**Wave Geometry No. 35**

# Cosmology in Terms of Wave Geometry (V) Universe with Born-Type Electromagnetism.

By

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## § 1. Introduction.

In wave geometry it has been shown that the equation of motion of a particle is given by[1]

[I] in space with matter only

$$u^i \nabla_i u^j = Q u^j \tag{1.1}$$

where $u^j \equiv \psi^\dagger A \gamma^j \psi$,

and

[II] in space with Born-type electromagnetism[2]

$$u^i \nabla_i u^j = 2(M \overset{1}{F}{}_i^{\cdot j} + N \overset{2}{F}{}_i^{\cdot j}) u^i + Q u^j \tag{1.2}$$

where $\overset{1}{F}_{ij}$ and $\overset{2}{F}_{ij}$ are antisymmetric tensors.

Cosmology in terms of wave geometry has been developed on the assumption that the assemblage of nebulae is the universe and each nebula moves along the path determined by (1.1) But if, besides nebulae, we take radiation as constituent of the universe, the equation of motion of a particle might be (1.2) instead of (1.1). Hence, for the further development of wave geometry, it is worth while to investigate the space in which a free particle moves along the path (1.2).

## § 2. The fundamental equation for $\psi$.[3]

Wave geometry is based on the vector- and spin parallel displace-

(1) T. Iwatsuki, Y. Mimura, and T. Sibata, this Journal, **8** (1938), 187, (W. G. No. 27).

(2) In this paper, since we take $\overset{1}{F}_{ij} = A_{[ij]}$ and $\overset{2}{F}_{ij} = A_{[ij]}^{\cdot\cdot 5}$, the factor 2 appears in (1.2). The term $Qu^j$ disappears when $u^j$ is suitably normalized.

(3) The method of finding the fundamental equation here adopted is closely analogous to T. Sibata's, this Journal **8** (1938), 199–204, (W. G. No. 29); thus the equations which are not important are omitted here. The notations used in this paper are the same as in Sibata's.

ments which make $ds\psi=0$ invariant. When the vector parallel displacement is taken as Riemannian and the spin parallel displacement as most general, the fundamental equation for $\psi$ becomes

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+\Sigma_i)\psi\,, \qquad (i=1,2,3,4) \tag{2.1}$$

where $\Gamma_i$ and $\Sigma_i$ are both 4–4 matrices introduced by the vector- and spin parallel displacements respectively. $\Sigma_i$ is arbitrary, and $\Gamma_i$ is defined by

$$\frac{\partial\gamma_j}{\partial x^i}=\{{}^{h}_{ij}\}\gamma_h+\Gamma_i\gamma_j-\gamma_j\Gamma_i\,. \tag{2.2}$$

Expanding $\Sigma_i$ in sedenion, we have

$$\Sigma_i=A_i^{\cdot pq}\gamma_p\gamma_q+A_i+A_i^{\cdot 5}\gamma_5+A_i^{\cdot l}\gamma_l+A_i^{\cdot l5}\gamma_l\gamma_5\,,\ (A_i^{\cdot pq}=-A_i^{\cdot qp}) \tag{2.3}$$

assuming that the coefficients of expansion $A_i^{\cdot pq}$, $A_i$, $A_i^{\cdot 5}$, $A_i^{\cdot l}$ and $A_i^{\cdot l5}$ are real, and $\gamma_5^2=-1$. Then we have

$$u^i\nabla_i u^j=(4A_i^{\cdot jq}u_q+2A_iu^j+2A_i^{\cdot j}M+2A_i^{\cdot j5}N)u^i\,. \tag{2.4}$$

From (2.4) and (1.2) we have

$$\{4A_{i\cdot q}^{\cdot j}u^q+2(A_iu^j+A_i^{\cdot j}M+A_i^{\cdot j5}N)\}u^i=2(M\overset{1}{F}{}_i^{\cdot j}+N\overset{2}{F}{}_i^{\cdot j})u^i+Qu^j\,. \tag{2.5}$$

If we restrict ourselves to the case in which the fundamental equation (2.1) is completely integrable, (2.5) must hold good identically for all values of $u^i$ under the restriction $u^iu^jg_{ij}\geqq 0$. Hence we have

$$A_{i\cdot q}^{\cdot j}+A_{q\cdot i}^{\cdot j}=0\,, \tag{2.6}$$

$$A_{(ij)}=\alpha g_{ij}\,, \qquad A_{(ij)}^{\cdot\cdot 5}=\beta g_{ij}\,, \tag{2.7}$$

$$A_{[ij]}=\overset{1}{F}_{ij}\,, \qquad A_{[ij]}^{\cdot\cdot 5}=\overset{2}{F}_{ij}\,. \tag{2.8}$$

From (2.3) and (2.6), we have

$$A_{ijk}=\epsilon_{ijkl}D\varphi^l\,. \tag{2.9}$$

Hence (2.1) becomes:

$$\nabla_i\psi=\{\epsilon_{ijkl}D\varphi^l\gamma^j\gamma^k+A_i+A_i^{\cdot 5}\gamma_5+\alpha\gamma_i+\beta\gamma_i\gamma_5+\overset{1}{F}_{ij}\gamma^j+\overset{2}{F}_{ij}\gamma^j\gamma_5\}\psi\,. \tag{2.10}$$

So we have the result: *In order that the vector* $u^i\equiv\psi^\dagger A\gamma^i\psi$ *shall satisfy the equation of the form (1.2), the fundamental equation for* $\psi$ *must be*

$$\nabla_i\psi=\{\epsilon_{ijkl}D\varphi^l\gamma^j\gamma^k+A_i+A_i^{\cdot 5}\gamma_5+\alpha\gamma_i+\beta\gamma_i\gamma_5+\overset{1}{F}_{ij}\gamma^j+\overset{2}{F}_{ij}\gamma^j\gamma_5\}\psi\,, \quad (2.10)$$

*so far as we are dealing with the case of complete integrability for* $\psi$.

Naturally, in all the results in this section, if we put $\overset{a}{F}_{ij}=0$ $(a=1,2)$, we get Sibata's result.(1)

## § 3. The condition of integrability of the fundamental equation.

To obtain the condition of integrability of (2.10), we operate $\nabla_h\equiv\frac{\partial}{\partial x^h}-\Gamma_h$ on (2.10) and subtract from it the equation obtained by interchanging the suffixes $h$ and $i$; and using the relation (2.2) i.e. $\nabla_i\gamma_j=0$, we have

$$\begin{aligned}\nabla_{[h}\nabla_{i]}\psi=[&\nabla_{[h}A_{i]jk}\gamma^j\gamma^k+\nabla_{[h}A_{i]}+\nabla_{[h}A_{i]}^{\cdot 5}\gamma_5+(\nabla_{[h}\alpha)\gamma_{i]}+(\nabla_{[h}\beta)\gamma_{i]}\gamma_5\\&+\nabla_{[h}\overset{1}{F}_{i]j}\gamma^j+\nabla_{[h}\overset{2}{F}_{i]j}\gamma^j\gamma_5+\{A_{[i|jk|}\gamma^j\gamma^k+A_{[i}+A_{[i}^{\cdot 5}\gamma_{|5|}+\alpha\gamma_{[i}+\beta\gamma_{[i}\gamma_{|5|}\\&+\overset{1}{F}_{[i|j|}\gamma^j+\overset{2}{F}_{[i|j}\gamma^j\gamma_{5|}\}\times\{A_{h]lm}\gamma^l\gamma^m+A_{h]}+A_{h]}^{\cdot 5}\gamma_5+\alpha\gamma_{h]}+\beta\gamma_{h]}\gamma_5\\&+\overset{1}{F}_{h]l}\gamma^l+\overset{2}{F}_{h]l}\gamma^l\gamma_5\}]\psi\,. \end{aligned}\quad (3.1)$$

On the other hand, since

$$\nabla_{[h}\nabla_{i]}\psi=\frac{1}{8}K_{hi}^{\cdot\cdot lm}\gamma_l\gamma_m\psi\,, \quad (3.2)$$

(3.1) becomes

$$\frac{1}{8}K_{hi}^{\cdot\cdot lm}\gamma_l\gamma_m\psi=P_{hi}\psi \quad (3.3)$$

where

$$\begin{aligned}P_{hi}\equiv\{&\nabla_{[h}A_{i]jk}\gamma^j\gamma^k+\nabla_{[h}A_{i]}+\nabla_{[h}A_{i]}^{\cdot 5}\gamma_5+(\nabla_{[h}\alpha)\gamma_{i]}+(\nabla_{[h}\beta)\gamma_{i]}\gamma_5\\&+(\nabla_{[h}\overset{1}{F}_{i]j})\gamma^j+(\nabla_{[h}\overset{2}{F}_{i]j})\gamma^j\gamma_5\}+\{A_{[i|lp|}A_{h]\cdot m}^{\cdot p}\gamma^l\gamma^m-4\alpha A_{[ih]l}\gamma^l\\&-4\beta A_{[ih]l}\gamma^l\gamma_5-2\alpha A_{[i}^{\cdot 5}\gamma_{h]}\gamma_5+2\beta A_{[i}^{\cdot 5}\gamma_{h]}+\alpha^2\gamma_{[i}\gamma_{h]}+\beta^2\gamma_{[i}\gamma_{h]}\}\\&+\{2\alpha\overset{1}{F}_{[i|l|}g_{h]m}+2\beta\overset{2}{F}_{[i|l|}g_{h]m}+\overset{1}{F}_{[i|l|}\overset{1}{F}_{h]m}+\overset{2}{F}_{[i|l|}\overset{2}{F}_{h]m}\}\gamma^{[l}\gamma^{m]}\\&+\{4A_{[h|pl|}\overset{1}{F}_{i]q}g^{pq}+2\overset{2}{F}_{[h|l|}A_{i]}^{\cdot 5}\}\gamma^l+\{4A_{[h|pl|}\overset{2}{F}_{i]q}g^{pq}+2\overset{1}{F}_{[i|l|}A_{h]}^{\cdot 5}\}\gamma^l\gamma_5\\&+2\{\alpha\overset{2}{F}_{hi}+\beta\overset{1}{F}_{ih}+\overset{1}{F}_{[i|l|}\overset{2}{F}_{h]j}g^{jl}\}\gamma_5\,. \end{aligned}\quad (3.4)$$

