Relation between Intuitionistic Logic and Lattice.

By

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A. Tarski[1] has recently obtained interesting results about the relations between the propositional calculi and topologies, which lead to the question: What are the propositional calculi in lattice terms? The classical propositional calculus is the Boolean Algebra. Here I shall show that the intuitionistic propositional calculus is a residuated lattice closed with respect to the lattice operation, meet, which has a null element. For convenience we refer to A. Tarski's set of postulates of propositional calculi, and the proof shall be effected by characterizing the implication and negation in lattice terms.

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§ 1. Let $\rightarrow$, $\wedge$, $\vee$, —, be the four fundamental operators in the propositional calculi, or logical constants, the first three of which are binary operators, but the last is unary. Let A, B, C, ... , be expressions formulated from the propositional variables and the above four operators. Following A. Tarski we shall here reproduce the postulates of the propositional calculi:[1]

(i) $A \rightarrow (B \rightarrow A)$.

(ii) $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$.

(iii) $(A \wedge B) \rightarrow A$.

(iv) $(A \wedge B) \rightarrow B$.

(v) $(C \rightarrow A) \rightarrow \{(C \rightarrow B) \rightarrow [C \rightarrow (A \wedge B)]\}$.

(vi) $A \rightarrow (A \vee B)$.

(vii) $B \rightarrow (A \vee B)$.

(viii) $(A \rightarrow C) \rightarrow \{(B \rightarrow C) \rightarrow [(A \vee B) \rightarrow C]\}$.

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(1) A. Tarski, Fundamenta Math. **31** (1939), 103–134.

(ix)	$\bar{A}\to(A\to B)$.
(x)	$(A\to\bar{A})\to\bar{A}$.
(x)′	$(\bar{A}\to A)\to A$.
(α)	When A and $A\to B$ are asserted, then B is asserted.

(i)—(x) and (α) are the postulates for the intuitionistic propositional calculus, and (i)—(ix), (x)′ and (α) the postulates for the classical or two-valued propositional calculus.

§ 2. When A is asserted, we write $\vdash A$.

Lemma 2.1. $\vdash A$ implies $\vdash B\to A$.

Proof.
$\vdash A\to(B\to A)$ by (i),
$\vdash B\to A$ by (α).

Lemma 2.2. $\vdash A\to B$ and $\vdash B\to C$ imply $\vdash A\to C$.

Proof.
$\vdash A\to(B\to C)$ by Lemma 2.1,
$\vdash (A\to B)\to(A\to C)$ by (ii) and (α),
$\vdash A\to C$ by (α).

Lemma 2.3. $\vdash A\to A$.

Proof.
$\vdash \{A\to[(A\to A)\to A]\}\to\{[A\to(A\to A)]\to(A\to A)\}$ by (ii),
$\vdash A\to A$ by (i) and (α).

When $\vdash A\to B$ holds good, we write $A\subset B$, or $B\supset A$. From this and the lemmas above we can draw the following conclusions:

L 1	$A\subset A$,	by lemma 2.3,
L 2	$A\subset B$ and $B\subset C$ imply $A\subset C$,	by lemma 2.2,
(*)	$\vdash A$ implies $B\subset A$ for any B,	by lemma 2.1.

By similar reasoning, based on postulates (iii)—(viii) and (α), we can draw the further conclusions:

L 3 (1) $A\wedge B\subset A$.
(2) $A\wedge B\subset B$.
(3) $C\subset A$ and $C\subset B$ imply $C\subset A\wedge B$.

L 4 (1) $A\subset A\vee B$.
(2) $B\subset A\vee B$.
(3) $A\subset C$ and $B\subset C$ imply $A\vee B\subset C$.

These conclusions show that $A, B, C, \ldots\ldots$, from a lattice. We write $A=B$ when $A\subset B$ and $B\subset A$ hold good, or $A\rightarrow B$ and $B\rightarrow A$ is asserted. Take any asserted expression, say, I; then, by (*), we have, when $A=I$, $B\subset A=I$ for all B. Hence I is a unit element in the lattice.

§ 3. For the further investigation of the propositional calculi it is convenient to introduce another symbol for the implication. We shall write $A:B$, instead of $B\rightarrow A$, in case of need. From the definition of the binary relation $\subset$, we have.

L 5 $\quad A:B=I$ implies, and is implied by, $A\supset B$.

With our present notation A. Tarski's set of postulates can be written in the form:

(I) $\quad A\subset A:B$.

(II) $\quad (C:B):A\subset(C:A):(B:A)$.

(III) $\quad A\wedge B\subset A$.

(IV) $\quad A\wedge B\subset B$.

(V) $\quad A:C\subset(A\wedge B:C):(B:C)$.

(VI) $\quad A\subset A\vee B$.

(VII) $\quad B\subset A\vee B$.

(VIII) $\quad C:A\subset(C:A\vee B):(C:B)$.

(IX) $\quad (B:A):\bar{A}=I$.

(X) $\quad A:A\subset\bar{A}$.

