in \mathfrak{B})$, such that all $\mathfrak{q}_\beta(U)$ are defined in the same $\mathfrak{M}\mathfrak{Z}\Omega$ and $\mathfrak{q}_\beta(U)$ and $\mathfrak{q}_{\beta'}(U)$ are orthogonal for $\beta \neq \beta'$. $\mathfrak{B}$ is the set of indices which may be non-enumerably infinite. To treat such an orthogonal system is very inconvenient. But *we can reduce such an orthogonal systsm to an orthogonal system which is composed of only one vector valued differential set function.* For this purpose, we label the space Ω and the differential set system $\mathfrak{M}\mathfrak{Z}\Omega$, so that $\mathfrak{q}_\beta(U)$ is defined in $\mathfrak{M}_\beta\mathfrak{Z}_\beta\Omega_\beta$, which is the aggregate of all elements of $\mathfrak{M}_\beta V_{\beta\alpha}$, for all $\alpha \in \mathfrak{A}$. Put

$$\Omega_0 = \sum_{\substack{\alpha \in \mathfrak{A} \\ \beta \in \mathfrak{B}}} V_{\beta\alpha},$$

and denote by $\mathfrak{M}_0\mathfrak{Z}_0\Omega_0$ the aggregate of all elements of $\mathfrak{M}_\beta V_{\beta\alpha}$ for all $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$. Then $\mathfrak{M}_0\mathfrak{Z}_0\Omega_0$ is a differential set system, and we have a completely additive vector valued differential set function $\mathfrak{q}(U)$ defined in $\mathfrak{M}_0\mathfrak{Z}_0\Omega_0$, which has $\mathfrak{q}_{\beta\alpha}(U)$ for its components. Then $\{\mathfrak{q}(U)\}$ is the required complete orthogonal system.

The possibility of this procedure is due to the fact that the domain of the index U is the *differential set system.* When $\mathfrak{H}$ is separable, it is enough to treat only the *enumerable* differential set system. But when $\mathfrak{H}$ is non-separable, we must introduce the *non-enumerable* differential set system.

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(1) F. Maeda, loc. cit., this journal, **6** (1936), 35–37.

Orthogonal System of Closed Linear Manifolds.

4. Let $\{\mathfrak{M}_U\}$ be a system of closed linear manifolds whose index U is the set in $\mathfrak{M}\mathfrak{Z}\Omega$. We may assume that $\mathfrak{M}_U$ is defined for all sets U in $\mathfrak{M}\mathfrak{Z}\Omega$. For, if not, let $\mathfrak{M}_U$ be a closed linear manifold composed of only the null element. When $\{\mathfrak{M}_U\}$ satisfies the following conditions, then we say that $\{\mathfrak{M}_U\}$ is an *orthogonal system of closed linear manifolds.*[1]

(α) $\mathfrak{M}_U \perp \mathfrak{M}_{U'}$ when $UU' = 0$,

(β) $\mathfrak{M}_U = \sum_n (\mathfrak{M}_{U_n};\ \oplus)$

for any decomposition of U in $\mathfrak{M}V_a$ (for any $a \in \mathfrak{A}$)

$$U = \sum_n U_n .$$

When an orthogonal system $\{\mathfrak{M}_U\}$ satisfies the following condition

(γ) $\mathfrak{H} = \sum_{a \in \mathfrak{A}} (\mathfrak{M}_{V_a};\ \oplus)$,

then we say that $\{\mathfrak{M}_U\}$ is *complete* in $\mathfrak{H}$.

When $\{\mathfrak{M}_U\}$ is a complete orthogonal system in $\mathfrak{H}$, the projecting operator $E(U)$ on $\mathfrak{M}_U$ ($U \in \mathfrak{M}\mathfrak{Z}\Omega$) is called a *resolution of identity.*

From (α) and (β) we have

(α') $E(U)E(U') = 0$ when $UU' = 0$,

(β') $E(U) = \sum_n E(U_n)$ when $U = \sum_n U_n$.

We can replace the relation (α') by the following:

(α'') $E(U)E(U') = E(UU')$.