From the condition that (2.10) is completely integrable, we have

(1) T. Sibata, this Journal **8** (1938), 204, (W. G. No. 29)

$$\frac{1}{8}K_{hi}^{\cdot\cdot lm}\gamma_l\gamma_m=P_{hi}\,. \tag{3.5}$$

Comparing the coefficients of each base of sedenion, we have the following relations, (3.6)-(3.10)

$$\frac{1}{8}K_{hi}^{\cdot\cdot lm}=\nabla_{[h}A_{i]}^{\cdot lm}+4A_{[i}^{\cdot lu}A_{h]}^{\cdot vm}g_{uv}+\alpha^2\delta_{[i}^l\delta_{h]}^m+\beta^2\delta_{[i}^l\delta_{h]}^m+2\alpha\overset{1}{F}{}_{[i}^{\cdot[l}\delta_{h]}^{m]}$$
$$+2\beta\overset{2}{F}{}_{[i}^{\cdot[l}\delta_{h]}^{m]}+\overset{1}{F}{}_{[i}^{\cdot l}\overset{1}{F}{}_{h]}^{\cdot m}+\overset{2}{F}{}_{[i}^{\cdot l}\overset{2}{F}{}_{h]}^{\cdot m}\,, \tag{3.6}$$

(coefficients of $\gamma_{[l}\gamma_{m]}$)

$$\delta_{[i}^l\nabla_{h]}\alpha+\nabla_{[h}\overset{1}{F}{}_{i]}^{\cdot l}-4\alpha A_{[ih]}^{\cdot\cdot l}+2\beta A_{[i}^{\cdot 5}\delta_{h]}^l+4A_{[h}^{\cdot pl}\overset{1}{F}{}_{i]p}+2\overset{2}{F}{}_{[h}^{\cdot l}A_{i]}^{\cdot 5}=0\,, \tag{3.7}$$

(coefficients of $\gamma_l$)

$$\delta_{[i}^l\nabla_{h]}\beta+\nabla_{[h}\overset{2}{F}{}_{i]}^{\cdot l}-4\beta A_{[ih]}^{\cdot\cdot l}-2\alpha A_{[i}^{\cdot 5}\delta_{h]}^l+4A_{[h}^{\cdot pl}\overset{2}{F}{}_{i]p}-2\overset{1}{F}{}_{[h}^{\cdot l}A_{i]}^{\cdot 5}=0\,, \tag{3.8}$$

(coefficients of $\gamma_l\gamma_5$)

$$\nabla_{[h}A_{i]}^{\cdot 5}+2\beta\overset{1}{F}_{ih}+2\alpha\overset{2}{F}_{hi}+2\overset{1}{F}{}_{[i}^{\cdot s}\overset{2}{F}_{h]s}=0\,, \tag{3.9}$$

(coefficients of $\gamma_5$)

$$\nabla_{[h}A_{i]}=0\,. \tag{3.10}$$

(coefficients of I)

*These five equations give the conditions of complete integrability of (2.10).* From (3.10) we know that the vector $A_i$ is a gradient vector. Hence, without loss of generality, we can take $A_i=0$.

## § 4. The reduction of the condition (3.6).

In this section we shall study the condition (3.6) more closely. Substituting (2.9), namely $A_{ijk}=\epsilon_{ijkl}D\varphi^l$, into (3.6), we have

$$K_{ijlm}+8\epsilon_{[i|lmk|}D\nabla_{j]}\varphi^k+32\epsilon_{[i|lpu|}\epsilon_{j]qmv}D^2\varphi^u\varphi^v g^{pq}-8(\alpha^2+\beta^2)g_{[i|m|}g_{j]l}$$
$$-8\{g_{[i|m|}F_{j]l}-g_{[i|l|}F_{j]m}\}-8\{\overset{1}{F}_{[i|m|}\overset{1}{F}_{j]l}+\overset{2}{F}_{[i|m|}\overset{2}{F}_{j]l}\}=0 \tag{4.1}$$

where
$$\alpha\overset{1}{F}_{ij}+\beta\overset{2}{F}_{ij}\equiv F_{ij}\,. \tag{4.2}$$

From (4.1) and $K_{[ijl]m}=0$, we have

$$\sum_{i,j,l}[2D\{\epsilon_{ilmk}\nabla_j\varphi^k-\epsilon_{jlmk}\nabla_i\varphi^k\}+8\{\epsilon_{ilpu}\epsilon_{jqmv}-\epsilon_{jlpu}\epsilon_{iqmv}\}D^2\varphi^u\varphi^v g^{pq}$$
$$-2g_{im}F_{jl}+2g_{jm}F_{il}-2(\overset{1}{F}_{im}\overset{1}{F}_{jl}-\overset{1}{F}_{jm}\overset{1}{F}_{il}+\overset{2}{F}_{im}\overset{2}{F}_{jl}-\overset{2}{F}_{jm}\overset{2}{F}_{il})]=0 \tag{4.3}$$

where $\sum_{i,j,l}$ denotes the summation in cyclic order with respect to $i,j,l$. Subtracting (4.3) from (4.1), we have

$$K_{ijlm}=4D\epsilon_{jimk}\nabla_l\varphi^k+16D^2\epsilon_{jipu}\epsilon_{lqmv}\varphi^u\varphi^v g^{pq}+8(\alpha^2+\beta^2)g_{[i|m|}g_{j]l}$$
$$-4g_{lm}F_{ij}-4g_{il}F_{jm}+4g_{jl}F_{im}-4(\overset{1}{F}_{ij}\overset{1}{F}_{lm}+\overset{2}{F}_{ij}\overset{2}{F}_{lm}). \tag{4.4}$$

Multiplying (4.4) by $g^{lm}$ and contracting it by $l$ and $m$, we have

$$2F_{ij}=Dg^{lm}\epsilon_{jimk}\nabla_l\varphi^k. \tag{4.5}$$

Next, from (4.4) and the identity $K_{ijlm}=K_{lmij}$, we have

$$D\epsilon_{jimk}\nabla_l\varphi^k-g_{lm}F_{ij}-g_{il}F_{jm}+g_{jl}F_{im}$$
$$=D\epsilon_{mljk}\nabla_i\varphi^k-g_{ij}F_{lm}-g_{li}F_{mj}+g_{mi}F_{lj}. \tag{4.6}$$

Multiplying (4.6) by $g^{jl}$ and contracting it by $j$ and $l$, we have (4.5). Also, multiplying (4.6) by $\frac{1}{D}\epsilon^{mjpq}$ and contracting it by $m$ and $j$, we have

$$\delta_i^p\nabla_l\varphi^q-\delta_i^q\nabla_l\varphi^p+\delta_l^p\nabla_i\varphi^q-\delta_l^q\nabla_i\varphi^p+2g_{il}\overset{*}{F}{}^{pq}$$
$$+\frac{1}{D}\epsilon^{mjpq}(g_{jl}F_{im}-g_{mi}F_{lj})=0 \tag{4.7}$$

where $\overset{*}{F}{}^{pq}=\frac{1}{2D}F_{ij}\epsilon^{ijpq}$ i.e. $F_{ab}=\frac{D}{2}\epsilon_{pqab}\overset{*}{F}{}^{pq}$. (4.8)

If we contract (4.7) by $q$ and $i$, we have

$$\nabla_l\varphi_p=g_{pl}\sigma-\overset{*}{F}_{lp} \tag{4.9}$$

where $$\sigma=\frac{1}{4}\nabla_s\varphi^s. \tag{4.10}$$

From (4.9), we have $\overset{*}{F}_{lp}=-\nabla_{[l}\varphi_{p]}$, from which it follows that $\nabla_s F_l^{\cdot s}=0$. This shows that: *$\overset{*}{F}_{ij}$ is a rotation, that is the divergence of $F_{ij}$ vanishes.* If we substitute (4.9) into (4.5), (4.6), and (4.7), we see that they are all satisfied identically.