(X)′ $\quad A:\bar{A}\subset A$.

Lemma 3.1. $\quad A\subset B$ implies $A:C\subset B:C$.

Proof. $\quad (B:A):C\subset(B:C):(A:C)$ by (II),

$1\subset(B:C):(A:C)$ by (I) and L 5,

$B:C\supset A:C$ by L 5.

Lemma 3.2. $\quad A:1=A$

Proof. By (I) it is sufficient to show that $A:1\subset A$.

$(A:1):(A:1)\subset[A:(A:1)]:[1:(A:1)]$ by (II),

$1\subset[A:(A:1)]:1$ by L 5,

$A:(A:1)\supset 1$ by L 5,

$A\supset A:1$ by L 5.

Lemma 3.3. $B \subset A$ implies $C:B \supset C:A$.

Proof. $C:A \subset (C:A):B$ by (I),

$C:A \subset (C:B):(A:B)$ by (II),

$C:A \subset C:B$ by lemma 3.2.

Lemma 3.4. $(A:B):C=(A:C):(B:C)=(A:C):B$.

Proof. $(A:B):C \subset (A:C):(B:C)$ by (II),

$\subset (A:C):B$ by lemma 3.3.

By symmetry

$$(A:C):B \subset (A:B):C,$$

so that the lemma is established.

Lemma 3.5. $A \wedge B:B=A:B$

Proof. $A:B \subset (A \wedge B:B):(B:B)$ by (V),

$A:B \subset A \wedge B:B$ by lemma 3.2.

On the otherhand

$A:B \supset A \wedge B:B$ by lemma 3.1,

so that the lemma is established.
The important properties of the implications are

L 6 $(A:B):C=A:B \wedge C$

Proof. $A:B \wedge C=(A:B \wedge C):(B:B \wedge C)$ by lemma 3.2.

$=(A:B):(B \wedge C:B)$ by lemma 3.4,

$=(A:B):(C:B)$ by lemma 3.5,

$=(A:B):C$ by lemma 3.4.

From L 5 and L 6 we can see that

L 7 (1) $A \supset B \wedge (A:B)$

(2) $A \supset B \wedge C$ implies $A:B \supset C$

Proof ad (1) $A:B \wedge (A:B)=(:B).(A:B)$ by L 6,

$=1$ by L 5.

Proof ad (2) $(A:B):C=A:B \wedge C$ by L 6,

$=1$ by L 5.

The element $A:B$, which has the properties mentioned in L 7, is called a residual of A to B with respect to the lattice operation, meet; or more simply, a residual of A to B. The implication in the propositional calculi is the residuation in lattice terms. These established that

$A, B, C, \ldots$, form a residuated lattice. Such a lattice is characterised by L 1—L 6, the proof of which presents no difficulties. M. Ward has investigated residuated lattices and shown that they have a unit element and are subject to the distributive law, with the relations[1]

R 1 $\qquad A : B \vee C = (A : B) \wedge (A : C)$

R 2 $\qquad A \wedge B : C = (A : C) \wedge (B : C)$

R 3 $\qquad A : [A : (A : B)] = A : B.$

These can be verified without much labour.

§ 4. Now we shall turn to the study of the negation. Postulate (IX) tells us that

$$B : A \wedge \bar{A} = (B : A) : \bar{A} \qquad \text{by L 6,}$$
$$= 1 \qquad \text{by (IX).}$$

Hence $B \supset A \wedge \bar{A}$ for all B. Therefore there exists the null element 0 such that $A \wedge \bar{A} = 0$.

In the intuitionistic propositional calculus we have, from (I) and (X).

$$\bar{A} = \bar{A} : A$$
$$= \bar{A} \wedge A : A \qquad \text{by lemma 3.2,}$$

so that we have

L 8 $\qquad \bar{A} = 0 : A.$

Thus the negation is the residuation to the null element. From this and the preceding paragraphs we can draw the conclusions that the intuitionistic propositional calculus is a free residuated lattice closed with respect to the lattice operation, meet, which has a null element, the generating elements of which are the propositional variables. For we can verify without much labour that such a lattice satisfies A. Tarski's set of postulates, provided the negation is interpreted as the residuation to the null element.

In the classical propositional calculus, by a similar argument, based on postulate (X)′, we have

(1) $\qquad A = 0 : \bar{A}$

(2) $\qquad A = 0 ; (0 : A) \qquad$ by (1) and R 3,

(3) $\qquad \bar{A} = 0 : A \qquad$ by (2) and R 2.

(1) M. Ward, Annals of Math. **39** (1938), 558-568.

In order to show that the calculus is a Boolean Algebra it is sufficient to show that

$$A \vee \bar{A}=1$$

Proof. $A \vee \bar{A}=0:[0:A \vee \bar{A}]$ by (2),

$=0:(0:A) \wedge (0:\bar{A})$ by R 1,

$=(0:\bar{A}):(0:\bar{A})$ by L 5,

$=1$ by L 5.