For when $U, U' \in \mathfrak{M}V_a$, consider two decompositions of V_a such that

$$V_a = U + \sum_n U_n , \qquad V_a = U' + \sum_m U'_m .$$

Then $$U' = UU' + \sum_n U_n U' , \qquad U = UU' + \sum_m UU'_m .$$

Hence by (β')

$$E(U') = E(UU') + \sum_n E(U_n U') , \qquad E(U) = E(UU') + \sum_m E(UU'_m) .$$

Therefore by (α'), we have

(1) K. Friedrichs considered the orthogonal system of eigenmanifolds whose index is the interval. Jahrber. Deutch. Math. Ver., **45** (1935), 2. Abt. 81.

$$E(U)E(U') = E(UU')E(UU') = E(UU').$$

Thus the definition of the resolution of identity in the present paper is a generalization of that of the preceding paper,(1) in which the resolution of identity $E(U)$ is defined as a self-adjoint operator which satisfies (α''), (β') and

(γ') $E(\Omega) = \mathbf{1}$,

the index U being the set in $\mathfrak{M}\Omega$.

If $\mathfrak{q}(U)$ is a completely additive vector valued differential set function defined in $\mathfrak{M}\mathfrak{Z}\Omega$, which satisfies

$$E(U')\mathfrak{q}(U) = \mathfrak{q}(U'U),$$

then I say that $\mathfrak{q}(U)$ is *generated* by $E(U)$.(2) Evidently all the vector valued differential set functions of the form

$$\mathfrak{q}(U) = E(U)\mathfrak{f}$$

are generated by $E(U)$.

Relation between the Orthogonal System of Elements and That of Closed Linear Manifolds.

5. Let $\{\mathfrak{q}(U)\}$ be a complete orthogonal system of elements in $\mathfrak{H}$. For a definite set U in $\mathfrak{M}\mathfrak{Z}\Omega$, define $\mathfrak{M}_U$ by the closed linear manifold determined by all $\mathfrak{q}(E)$, where E runs over all the subsets of U, which belong to $\mathfrak{M}\mathfrak{Z}\Omega$. I will show that *$\{\mathfrak{M}_U\}$ is a complete orthogonal system of closed linear manifolds in $\mathfrak{H}$.*

First the condition (α):

$$\mathfrak{M}_U \perp \mathfrak{M}_{U'} \qquad \text{when } UU' = 0,$$

is evident.

Next, let $\mathfrak{f}$ be any element in $\mathfrak{M}_U$, and put

$$\xi(E) = \left(\mathfrak{f}, \mathfrak{q}(E)\right).$$

Since $\mathfrak{f}$ is orthogonal to every $\mathfrak{q}(E)$ such that E is disjoint to U,

$$\xi(E) = 0 \qquad \text{when } EU = 0.$$

(1) F. Maeda, loc. cit., this journal, **6** (1936), 38.
(2) Ibid., 38.

Therefore, when $U=\sum_n U_n$,

$$\mathfrak{f}=\int_\Omega \frac{\xi(dE)\mathfrak{q}(dE)}{\sigma(dE)}=\int_U \frac{\xi(dE)\mathfrak{q}(dE)}{\sigma(dE)}=\sum_n \int_{U_n} \frac{\xi(dE)\mathfrak{q}(dE)}{\sigma(dE)}.$$

Since each element of the summation belongs to $\mathfrak{M}_{U_n}$ respectively, we have

$$\mathfrak{M}_U \leqq \sum_n (\mathfrak{M}_{U_n};\oplus).$$

Next, let $\mathfrak{f}$ be any element in $\sum_n (\mathfrak{M}_{U_n};\oplus)$, and put

$$\xi(E)=\big(\mathfrak{f},\mathfrak{q}(E)\big).$$

Since $\mathfrak{f}$ is orthogonal to every $\mathfrak{q}(E)$ such that E is disjoint to all U_n $(n=1,2,\dots.)$, that is, disjoint to $U=\sum_n U_n$,

$$\xi(E)=0 \qquad \text{when } EU=0.$$

Hence $$\mathfrak{f}=\int_\Omega \frac{\xi(dE)\mathfrak{q}(dE)}{\sigma(dE)}=\int_U \frac{\xi(dE)\mathfrak{q}(dE)}{\sigma(dE)}.$$

That is $\mathfrak{f}\in\mathfrak{M}_U$. Consequently

$$\sum_n (\mathfrak{M}_{U_n};\oplus)\leqq \mathfrak{M}_U.$$

Thus the condition (β) is proved.