By (4.9), the first term of the right-hand side of (4.4) becomes

$$4D\epsilon_{jimk}(\sigma\delta_l^k-\overset{*}{F}{}_l^{\cdot k})=4D\sigma\epsilon_{ijlm}-2g_{ls}F_{ab}\times 6\delta_j^{[a}\delta_i^b\delta_m^{s]}$$
$$=4D\sigma\epsilon_{ijlm}+4g_{lm}F_{ij}-4g_{li}F_{mj}-4g_{lj}F_{im}.$$

And thus (4.4) becomes:

$$K_{ijlm}=4D\sigma\epsilon_{ijlm}+16D^2\epsilon_{ijpu}\epsilon_{lmqv}\varphi^u\varphi^v g^{pq}+8(\alpha^2+\beta^2)g_{[i|m|}g_{j]l}$$
$$-4(\overset{1}{F}_{ij}\overset{1}{F}_{lm}+\overset{2}{F}_{ij}\overset{2}{F}_{lm}). \tag{4.11}$$

Putting together the above-obtained results, we have: *The equation (3.6) is reducible to the two equations (4.9) and (4.11).*

## § 5. The actual form of the equation (4.11) when the metric is spherically symmetric.

The most general spherically symmetric line-element with the signature $- - - +$ is given by[(1)]

$$ds^2 = -A(r,t)dr^2 - B(r,t)(d\theta^2 + \sin^2\theta d\phi^2) + C(r,t)dt^2 \qquad (5.1)$$

where $A$, $B$, and $C$, are any positive functions. The components of the curvature tensor made from this line-element are given by[(2)]

$$\left.\begin{aligned}
&K_{1212} = \frac{2B''B - B'^2}{4B} - \frac{A'B'}{4A} - \frac{\dot{A}\dot{B}}{4C}\\
&K_{1414} = \frac{2\ddot{A}C - \dot{A}\dot{C}}{4C} - \frac{2C''C - C'^2}{4C} + \frac{A'C' - \dot{A}^2}{4A}\\
&K_{2323} = -\sin^2\theta\left(B - \frac{B'^2}{4A} + \frac{\dot{B}^2}{4C}\right)\\
&K_{2424} = \frac{2\ddot{B}C - \dot{B}\dot{C}}{4C} - \frac{B'C'}{4A} - \frac{\dot{B}^2}{4B}\\
&K_{1224} = -\frac{2\dot{B}'C - \dot{B}C'}{4C} + \frac{\dot{B}B'}{4B} + \frac{\dot{A}B'}{4A}\\
&K_{1313} = \sin^2\theta K_{1212},\quad K_{3434} = \sin^2\theta K_{2424},\quad K_{1334} = \sin^2\theta K_{1224}\\
&K_{1212},\ K_{2424},\ K_{1224},\ K_{1414} \text{ are functions of } r \text{ and } t\\
&K_{2323} = f(r,t)\sin^2\theta,\quad \text{other } K_{ijlm} = 0
\end{aligned}\right\} \qquad (5.2)$$

where accents and dots denote differentiations with respect to $r$ and $t$ respectively. For later convenience, we set down the values of Christoffel's symbol got from the line-element (5.1).

$$\left\{{1 \atop 11}\right\} = \frac{A'}{2A},\ \left\{{4 \atop 11}\right\} = \frac{\dot{A}}{2C},\ \left\{{2 \atop 12}\right\} = \frac{B'}{2B} = \left\{{3 \atop 13}\right\},\ \left\{{1 \atop 14}\right\} = \frac{\dot{A}}{2A},\ \left\{{4 \atop 14}\right\} = \frac{C'}{2C},\ \left\{{1 \atop 22}\right\} = -\frac{B'}{2A},$$

(1) H. Takeno, this Journal, 8 (1938), 272, (W. G. No. 33).
(2) H. Takeno; ibid., 277.

$$\left\{ {4 \atop 22} \right\} = \frac{\dot{B}}{2C},\ \left\{ {3 \atop 23} \right\} = \cot\theta,\ \left\{ {2 \atop 24} \right\} = \frac{\dot{B}}{2B},\ \left\{ {1 \atop 33} \right\} = -\frac{B'}{2A}\sin^2\theta,\ \left\{ {2 \atop 33} \right\} = -\sin\theta\cos\theta,$$

$$\left\{ {4 \atop 33} \right\} = \frac{\dot{B}}{2C}\sin^2\theta,\ \left\{ {3 \atop 34} \right\} = \frac{\dot{B}}{2B},\ \left\{ {1 \atop 44} \right\} = \frac{C'}{2A},\ \left\{ {4 \atop 44} \right\} = \frac{\dot{C}'}{2C},\ \text{other } \left\{ {k \atop ij} \right\} = 0\,. \tag{5.3}$$

In the equation (4.11), if we substitute 1, 2, 3, 4, into $i, j, l$, and $m$, and use (5.2), we have the following equations:

$$\begin{cases} K_{1212} = 16D^2\{g^{33}(\varphi^4)^2 + g^{44}(\varphi^3)^2\} - 4(\alpha^2+\beta^2)g_{11}g_{22} - 4\{(\overset{1}{F}_{12})^2 + (\overset{2}{F}_{12})^2\} & \text{(5.4a)} \\ K_{1313} = 16D^2\{g^{22}(\varphi^4)^2 + g^{44}(\varphi^2)^2\} - 4(\alpha^2+\beta^2)g_{11}g_{33} - 4\{(\overset{1}{F}_{13})^2 + (\overset{2}{F}_{13})^2\} & \text{(5.4b)} \\ K_{2424} = 16D^2\{g^{11}(\varphi^3)^2 + g^{33}(\varphi^1)^2\} - 4(\alpha^2+\beta^2)g_{22}g_{44} - 4\{(\overset{1}{F}_{24})^2 + (\overset{2}{F}_{24})^2\} & \text{(5.4c)} \\ K_{3434} = 16D^2\{g^{11}(\varphi^2)^2 + g^{22}(\varphi^1)^2\} - 4(\alpha^2+\beta^2)g_{33}g_{44} - 4\{(\overset{1}{F}_{34})^2 + (\overset{2}{F}_{34})^2\} & \text{(5.4d)} \\ K_{1414} = 16D^2\{g^{22}(\varphi^3)^2 + g^{33}(\varphi^2)^2\} - 4(\alpha^2+\beta^2)g_{11}g_{44} - 4\{(\overset{1}{F}_{14})^2 + (\overset{2}{F}_{14})^2\} & \text{(5.4e)} \\ K_{2323} = 16D^2\{g^{11}(\varphi^4)^2 + g^{44}(\varphi^1)^2\} - 4(\alpha^2+\beta^2)g_{22}g_{33} - 4\{(\overset{1}{F}_{23})^2 + (\overset{2}{F}_{23})^2\} & \text{(5.4f)} \end{cases}$$

$$\left.\begin{aligned} K_{1234} &= 0 = 4D\sigma - 4(\overset{1}{F}_{12}\overset{1}{F}_{34} + \overset{2}{F}_{12}\overset{2}{F}_{34})\,, \\ K_{1324} &= 0 = -4D\sigma - 4(\overset{1}{F}_{13}\overset{1}{F}_{24} + \overset{2}{F}_{13}\overset{2}{F}_{24}) \\ K_{1423} &= 0 = 4D\sigma - 4(\overset{1}{F}_{14}\overset{1}{F}_{23} + \overset{2}{F}_{14}\overset{2}{F}_{23}) \end{aligned}\right\} \tag{5.5}$$

$$\left.\begin{aligned} K_{1213} &= -16D^2\varphi^2\varphi^3g^{44} - 4(\overset{1}{F}_{12}\overset{1}{F}_{13} + \overset{2}{F}_{12}\overset{2}{F}_{13}) = 0 \\ K_{2434} &= -16D^2\varphi^2\varphi^3g^{11} - 4(\overset{1}{F}_{24}\overset{1}{F}_{34} + \overset{2}{F}_{24}\overset{2}{F}_{34}) = 0 \end{aligned}\right\} \tag{5.6a}$$

$$\left.\begin{aligned} K_{1214} &= -16D^2\varphi^2\varphi^4g^{33} - 4(\overset{1}{F}_{12}\overset{1}{F}_{14} + \overset{2}{F}_{12}\overset{2}{F}_{14}) = 0 \\ K_{2334} &= +16D^2\varphi^2\varphi^4g^{11} - 4(\overset{1}{F}_{23}\overset{1}{F}_{34} + \overset{2}{F}_{23}\overset{2}{F}_{34}) = 0 \end{aligned}\right\} \tag{5.6b}$$

$$\left.\begin{aligned} K_{1314} &= -16D^2\varphi^3\varphi^4g^{22} - 4(\overset{1}{F}_{13}\overset{1}{F}_{14} + \overset{2}{F}_{13}\overset{2}{F}_{14}) = 0 \\ K_{2324} &= -16D^2\varphi^3\varphi^4g^{11} - 4(\overset{1}{F}_{23}\overset{1}{F}_{24} + \overset{2}{F}_{23}\overset{2}{F}_{24}) = 0 \end{aligned}\right\} \tag{5.6c}$$

$$\left.\begin{aligned} K_{1223} &= +16D^2\varphi^1\varphi^3g^{44} - 4(\overset{1}{F}_{12}\overset{1}{F}_{23} + \overset{2}{F}_{12}\overset{2}{F}_{23}) = 0 \\ K_{1434} &= -16D^2\varphi^1\varphi^3g^{22} - 4(\overset{1}{F}_{14}\overset{1}{F}_{34} + \overset{2}{F}_{14}\overset{2}{F}_{34}) = 0 \end{aligned}\right\} \tag{5.6d}$$

$$\left.\begin{aligned} K_{1323} &= -16D^2\varphi^1\varphi^2g^{44} - 4(\overset{1}{F}_{13}\overset{1}{F}_{23} + \overset{2}{F}_{13}\overset{2}{F}_{23}) = 0 \\ K_{1424} &= -16D^2\varphi^1\varphi^2g^{33} - 4(\overset{1}{F}_{14}\overset{1}{F}_{24} + \overset{2}{F}_{14}\overset{2}{F}_{24}) = 0 \end{aligned}\right\} \tag{5.6e}$$

$$\left.\begin{aligned} K_{1224} &= +16D^2\varphi^1\varphi^4g^{33} - 4(\overset{1}{F}_{12}\overset{1}{F}_{24} + \overset{2}{F}_{12}\overset{2}{F}_{24}) \\ K_{1334} &= +16D^2\varphi^1\varphi^4g^{22} - 4(\overset{1}{F}_{13}\overset{1}{F}_{34} + \overset{2}{F}_{13}\overset{2}{F}_{34}) = \sin^2\theta K_{1224} \end{aligned}\right\} \tag{5.6f}$$