§ 5. We shall here give the relations between two propositional calculi, shown by V. Glivenko.(1) We write $\vdash^* A$ when $\bar{\bar{A}}$ is asserted in the intuitionistic propositional calculus. If we can show that $\vdash^* (\bar{A} \rightarrow A) \rightarrow A$, and $\vdash^* A$ and $\vdash^* A \rightarrow B$ impiy $\vdash^* B$, we can draw the conclusion that $\bar{\bar{A}}$ is asserted in the intuitionistic propositional calculus when, and only when, it is asserted in the classical one, and $\bar{A}$ is asserted in both or neither, since $\bar{A} \equiv \bar{\bar{\bar{A}}}$.

Lemma 5.1. $\bar{\bar{A}}=1$ and $\overline{\overline{A \rightarrow B}}=1$ imply $\bar{\bar{B}}=1$.

Proof. $0:A=0:[0:(0:A)]$ by R 3,

$=0:\bar{\bar{A}}$ by L 8,

$=0:1$,

$=0$.

Similarly we get

$$0:(B:A)=0.$$

From these we have

$(0:A):(B:A)=0$,

$(0:A):B=0$ by lemma 3.4,

$0:B=0$,

$0:(0:B)=1$,

so that we have

$$\bar{\bar{B}}=1.$$

Lemma 5.2. $\overline{\overline{A:(A:\bar{A})}}=1$

Proof. $A:(A:\bar{A})=A:(A \wedge \bar{A}:\bar{A})=A:(0:\bar{A})=A:\bar{\bar{A}}$,

$0:(A:\bar{\bar{A}}) \subset 0:A$, and $0:(0:\bar{\bar{A}})$,

(1) V. Glivenko, Acad. r de Belgique, 5 série **15** (1929), 183.

hence we have

$$0:(A:\bar{\bar{A}})\subset \bar{A}\wedge\bar{\bar{A}},$$

so that

$$0:[0:(A:\bar{\bar{A}})]\supset 0:\bar{A}\wedge\bar{\bar{A}}=0:0=1.$$

§ 6. Let L be any residuated lattice with a null element, whose elements we shall denote by $a, b, c, \ldots.$ Let A be any expression formulated from the propositional variables $x, y, z, \ldots,$ and fundamental operators, which is written in the form

$$\varphi(x, y, z, \ldots).$$

If we substitute in $\varphi, a, b, c, \ldots$ instead of $x, y, z, \ldots$ and translate the fundamental operators into the lattice operators, we get an element in L which we write $\varphi(a, b, c, \ldots)$. If A is asserted in the intuitionistic propositional calculus, since this is a free residuated lattice with a null element $\varphi(a, b, c, \ldots)$ denotes a unit element. Thus the intuitionistic propositional calculus satisfies any residuated lattice with a null element.

Conversely, let L be any set of elements $a, b, c, \ldots$ with four operators corresponding to those in the propositional calculi, with respect to which it is closed. If the intuitionistic propositional calculus satisfies this system, it forms a residuated lattice with a null element, for it has to satisfy similar postulates to A. Tarski's. Thus the value system of the intuitionistic propositional calculus is the same as the residuated lattice with a null element.

Contrariwise we can conclude that in the case of classical one the value system is the same as the Boolean Algebra.

Since any finite distributive lattice is a residuated one with a null element, we can see by Jaśkowski's theorem that any propositional calculus with four fundamental operators is the intuitionistic one when, and only when, it satisfies any finite distributive lattice.(1)

In order to translate Jaśkowski's theorem into lattice terms, we have to introduce operations on lattices: direct sum and adjunction. Let $L=(a, b, c, \ldots)$ $L'=(a', b', c', \ldots)$, be any set of lattices: then the set of ordered elements $(a, a', \ldots)$ forms a lattice if we define the binary relation $\subset$ by

$$(a,\ a', \ldots)\subset(a,\ b', \ldots)\quad\text{when}\quad a\subset b\quad\text{and}\quad a\subset b, \ldots,$$

(1) A. Tarski, loc. cit., 108.

which is called the direct sum of $L, L', \ldots .$ If the number of the lattices is finite, say n, and each is isomorphic to L, we shall denote the direct sum by L^n. If L is a residuated lattice with a null element, then so is L^n.

Let $L=(a, b, c, \ldots .)$ be any lattice and i be a new element; the set of elements $L+(i)$ forms a lattice if we define

$a \subset b$ in $L+(i)$ when $a, b, \in L$ and $a \subset b$ hold good in L,
$a \subset i$ in $L+(i)$ for any element a in L,

which we denote by L^*. If L is a residuated lattice with a null element, then so is L^*.

Let Z be a Boolean Algebra with two elements. We define successively

$$Z_1 = Z, \qquad Z_{n+1} = \left((Z_n)^n\right)^*$$

Z_n is a finite distributive lattice. Jaśkowski's theorem states that any propositional calculus with the four fundamental operators is the intuitionistic one if, and only if, it satisfies Z_n, $n=1, 2, \ldots .$[1]

(1) A. Tarski, loc. cit., 108.