Since $\{\mathfrak{q}(U)\}$ is complete in $\mathfrak{H}$, by the definition of sec. 3,

$$\mathfrak{H}=\sum_{a\in\mathfrak{A}}\big(\mathfrak{M}(\mathfrak{q}_a);\oplus\big).$$

But $\mathfrak{M}(\mathfrak{q}_a)=\mathfrak{M}_{V_a}$. Hence the condition (γ) holds.

Thus we have constructed $\{\mathfrak{M}_U\}$ from $\{\mathfrak{q}(U)\}$. Let $E(U)$ be the projecting operator on $\mathfrak{M}_U$. Then $\mathfrak{q}(U)$ *is generated by* $E(U)$. For, it is evident that for a definite set U,

(i) $$E(U)\mathfrak{q}(E)=\mathfrak{q}(E) \qquad \text{when } E\leqq U,$$

(ii) $$E(U)\mathfrak{q}(E)=0 \qquad \text{when } EU=0.$$

Hence when $U\in\mathfrak{M}V_a$, it is sufficient to consider the case when $E\in\mathfrak{M}V_a$. Consider the decomposition

$$V_a=U+\sum_n U_n,$$

then
$$E = EU + \sum_n EU_n .$$
Hence
$$\mathfrak{q}(E)\,[=]\,\mathfrak{q}(EU) + \sum_n \mathfrak{q}(EU_n) .$$
Consequently from (i) and (ii)
$$E(U)\mathfrak{q}(E) = \mathfrak{q}(EU) .$$

The above consideration may also be stated as follows:

When $\{\mathfrak{q}(U)\}$ *is complete in* $\mathfrak{H}$, *there exists a resolution of identity which generates* $\mathfrak{q}(U)$.

This is already proved in my previous paper,(1) not using the orthogonal system $\{\mathfrak{M}_U\}$.

The above method can also be applied with slight modification to the case where $\{\mathfrak{q}_\beta(U)\}$ $(\beta \in \mathfrak{B})$ is complete in $\mathfrak{H}$.

6. Now the converse problem. Let $\{\mathfrak{M}_U\}$ be a complete orthogonal system of closed linear manifolds in $\mathfrak{H}$, and $E(U)$ be the corresponding resolution of identity. Then by the following method, we can find an aggregate $\mathfrak{P}$ of elements in $\mathfrak{H}$, such that $\{\mathfrak{q}_\mathfrak{b}(U)\}$ $(\mathfrak{b} \in \mathfrak{P})$ is a complete orthogonal system in $\mathfrak{H}$, where
$$\mathfrak{q}_\mathfrak{b}(U) = E(U)\mathfrak{b} .$$

Give to $\mathfrak{H}$ a normal order-type (Wohlordnung). We find the elements of $\mathfrak{P}$ by the transfinite induction as follows: $\mathfrak{b}$ belongs to $\mathfrak{P}$ when and only when $\|\mathfrak{b}\| > 0$ and $\mathfrak{b}$ is orthogonal to $\mathfrak{M}(\mathfrak{q}_\mathfrak{a})$ for all $\mathfrak{a}(\in \mathfrak{P})$ which have lower ranks than $\mathfrak{b}$ in the normal order-type. Since
$$\big(E(U')\mathfrak{a}, E(U)\mathfrak{b}\big) = \big(E(UU')\mathfrak{a}, \mathfrak{b}\big) = 0 .$$
$\{\mathfrak{q}_\mathfrak{b}(U)\}$ is an orthogonal system. The completeness is evident.

Thus *we can find a complete orthogonal system of elements* $\{\mathfrak{q}_\beta(U)\}$ $(\beta \in \mathfrak{B})$ *in* $\mathfrak{H}$, *such that* $\mathfrak{q}_\beta(U)$ *are generated by* $E(U)$. But such orthogonal systems are not unique.

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(1) F. Maeda, "Theory of Vector Valued Set Functions," this journal, **4** (1934), 81.