## § 6. The case when $B(r, t)$ = constant.

In this section we shall first find the tensors $g_{ij}$, $\overset{a}{F}_{ij}$ $(a=1,2)$, $\varphi_j$, $\alpha$, $\beta$, and $A_i^{;5}$, by solving the equations of conditions of integrability when the function $B(r, t)$ in (5.1) is constant. Although such a line-element has never been treated in the theory of relativity, we must study it carefully, because it appears in the theory of the hydrogen atom in terms of wave geometry.[1]

When $B$=constant, from (5.2) we see that

$$K_{1212}=K_{1313}=K_{2424}=K_{3434}=K_{1224}=K_{1334}=0\,. \tag{6.1}$$

In this case, from (5.4), i.e. (5.4a) $\times g_{44}-$(5.4c) $\times g_{11}$, we have

$$16D^2g^{33}\{g_{44}(\varphi^4)^2-g_{11}(\varphi^1)^2\}-4g_{44}\{(\overset{1}{F}_{12})^2+(\overset{2}{F}_{12})^2\}+4g_{11}\{(\overset{1}{F}_{24})^2+(\overset{2}{F}_{24})^2\}=0 \tag{6.2}$$

But since $D\varphi^i$ is real, $g_{11}<0$, $g^{33}<0$, and $g_{44}>0$, it must be true that

$$\varphi^1=\varphi^4=\overset{a}{F}_{12}=\overset{a}{F}_{24}=0\,. \qquad (a=1,2) \tag{6.3}$$

Similarly, from (5.4b) and (5.4d), we have

$$\overset{a}{F}_{13}=\overset{a}{F}_{34}=0\,. \qquad (a=1,2) \tag{6.4}$$

So, from (5.6a), we have $\varphi^2\varphi^3=0$. Also, from this relation together with (6.1) and (5.4a, b), we have

$$\alpha=\beta=\varphi^2=\varphi^3=0\,, \qquad \text{so} \qquad \varphi^i=0\,. \tag{6.5}$$

Substituting (6.3), (6.4), and (6.5), into (5.4), (5.5), and (5.6), we have

$$K_{1414}=-4\{(\overset{1}{F}_{14})^2+(\overset{2}{F}_{14})^2\}\,, \qquad K_{2323}=-4\{(\overset{1}{F}_{23})^2+(\overset{2}{F}_{23})^2\}\,, \tag{6.6}$$

$$\sigma=0=\overset{1}{F}_{14}\overset{1}{F}_{23}+\overset{2}{F}_{14}\overset{2}{F}_{23}\,, \tag{6.7}$$

the others being satisfied identically. Moreover we can see that (4.9) is also satisfied identically, the remaining conditions of integrability being left the same as (3.7)–(3.9).

As is easily seen, $\overset{1}{F}_{[i}{}^{;s}\overset{2}{F}_{h]s}=0$. Hence from (3.9) it follows that $A_i^{;5}$ is a gradient vector. Therefore we can take $A_i^{;5}=0$ without loss of

(1) K. Morinaga, this Journal, **7** (1937), 282, (W. G. No. 22).

generality.[1] Further, from (6.5) the equations (3.7) and (3.8) become

$$\nabla_{[h}\overset{a}{F}_{il]}=0\,, \qquad (a=1,2) \tag{6.8}$$

from which we have $\sum\limits_{h,i,l}\nabla_h\overset{a}{F}_{il}=0$, where $\sum\limits_{h,i,l}$ denotes the cyclic summation with respect to $h$, $i$, and $l$. Hence, from (6.8), we have

$$\nabla_h\overset{a}{F}_{ij}=0\,. \qquad (a=1,2) \tag{6.9}$$

Further, in order to obtain the concrete forms of $g_{ij}$ and $\overset{a}{F}_{ij}$, as in the theory of the hydrogen atom,[2] we shall take the coordinate system such that $A(r,t)=C(r,t)$. In this case, from (5.3), we have

$$\left.\begin{aligned}&\left\{{}^{1}_{11}\right\}=\left\{{}^{4}_{14}\right\}=\left\{{}^{1}_{44}\right\}=\frac{A'}{2A}\,, \qquad \left\{{}^{4}_{11}\right\}=\left\{{}^{1}_{14}\right\}=\left\{{}^{4}_{44}\right\}=\frac{\dot{A}}{2A}\,,\\ &\left\{{}^{3}_{23}\right\}=\cot\theta\,, \quad \left\{{}^{2}_{33}\right\}=-\sin\theta\cos\theta, \quad \text{other}\quad \left\{{}^{k}_{ij}\right\}=0\,.\end{aligned}\right\} \tag{6.10}$$

Substituting (6.3), (6.4), and (6.10), into (6.9), and after some calculations, we have the following solution:

$$\left.\begin{aligned}&\overset{a}{F}_{14}=\overset{a}{c}A\,, \qquad \overset{a}{F}_{23}=\overset{a}{b}\sin\theta\,, \qquad (a=1,2)\\ &\overset{a}{c}\text{ and }\overset{a}{b}\text{ are integration constants.}\end{aligned}\right\} \tag{6.11}$$

On the other hand, from (6.9) we know that $\overset{a}{F}_{ij}$ $(a=1,2)$ are both rotations, i. e.

$$\frac{\partial\overset{a}{F}_{ij}}{\partial x^k}+\frac{\partial\overset{a}{F}_{jk}}{\partial x^i}+\frac{\partial\overset{a}{F}_{ki}}{\partial x^j}=0\,. \qquad (a=1,2) \tag{6.12}$$

Putting together all the results obtained above, we have: *When $B$ is constant, the spherical symmetry of $g_{ij}$ necessarily implies the spherical symmetry of $\overset{a}{F}_{ij}$, and then $\overset{a}{F}_{ij}$ become both rotations.*

The remaining conditions to be satisfied by $g_{ij}$ and $\overset{a}{F}_{ij}$ are merely (6.6) and (6.7). Substituting (5.2) and (6.11) into (6.6) and (6.7), we have

$$\left.\begin{aligned}&\frac{1}{2A^3}\{(\ddot{A}A-\dot{A}^2)-(A''A-A'^2)\}+4\{(\overset{1}{c})^2+(\overset{2}{c})^2\}=0\,,\\ &B=4\{(\overset{1}{b})^2+(\overset{2}{b})^2\}\,,\end{aligned}\right\} \tag{6.13}$$

$$\overset{11}{cb}+\overset{22}{cb}=0\,. \tag{6.14}$$

(1) T. Sibata, this Journal, **8** (1938), 209, (W. G. No. 29).
(2) H. Takeno, this Journal, **8** (1938), 284, (W. G. No. 33).

The first equation of (6.13) has already been solved by the present writer.[1] If we assume that the metric $ds^2=-Adr^2-B(d\theta^2+\sin^2\theta d\phi^2)+Adt^2$ is static, i. e. $A(r, t)$ does not contain $t$, the solution of this equation is as follows:

**The first case.** When $(\overset{1}{c})^2+(\overset{2}{c})^2\neq 0$.

$$
\text{III}\quad\left\{\begin{array}{l}
ds^2=\dfrac{c^2}{4p\sinh^2\dfrac{\alpha}{2}}(-dr^2+dt^2)-B(d\theta^2+\sin^2\theta d\phi^2)\,, \\[2ex]
\overset{a}{F}_{14}=\overset{a}{c}A,\quad \overset{a}{F}_{23}=\overset{a}{b}\sin\theta,\quad \text{other}\ \ \overset{a}{F}_{ij}=0,\quad (a=1,2) \\[1ex]
\text{where } \alpha=cr+c_1, \text{ and } B, c, c_1, p, \overset{a}{c}, \text{ and } \overset{a}{b} \text{ are real constants} \\
\quad\text{satisfying the equations} \\[1ex]
B=4\{(\overset{1}{b})^2+(\overset{2}{b})^2\}\,,\qquad p=4\{(\overset{1}{c})^2+(\overset{2}{c})^2\}\,,\qquad \overset{11}{cb}+\overset{22}{cb}=0\,.
\end{array}\right.
$$

In the special case in which $k=0$, $k$ being an arbitrary integration constant,[2] the line-element becomes

$$
ds^2=\frac{1}{(\sqrt{p}\,r+c_1)^2}(-dr^2+dt^2)-B(d\theta^2+\sin^2\theta d\phi^2)\,. \qquad (6.15)
$$

**The second case.** When $\overset{1}{c}=\overset{2}{c}=p=0$.

$$
\text{IV}\quad\left\{\begin{array}{l}
ds^2=e^{cr+c_1}(-dr^2+dt^2)-B(d\theta^2+\sin^2\theta d\phi^2)\,, \\
\overset{a}{F}_{23}=\overset{a}{b}\sin\theta,\quad \text{other}\ \ \overset{a}{F}_{ij}=0\quad (a=1,2),\quad B=4\{(\overset{1}{b})^2+(\overset{2}{b})^2\}\,.
\end{array}\right.
$$

Summarizing the above results, we have: *If we assume that $B=$ constant, then necessarily*

$$
\alpha=\beta=\varphi_i=0\,,
$$

$\overset{a}{F}_{ij}$ *then become rotations and are spherically symmetric, and $g_{ij}$ and $\overset{a}{F}_{ij}$ are given by* III *or* IV.

Observing carefully the calculations in this section, we see that the equations

$$
K_{ijlm}=8(\alpha^2+\beta^2)g_{[i|m|}g_{j]l} \qquad (6.16)
$$

and

$$
K_{ijlm}=16D^2\epsilon_{ijpu}\epsilon_{lmqv}\varphi^u\varphi^v g^{pq} \qquad (6.17)
$$

(1), (2) H. Takeno, ibid., 285; (in that paper I carelessly omitted the second case IV).

neither of them supply as their solutions the spherically symmetric line element such that $B=\text{constant}$. Hence: *Under the assumption that the solution $g_{ij}$ should be spherically symmetric in the sense used by Eiesland,*[1] *the equations (6.16) and (6.17) give as their unique solutions the line element of the de Sitter type and that of the Einstein type respectively.* In previous papers we have solved (6.16) and (6.17), assuming that $B$ is not constant, but now we see that this assumption was not necessary.

## § 7. The case where $B(r,t)\neq$ constant. i. When $F_{ij}=0$.

In sections 7 and 8 we shall solve $g_{ij}$ and $\overset{a}{F}_{ij}$ when $B(r,t)$ in (5.1) is not constant. In the following, as in the theory of relativity, we shall consider the problem in the coordinate system in which $B(r,t)=r^2$, because such selection of coordinate system is always possible without disturbing the spherically symmetric character.

First, we shall consider the case in which

$$F_{ij}\equiv\alpha\overset{1}{F}_{ij}+\beta\overset{2}{F}_{ij}=0\,. \tag{7.1}$$

**[Case $\alpha$]** The case when $\alpha^2+\beta^2\neq 0$, i. e. $\alpha$ or $\beta\neq 0$.

In this case we assume that $\beta\neq 0$, since the case in which $\alpha\neq 0$ and $\beta=0$ can be treated in the same way.

Eliminating $\overset{2}{F}_{ij}$ from (7.1), (5.1), (5.5), and (5.6), we have

$$\overset{1}{F}_{12}\overset{1}{F}_{34}=-\overset{1}{F}_{13}\overset{1}{F}_{24}=\overset{1}{F}_{14}\overset{1}{F}_{23} \tag{7.2}$$

$$C\overset{1}{F}_{12}\overset{1}{F}_{13}=-A\overset{1}{F}_{24}\overset{1}{F}_{34} \quad \text{(7.3a)}, \qquad r^2\sin^2\theta\overset{1}{F}_{12}\overset{1}{F}_{14}=-A\overset{1}{F}_{23}\overset{1}{F}_{34} \quad \text{(7.3b)}$$

$$r^2\overset{1}{F}_{13}\overset{1}{F}_{14}=A\overset{1}{F}_{23}\overset{1}{F}_{24} \quad \text{(7.3c)}, \qquad C\overset{1}{F}_{12}\overset{1}{F}_{23}=r^2\overset{1}{F}_{14}\overset{1}{F}_{34} \quad \text{(7.3d)}$$

$$C\overset{1}{F}_{13}\overset{1}{F}_{23}=-r^2\sin^2\theta\overset{1}{F}_{14}\overset{1}{F}_{24} \quad \text{(7.3e)}, \qquad \sin^2\theta\overset{1}{F}_{12}\overset{1}{F}_{24}=\overset{1}{F}_{13}\overset{1}{F}_{34} \quad \text{(7.3f)}$$

Using these equations we can prove that

$$\overset{a}{F}_{12}=\overset{a}{F}_{13}=\overset{a}{F}_{24}=\overset{a}{F}_{34}=0\,, \qquad (a=1,2) \tag{7.4}$$

Proof. If we assume that $\overset{1}{F}_{12}\neq 0$, from (7.2) and (7.3b), eliminating $\overset{1}{F}_{12}$ and $\overset{1}{F}_{23}$, we have

$$A(\overset{1}{F}_{34})^2+r^2\sin^2\theta(\overset{1}{F}_{14})^2=0\,.$$

(1) H. Takeno, ibid., 272.

Since $A>0$, we have $\overset{1}{F}_{34}\overset{1}{F}_{14}=0$, so that from (7.3), $\overset{1}{F}_{13}=\overset{1}{F}_{23}=\overset{1}{F}_{24}=0$. Therefore, from (7.1), (5.5), and (5.6), we have

$$\sigma=\varphi^2\varphi^3=\varphi^2\varphi^4=\varphi^3\varphi^4=\varphi^1\varphi^3=\varphi^1\varphi^2=0\,. \tag{7.5}$$

From this we can show that $\varphi^2=0$; for if $\varphi^2\neq 0$, we must have $\varphi^1=\varphi^3=\varphi^4=0$; but on the other hand, from (4.9) and (7.1) we have

$$\nabla_i\varphi_j=0\,, \tag{7.6}$$

and if we put $(i,j)=(3,3)$ in this equation, we obtain $\{{}^{\,2}_{33}\}=0$, which contradicts (5.3); hence we must have $\varphi^2=0$.

Similarly, by putting $(i,j)=(3,2)$ in (7.6), we can show that $\varphi^3=0$.

Therefore, if we substitute $\varphi^2=\varphi^3=0$ and (5.4) into (5.2), i. e.

$$\sin^2\theta K_{1212}=K_{1313}\,, \tag{7.7}$$

we can easily see that $\overset{a}{F}_{12}=0$ $(a=1,2)$, which contradicts the assumption $\overset{1}{F}_{12}\neq 0$. Hence we must have $\overset{1}{F}_{12}=0$; so $\overset{2}{F}_{12}=0$. In the same way we can show that $\overset{a}{F}_{13}=\overset{a}{F}_{24}=\overset{a}{F}_{34}=0$ $(a=1,2)$. Q. E. D.

From (7.4), we see at once that $\sigma=\varphi^2=\varphi^3=0$; again, from (4.9) we have $\nabla_2\varphi_2=0$ and $\nabla_4\varphi_1=0$; therefore we get $\varphi_1=0$ and $\varphi_4C'=0$, respectively. Hence, if $\varphi_4\neq 0$, from (5.2) we have $K_{2424}=0$, which, from (5.4), is inconsistent with the assumption $\alpha$ or $\beta\neq 0$. So that we have

$$\varphi^i=0\,. \tag{7.8}$$

Next we shall prove that $\overset{a}{F}_{14}=0$. *Proof.* Let us assume that $\overset{1}{F}_{14}\neq 0$. From (7.2) and (7.1), we have $\overset{a}{F}_{23}=0$ $(a=1,2)$; and from (5.2) and $K_{1224}=0$, which is obtained from (5.6f), we have $\dot{A}=0$, i. e. $A=A(r)$. So, from (5.4f) and (5.2), we have $A=\dfrac{1}{1-\kappa r^2}$, where we have put $\kappa=\kappa(r)\equiv 4(\alpha^2+\beta^2)$. Substituting this into (5.4a), we have $\kappa=$constant. On the other hand, from (5.4a, b) we have $C=\lambda(t)(1-\kappa r^2)$. Substituting these results into (5.4e), we have

$$(\overset{1}{F}_{14})^2+(\overset{2}{F}_{14})^2=0\,,$$

which is inconsistent with the assumption $\overset{1}{F}_{14}\neq 0$. So that we have $\overset{a}{F}_{14}=0$. Q. E. D.

Next, we shall show that $\overset{a}{F}_{23}=0$ $(a=1,2)$. *Proof.* If we substitute

(1, 1, 2), (2, 1, 2), (1, 1, 3), and (1, 1, 4), into $(l, i, h)$ in (3.7) in turns, and use (5.3), we can easily obtain

$$\nabla_i \alpha = 2\beta A_i^{\cdot 5}. \quad \text{Similarly, from (3.8),} \quad \nabla_i \beta = -2\alpha A_i^{\cdot 5}, \tag{7.9}$$

from which we have $\nabla_{[h} A_{i]}^{\cdot 5} = 0$. Therefore, substituting this equation and (7.4) into (3.9), we have

$$\beta \overset{1}{F}_{ij} - \alpha \overset{2}{F}_{ij} = 0\,, \tag{7.10}$$

from which, and (7.1), we have $\overset{a}{F}_{ij} = 0$. Q. E. D.

From the investigations given above we arrive at the result: *If $\alpha^2 + \beta^2 \neq 0$, it must be true that $\varphi^i = \overset{a}{F}_{ij} = 0$. The equation (4.11) is then reduced to (6.16), which supplies de Sitter type space, as seen in the preceding section.*

**[Caes *b*]** When $\alpha^2 + \beta^2 = 0$, i. e. $\alpha = \beta = 0$.

First we shall prove the following:

Lemma 1. *If $\varphi_i = 0$, then $\overset{a}{F}_{ij} = 0$ $(a = 1, 2)$, and the space becomes euclidean.*

Proof. From (5.2), (5.4), (5.6), and $\varphi_i = 0$, we have

$$\{(\overset{1}{F}_{12})^2 + (\overset{2}{F}_{12})^2\} \sin^2\theta = (\overset{1}{F}_{13})^2 + (\overset{2}{F}_{13})^2, \quad \{(\overset{1}{F}_{24})^2 + (\overset{2}{F}_{24})^2\} \sin^2\theta = (\overset{1}{F}_{34})^2 + (\overset{2}{F}_{34})^2. \tag{7.11}$$

Then $\overset{a}{F}_{12} = 0$; for, if $\overset{1}{F}_{12}$ or $\overset{2}{F}_{12} \neq 0$, then $\overset{1}{F}_{13}$ or $\overset{2}{F}_{13} \neq 0$, so that, from (5.6a, d), we have $\overset{1}{F}_{13}\overset{2}{F}_{23} - \overset{2}{F}_{13}\overset{1}{F}_{23} = 0$, and, from this and (5.6e), we have $\overset{a}{F}_{23} = 0$; hence, from (5.4f), we have $A = 1$; hence again, in consequence of (5.2), we have $K_{1212} = 0$, which contradicts (5.4a); so it must be true that $\overset{a}{F}_{12} = 0$.

From this result we have also $\overset{a}{F}_{13} = 0$.

Next, lowering the suffix $l$ in (3.7) and (3.8), taking the summation cyclically with respect to $(l, i, h)$, and subtracting the original from the resulting equations, we have

$$\nabla_l \overset{1}{F}_{hi} + 2\overset{2}{F}_{ih} A_l^{\cdot 5} = 0 \quad \text{and} \quad \nabla_l \overset{2}{F}_{hi} - 2\overset{1}{F}_{ih} A_l^{\cdot 5} = 0 \tag{7.12}$$

respectively. If we put $(l, h, i) = (4, 1, 2)$ in (7.12), we have

$$\overset{a}{F}_{24} C' = 0\,.$$

From this, (5.4c), (5.2), and (7.11), we have $\overset{a}{F}_{24} = C' = \overset{a}{F}_{34} = 0$. On the other hand, from (5.6f), $\dot{A}=0$; therefore $K_{1414}=0$ and $\overset{a}{F}_{14}=0$. Lastly, putting $(l, h, i)=(3, 1, 2)$ in (7.12), we have at once $\overset{a}{F}_{23}=0$. Q. E. D.

Lemma 2. *There is no solution $g_{ij}$ such that $A$ involves $t$.*

Proof. By the above lemma and (4.9), we have

$$\nabla_{[m}\nabla_{l]}\varphi_p=g_{p[l}\nabla_{m]}\sigma\,, \tag{7.13}$$

or

$$\frac{1}{2}K_{mlps}\varphi^s=g_{p[l}\nabla_{m]}\sigma\,.$$

From this and (5.2), we have

$$\left.\begin{aligned}&K_{12}^{\cdot\cdot 12}\varphi_1=K_{13}^{\cdot\cdot 13}\varphi_1=K_{14}^{\cdot\cdot 14}\varphi_1=-\sigma_1\,,\ K_{12}^{\cdot\cdot 12}\varphi_2=K_{23}^{\cdot\cdot 23}\varphi_2=K_{24}^{\cdot\cdot 24}\varphi_2=-\sigma_2\,,\\ &K_{13}^{\cdot\cdot 13}\varphi_3=K_{23}^{\cdot\cdot 23}\varphi_3=K_{34}^{\cdot\cdot 34}\varphi_3=-\sigma_3\,,\ K_{14}^{\cdot\cdot 14}\varphi_4=K_{24}^{\cdot\cdot 24}\varphi_4=K_{34}^{\cdot\cdot 34}\varphi_4=-\sigma_4\,,\\ &K_{1224}\varphi^2=K_{1334}\varphi^3=0\,.\end{aligned}\right\}\tag{7.14}$$

When $A(r, t)$ is non-static, from (5.2), we have $K_{1224}\neq 0$; so that $\varphi^2=\varphi^3=0=\sigma_2=\sigma_3$. Hence, if we assume that $\varphi_1\neq 0$, then, since $K_{12}^{\cdot\cdot 12}=K_{14}^{\cdot\cdot 14}$, from (5.4) we have

$$g^{22}[16D^2g^{33}(\varphi^4)^2-4\{(\overset{1}{F}_{12})^2+(\overset{2}{F}_{12})^2\}]=-4g^{44}\{(\overset{1}{F}_{14})^2+(\overset{2}{F}_{14})^2\}\,;$$

but as $D\varphi^i$ is real, $g_{22}, g_{33}<0$, and $g_{44}>0$, we have $\varphi^4=\overset{a}{F}_{12}=\overset{a}{F}_{14}=0$, which contradicts $K_{1224}\neq 0$. Hence it must be true that $\varphi_1=0$.

Further, since $K_{1224}\neq 0$, from (5.6f), we have $(\overset{1}{F}_{12})^2+(\overset{2}{F}_{12})^2\neq 0$ and $(\overset{1}{F}_{24})^2+(\overset{2}{F}_{24})^2\neq 0$, i. e. $K_{2424}\neq 0$; and as seen in lemma 1, from (5,6), we have $\overset{a}{F}_{14}=0$; so $K_{1414}=0$. Therefore, by (7.14), $\varphi_4=0$; so that $\varphi^i=0$. Hence, using lemma 1, the present lemma is evident. Q. E. D.

Using lemmas 1 and 2, we can prove the following

Theorem. *If $\alpha=\beta=0$, we must have $\overset{a}{F}_{ij}=0$, and then the equation (4.11) is reduced to (6.17), which, as seen in the preceding section, supplies the Einstein type space in our cosmology.*

Proof. By the two lemmas above, we know that it is enough to study only the case where $\varphi^i\neq 0$ and $A=A(r)$. If we assume that $\varphi_2\neq 0$, then, from (7.14), we have $K_{12}^{\cdot\cdot 12}=K_{13}^{\cdot\cdot 13}=K_{24}^{\cdot\cdot 24}$. Substituting (5.2) into this, we can easily find[1]

$$A=\frac{1}{1+pr^2}\,,\quad C=\lambda(t)\,(1+pr^2),\quad (p \text{ being integration constant}).$$

(1) In this calculation we have used the relation $A\neq 1$. When $A=1$, as $K_{2323}=0=K_{2424}$, we have $C'=1$, so $K_{1414}=0$; and then, from (5.4e), we have $\varphi_2=0$.

So that, from (5.2), $K_{14}^{\cdot\cdot 14}=K_{12}^{\cdot\cdot 12}=p$; and, from (5.4), we have

$$16D^2g^{11}g^{44}\{g^{22}(\varphi^3)^2+g^{33}(\varphi^2)^2\}-4g^{11}g^{44}\{(\overset{1}{F}_{14})^2+(\overset{2}{F}_{14})^2\}$$
$$=16D^2g^{11}g^{22}\{g^{33}(\varphi^4)^2+g^{44}(\varphi^3)^2\}-4g^{11}g^{22}\{(\overset{1}{F}_{12})^2+(\overset{2}{F}_{12})^2\},$$

from which we have $\varphi^2=\overset{a}{F}_{14}=\varphi^4=\overset{a}{F}_{12}=0$; but this contradicts $\varphi_2\neq 0$. Hence we must have $\varphi_2=0$, and similarly $\varphi_3=0$.

Next, if we assume that $\varphi_1\neq 0$, then, as in lemma 2, we have $\varphi_4=\overset{a}{F}_{12}=\overset{a}{F}_{14}=0$; so that, from (5.5) and (4.9), we have

$$\sigma=0, \qquad \nabla_2\varphi_2=0,$$

which is absurd on account of $\nabla_2\varphi_2=\dfrac{r}{2A}\varphi_1\neq 0$. Hence it follows that $\varphi_1=0$ and $\varphi_4\neq 0$.

And then $\overset{a}{F}_{12}$ must vanish; for, if $\overset{1}{F}_{12}$ or $\overset{2}{F}_{12}\neq 0$, as in lemma 1 and 2 we have $\overset{a}{F}_{14}=\overset{a}{F}_{23}=0=K_{14}^{\cdot\cdot 14}=K_{24}^{\cdot\cdot 24}$; so, from (5.4), $\overset{a}{F}_{14}=\overset{a}{F}_{24}=\overset{a}{F}_{34}=0$ $(a=1, 2)$; hence, putting $(l, h, i)=(2, 2, 3)$ in (7.12), we have $\overset{a}{F}_{13}=0$; so that, from (7.11),(1) $\overset{a}{F}_{12}=0$; but this contradicts the assumption $\overset{1}{F}_{12}$ or $\overset{2}{F}_{12}\neq 0$. Hence, necessarily, $\overset{a}{F}_{12}=0$. In the same way we can easily obtain $\overset{a}{F}_{13}=\overset{a}{F}_{24}=\overset{a}{F}_{34}=0$. Hence, from this result, together with (5.4), and $K_{2424}=K_{3434}=0$, as in lemma 1, we have $\overset{a}{F}_{ij}=0$. Q. E. D.

## § 8. The case in which $B(r, t)\neq$ constant. ii. When $F_{ij}\neq 0$.

In the previous sections we saw that the spherical symmetry of $g_{ij}$ implies that of $\overset{a}{F}_{ij}$, but in the present case it is not easy to get the same result, though I think that it may be possible. Therefore at present we make the assumption:

Assumption. $\overset{a}{F}_{ij}$ are spherically symmetric,(2) that is,

$$\left.\begin{aligned}&\overset{a}{F}_{14}=\overset{a}{F}_{14}(r, t), \qquad \overset{a}{F}_{23}=\overset{a}{f}(r, t)\sin\theta \qquad (a=1, 2)\\ &\text{other} \qquad \overset{a}{F}_{ij}=0\end{aligned}\right\} \quad (8.1)$$

(1) (7.10) holds good when $\varphi^4\neq 0$ too.

(2) H. Takeno, this Journal, **8** (1938), 276, (W. G. No. 33).

Under this assumption as in section 7 we consider the problem in the coordinate system in which $B=r^2$.

Then, from (5.5) and (5.6), we have

$$\sigma=0=\overset{1}{F}_{14}\overset{1}{F}_{23}+\overset{2}{F}_{14}\overset{2}{F}_{23}, \tag{8.2}$$

$$\left.\begin{aligned}&\varphi^1\varphi^2=\varphi^1\varphi^3=\varphi^2\varphi^3=\varphi^2\varphi^4=\varphi^3\varphi^4=0,\\&K_{1224}=16D^2\varphi^1\varphi^4g^{33}.\end{aligned}\right\} \tag{8.3}$$

On the other hand, from (8.1) and (4.8), we have

$$F_{12}=F_{13}=F_{24}=F_{34}=\overset{*}{F}_{12}=\overset{*}{F}_{13}=\overset{*}{F}_{24}=\overset{*}{F}_{34}=0. \tag{8.4}$$

If we assume that $\varphi_2\neq0$, from (8.3) we have $\varphi_1=\varphi_3=\varphi_4=0$, and by (4.9) we have

$$\nabla_2\varphi_1=-\frac{1}{r}\varphi_2=0,$$

which is contradictory. Hence it must be true that $\varphi_2=0$; and moreover we can prove that $\varphi_3=0$ in a similar way.

Therefore, from (4.9),

$$\nabla_2\varphi_2=\frac{r}{A}\varphi_1=0, \qquad \nabla_3\varphi_2=0=-\overset{*}{F}_{32},$$

so we have $\varphi_1=\overset{*}{F}_{23}=0$. Hence, from (8.3) and (4.8), we have

$$\dot{A}=0 \quad \text{i.e.} \quad A=A(r), \quad \text{and} \quad F_{14}\equiv\alpha\overset{1}{F}_{14}+\beta\overset{2}{F}_{14}=0 \tag{8.5}$$

respectively. The remaining conditions of (4.9) become

$$\frac{\partial\varphi_4}{\partial x^2}=\frac{\partial\varphi_4}{\partial x^3}=\frac{\partial\varphi_4}{\partial x^4}-\frac{\dot{C}}{2C}\varphi_4=0, \quad -\overset{*}{F}_{14}=\frac{\partial\varphi_4}{\partial x^1}-\frac{C'}{2C}\varphi_4=\frac{C'}{2C}\varphi_4, \tag{8.6}$$

from which we have

$$\varphi_4=\varphi_4(r,t)=C^{\frac{1}{2}}\lambda_1(r)=C\lambda_2(t), \qquad \text{so} \qquad C=\lambda(t)\mu(r). \tag{8.7}$$

And by a suitable transformation of $t$ we can take $C=C(r)$. In such a coordinate system we have

$$\left.\begin{aligned}&\dot{C}=0\\&\dot{A}=0\end{aligned}\right\} \qquad \left.\begin{aligned}&\varphi_4=C\varphi^4, \quad (\varphi^4=\text{const.})\\&\text{other} \quad \varphi_i=0\end{aligned}\right\} \qquad \left.\begin{aligned}&\overset{*}{F}_{14}=-\frac{C'}{2}\varphi^4\\&\text{other } \overset{*}{F}_{ij}=0\end{aligned}\right\} \tag{8.8}$$

and all the other equations except (8.7) hold good as they are.

In this case, further, we investigate precisely conditions (3.7), (3.8), and (3.9). Lowering the suffix $l$, and substituting $1, \ldots, 4$ into $i$, $h$ and $l$, and using (5.3), we have, after some calculations, the systems of equations:

(a) (b)

$$\left.\begin{aligned} &\nabla_2\alpha-2\beta A_2^{;5}=0\\ &\nabla_3\alpha-2\beta A_3^{;5}=0\end{aligned}\right\} \qquad \left.\begin{aligned} &\nabla_2\beta+2\alpha A_2^{;5}=0\\ &\nabla_3\beta+2\alpha A_3^{;5}=0\end{aligned}\right\} \tag{8.9}$$

$$\left.\begin{aligned} &\nabla_2\alpha-2\beta A_2^{;5}+2\overset{2}{F}_{23}A^{35}=0\\ &\nabla_3\alpha-2\beta A_3^{;5}-2\overset{2}{F}_{23}A^{25}=0\end{aligned}\right\} \qquad \left.\begin{aligned} &\nabla_2\beta+2\alpha A_2^{;5}-2\overset{1}{F}_{23}A^{35}=0\\ &\nabla_3\beta+2\alpha A_3^{;5}+2\overset{1}{F}_{23}A^{25}=0\end{aligned}\right\} \tag{8.10}$$

$$\left.\begin{aligned} &\overset{2}{F}_{14}A_2^{;5}=0\\ &\overset{2}{F}_{14}A_3^{;5}=0\end{aligned}\right\} \qquad \left.\begin{aligned} &\overset{1}{F}_{14}A_2^{;5}=0\\ &\overset{1}{F}_{14}A_3^{;5}=0\end{aligned}\right\} \tag{8.11}$$

$$\left.\begin{aligned} &\nabla_1\alpha-2\beta A_1^{;5}-4D\varphi^4\overset{1}{F}{}^{23}=0\\ &-g_{44}(\nabla_1\alpha-2\beta A_1^{;5})+\partial_4\overset{1}{F}_{14}-2\overset{2}{F}_{14}A_4^{;5}=0\end{aligned}\right\} \qquad \left.\begin{aligned} &\nabla_1\beta+2\alpha A_1^{;5}-4D\varphi^4\overset{2}{F}{}^{23}=0\\ &-g_{44}(\nabla_1\beta+2\alpha A_1^{;5})+\partial_4\overset{2}{F}_{14}\\ &\qquad+2\overset{1}{F}_{14}A_4^{;5}=0\end{aligned}\right\} \tag{8.12}$$

$$\left.\begin{aligned} &g_{11}(\nabla_4\alpha-2\beta A_4^{;5})+\nabla_1\overset{1}{F}_{14}-2A_1^{;5}\overset{2}{F}_{14}=0\\ &g_{22}(\nabla_4\alpha-2\beta A_4^{;5})+\frac{r}{A}\overset{1}{F}_{14}=0\end{aligned}\right\} \qquad \left.\begin{aligned} &g_{11}(\nabla_4\beta+2\alpha A_4^{;5})+\nabla_1\overset{2}{F}_{14}\\ &\qquad+2A_1^{;5}\overset{1}{F}_{14}=0\\ &g_{22}(\nabla_4\beta+2\alpha A_4^{;5})+\frac{r}{A}\overset{2}{F}_{14}\\ &\qquad=0\end{aligned}\right\} \tag{8.13}$$

$$\left.\begin{aligned} &\frac{1}{r}\overset{1}{F}_{23}+4\alpha D\varphi^4=0\\ &\frac{1}{r}\overset{1}{F}_{23}-\frac{\partial\overset{1}{F}_{23}}{\partial x^1}-8\alpha D\varphi^4+2A_1^{;5}\overset{2}{F}_{23}=0\\ &\frac{\partial\overset{1}{F}_{23}}{\partial x^4}+4D\varphi^4\overset{1}{F}{}_4^{;1}-2\overset{2}{F}_{23}A_4^{;5}=0\end{aligned}\right\} \qquad \left.\begin{aligned} &\frac{1}{r}\overset{2}{F}_{23}+4\beta D\varphi^4=0\\ &\frac{1}{r}\overset{2}{F}_{23}-\frac{\partial\overset{2}{F}_{23}}{\partial x^1}-8\beta D\varphi^4\\ &\qquad-2A_1^{;5}\overset{1}{F}_{23}=0\\ &\frac{\partial\overset{2}{F}_{23}}{\partial x^4}+4D\varphi^4\overset{2}{F}{}_4^{;1}\\ &\qquad+2\overset{1}{F}_{23}A_4^{;5}=0\end{aligned}\right\} \tag{8.14}$$

From (8.9), (8.10), and (8.11), we have

$$A_2^{;5}=A_3^{;5}=0, \qquad \alpha=\alpha(r, t), \qquad \beta=\beta(r, t); \tag{8.15}$$

otherwise we have $\overset{a}{F}_{14}=\overset{a}{F}_{23}=0$ $(a=1,2)$; thus we arrive at a trivial case $F_{ij}=0$.

From (8.5), (8.12), (8.13) and (8.14), we have

$$\left.\begin{aligned}&\nabla_4(\alpha^2+\beta^2)=0\\&\nabla_1(\alpha^2+\beta^2)=8D\varphi^4(\alpha\overset{1}{F}{}^{23}+\beta\overset{2}{F}{}^{23})=32(\alpha^2+\beta^2)ACr(\varphi^4)^2\,.\end{aligned}\right\}\tag{8.16}$$

But, from (5.4)

$$K_{12}^{\cdot\cdot 12}-K_{24}^{\cdot\cdot 24}=16(\varphi^4)^2C\,,$$

and, substituting this into (5.2), we have

$$(\varphi^4)^2=-\frac{1}{32ACr}\left(\frac{A'}{A}+\frac{C'}{C}\right).\tag{8.17}$$

From (5.2) and (5.4c), it follows that

$$\alpha^2+\beta^2=-\frac{C'}{8ACr}\,.\qquad (C'<0)\tag{8.18}$$

Substituting (8.17) and (8.18) into the second equation of (8.16), we can easily find that

$$C(r)=a-k^2r^2.\quad (a \text{ and } k \text{ are both real constants})\tag{8.19}$$

But if we substitute (5.2), (8.14), (8.17), and (8.18), into (5.4f), we have

$$1-\frac{1}{A}=\frac{rA'}{2A^2}-\frac{r^2C'}{4AC}\left(\frac{A'}{A}+\frac{C'}{C}\right).$$

Here if we put $A=\dfrac{f(r)}{C}$ and use (8.19), we have

$$f^2-af=\frac{ar}{2}f',$$

which is a Riccati's equation whose special solution is given by $f=a$; hence the general solution is easily obtained as follows:

$$f=\frac{a}{1-\dfrac{r^2}{R^2}}\,;\quad\text{so}\quad A=\frac{a}{(a-k^2r^2)\left(1-\dfrac{r^2}{R^2}\right)}\,,\tag{8.20}$$

where $R^2$ is a positive integration constant. So we have

$$ds^2=-\frac{adr^2}{(a-k^2r^2)\left(1-\frac{r^2}{R^2}\right)}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(a-k^2r^2)\,dt^2.$$

When $k\to 0$ and $\frac{1}{R}\to 0$ in the above, the space tends to an euclidean, hence it is natural to choose constant $a$ so that the line-element may tend to the ordinary euclidean form: $ds^2=-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2$. For this, we have $a=1$.

Substituting $A$ and $C$ into (8.17) and (8.18), we get

$$\varphi^4=\frac{i\eta_2}{4R},\quad \alpha^2+\beta^2=\frac{k^2}{4}\left(1-\frac{r^2}{R^2}\right). \qquad (\eta_2=\pm 1) \qquad (8.21)$$

Next, in order to obtain $\overset{a}{F}_{14}$, if we substitute (5.2), (8.19), (8.20), and (8.21), into (5.4e), and use (8.5), we have[(1)]

$$-\frac{\overset{1}{F}_{14}}{\beta}=\frac{\overset{2}{F}_{14}}{\alpha}=\frac{\eta_1 r}{R\left(1-\frac{r^2}{R^2}\right)}. \qquad (\eta_1=\pm 1) \qquad (8.22)$$

And substituting (8.21) into the first equation of (8.14), we obtain

$$\frac{\overset{1}{F}_{23}}{\alpha}=\frac{\overset{2}{F}_{23}}{\beta}=-\frac{\eta_2 r^3\sin\theta}{R\sqrt{1-\frac{r^2}{R^2}}}, \qquad (8.23)$$

here taking the sign of the determinant of $\overset{i}{h}_j$ as negative, i. e.

$$D=-i\frac{r^2\sin\theta}{\sqrt{1-\frac{r^2}{R^2}}}. \qquad (8.24)$$

Substituting the above results into (8.12) and (8.13), we have

$$\left.\begin{aligned}A_1^{\cdot 5}&=\frac{1}{2\beta}\left(\frac{\partial\alpha}{\partial r}+\frac{\alpha r}{R^2\left(1-\frac{r^2}{R^2}\right)}\right)=-\frac{1}{2\alpha}\left(\frac{\partial\beta}{\partial r}+\frac{\beta r}{R^2\left(1-\frac{r^2}{R^2}\right)}\right)\\ A_4^{\cdot 5}&=\frac{1}{2\beta}\left\{\frac{\partial\alpha}{\partial t}+\frac{\beta\eta_1}{R}(1-k^2r^2)\right\}=-\frac{1}{2\alpha}\left\{\frac{\partial\beta}{\partial t}-\frac{\alpha\eta_1}{R}(1-k^2r^2)\right\}.\end{aligned}\right\} \qquad (8.25)$$

(1) Since $F_{ij}\neq 0$, $\alpha$ or $\beta$ is not zero.

And by direct calculation we can easily show that (8.22), (8.23), and (8.25), satisfy (3.10), i. e.

$$\nabla_{[h}A_{i]}^{\cdot 5}+2\beta \overset{1}{F}_{ih}+2\alpha \overset{2}{F}_{hi}=0\,. \tag{8.26}$$

Now, if we assume that $\alpha$ and $\beta$ are any functions of $r$ and $t$ satisfying (8.21), then $g_{ij}$, $\overset{a}{F}_{ij}$, and $A_i^{\cdot 5}$, are completely determined by (8.19), (8.20), (8.21), (8.22), (8.23), and (8.25), and by direct calculation it is easily shown that these tensors satisfy all the remaining equations (8.2), (8.9),...., (8.14).

So we have the result: *In the case where $B \neq$ constant and $F_{ij} \neq 0$, if we assume the spherical symmetry of $g_{ij}$ and $\overset{a}{F}_{ij}$ we have*

$$\text{V}\quad \begin{cases} ds^2=-\dfrac{dr^2}{(1-k^2r^2)\left(1-\dfrac{r^2}{R^2}\right)}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1-k^2r^2)dt^2 \\ -\dfrac{\overset{1}{F}_{14}}{\beta}=\dfrac{\overset{2}{F}_{14}}{\alpha}=\dfrac{\eta_1 r}{R\left(1-\dfrac{r^2}{R^2}\right)},\quad \dfrac{\overset{1}{F}_{23}}{\alpha}=\dfrac{\overset{2}{F}_{23}}{\beta}=-\dfrac{\eta_2 r^3\sin\theta}{R\sqrt{1-\dfrac{r^2}{R^2}}}, \\ \text{other}\quad \overset{a}{F}_{ij}=0\quad (a=1,2) \qquad\qquad (8.27) \\ \left.\begin{array}{l} \varphi^4=\dfrac{i\eta_2}{4R}\quad \text{other}\quad \varphi^i=0,\quad \alpha^2+\beta^2=\dfrac{k^2}{4}\left(1-\dfrac{r^2}{R^2}\right) \\ A_1^{\cdot 5},\ A_4^{\cdot 5}\ \text{given by (8.25), other}\ A_i^{\cdot 5}=0 \end{array}\right\}\ (8.28) \end{cases}$$

*where $k$ and $R$ are any constants, $\alpha$ and $\beta$ are any functions of $r$ and $t$ satisfying (8.28), and $\eta_1^2=\eta_2^2=1$.*

## § 9. Summary.

Summarizing the results obtained above we have: *In order that the vector $u^i \equiv \Psi^\dagger A\gamma^i\Psi$ may satisfy the equation of the form (1.2), the fundamental equation for $\Psi$ must be (2.10), whose conditions of integrability supply the following five kinds of metric:*

$$\text{I}\quad \begin{cases} ds^2=-\dfrac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1-k^2r^2)dt^2 \\ \overset{a}{F}_{ij}=0 \end{cases}$$

II
$$\begin{cases} ds^2 = -\dfrac{dr^2}{1-\dfrac{r^2}{R^2}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2 \\ \overset{a}{F}_{ij} = 0 \end{cases}$$

III
$$\begin{cases} ds^2 = \dfrac{c^2}{4p\sinh^2\dfrac{\alpha}{2}}(-dr^2+dt^2) - B(d\theta^2+\sin^2\theta d\phi^2) \\ \text{where } \alpha = cr+c_1 \text{ and } \overset{a}{F}_{ij} \neq 0 \end{cases}$$

IV
$$\begin{cases} ds^2 = e^{\alpha}(-dr^2+dt^2) - B(d\theta^2+\sin^2\theta d\phi^2) \\ \text{where } \alpha = cr+c_1 \text{ and } \overset{a}{F}_{ij} \neq 0 \end{cases}$$

V
$$\begin{cases} ds^2 = -\dfrac{dr^2}{(1-k^2r^2)\left(1-\dfrac{r^2}{R^2}\right)} - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 + (1-k^2r^2)dt^2 \\ \overset{a}{F}_{ij} \neq 0 \end{cases}$$

In the case V, if $\dfrac{1}{R} \to 0$, we have

$$\left.\begin{array}{l} ds^2 \to \text{de Sitter type } ds^2 \quad (\text{Case I}) \\ \alpha^2+\beta^2 \to \dfrac{k^2}{4}, \qquad \varphi^i \to 0, \qquad \overset{a}{F}_{ij} \to 0, \end{array}\right\}$$

from which we see that if $\dfrac{1}{R} \to 0$, then V becomes I, and also if $k \to 0$, we have $\alpha^2+\beta^2 \to 0$, and accordingly

$$\left.\begin{array}{l} ds^2 \to \text{Einstein type } ds^2 \quad (\text{Case II}) \\ \alpha \to 0, \qquad \beta \to 0, \end{array}\right\}$$

which shows that if $k \to 0$, then V becomes II. From this consideration we see that the space whose line-element is given by V is intermediate between that of the de Sitter and the Einstein type. Further, because of $\overset{a}{F}_{ij} \neq 0$ in case V, we can say that the universe in which matter and radiation coexist might be the one characterized by V.

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