Observations on Condon's Paper on the Fourier Transform

By

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Let L_2 be the class of complex-valued measurable function $f(x)$ defined over $(-\infty, +\infty)$ for which $\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$. Then L_2 is a Hilbert space with the inner product $(f, g) = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\,dx$. Let $\mathfrak{D}$ be the group of rotations of a plane about a fixed point. The classical Plancherel-Fourier transform is a unitary operator in L_2 and generates a cyclical group of order 4 which is isomorphic with the sub-group of $\mathfrak{D}$ through multiples of a right angle. E. U. Condon(1) proposed to find explicitly a family of unitary operators which is isomorphic with $\mathfrak{D}$ and immerses the Fourier transform. But his treatments are not rigorous, because of the introduction of an improper function $\delta(x)$ defined by the properties $\int_{-\infty}^{+\infty} \delta(x)dx = 1$ and $\delta(x) = 0$ for $x \neq 0$. Also, he makes some erroneous assertions, owing to his miscalculation.(2) Here I shall deal with the same problem and precise his results through the theory of differential set functions developed by F. Maeda. As we shall use the definitions and notations of that theory, the reader is refered to Maeda's papers(3) in this Journal.

I. Let $\beta(E)$ be a completely additive, non-negative differential set function in an abstract space Ω. We shall confine ourselves to $\mathfrak{L}_2(\beta)$, which is a Hilbert space. Let $\mathfrak{K}_\theta(E, E')$, $-\infty < \theta < +\infty$, belonging to $\mathfrak{L}_2(\beta, \beta)$.(4) If U_θ stand for operators which have $\mathfrak{K}_\theta(E, E')$ as the

(1) E. U. Condon, Proc. Nat. Acad. Sci. **23** (1937), 158–164.

(2) He obtains a double-valued representation of the group $\mathfrak{D}$, and asserts that the functional transform does not approach the identity as $\theta \to 0$; but this is not true. Cf. Condon, ibid., 163–164.

(3) e.g. F. Maeda, this Journal, 6 (1936), 19–45.

(4) F. Maeda, ibid., 31–32.

kernels of transformations, then we have the theorem:—

U_θ are a family of unitary operators such that

$$(\alpha)\quad U_\theta U_{\theta'}=U_{\theta+\theta'}\,,\qquad (\beta)\quad U_0=1\,,\qquad (\gamma)\quad \lim_{h\to 0} U_{\theta+h}=U_\theta\,,$$

if, and only if, there obtain the relations

$$(a)\quad \mathfrak{K}_\theta\mathfrak{K}_{\theta'}(E,E')=\mathfrak{K}_{\theta+\theta'}(E,E')\,,\qquad (b)\quad \mathfrak{K}_0(E,E')=\beta(EE')\,,$$

$$(c)\quad \mathfrak{K}_\theta^*(E,E')=\mathfrak{K}_{-\theta}(E,E')\,,$$

$$(d)\quad \lim_{\theta\to\theta_0}\mathfrak{K}_\theta(E,E')=\mathfrak{K}_{\theta_0}(E,E') \text{ for some } \theta_0\,.$$

First assume that (*a*), (*b*), (*c*), and (*d*) hold good.

Since $\beta(EE')$ is the kernel of the identical operator,(1) (*b*) gives (β).

From (*a*), (*b*), and (*c*) we have

$$(1)\qquad \mathfrak{K}_\theta\mathfrak{K}_\theta^*(E,E')=\mathfrak{K}_\theta^*\mathfrak{K}_\theta(E,E')=\beta(EE')\,,$$

which states that U_θ are unitary. Then, since U_θ become bounded, (*a*) gives (α).

As U_θ are unitary, (γ) holds good for any θ if it is true of some θ_0. To this end we have only to show that $\lim_{\theta\to\theta_0} U_\theta\xi(E)=U_{\theta_0}(E)$(2); that is,

$$(2)\qquad \lim_{\theta\to\theta_0}\left(\mathfrak{K}_\theta(_{(E)},E'),\overline{\xi(E')}\right)=\left(\mathfrak{K}_{\theta_0}(_{(E)},E'),\overline{\xi(E')}\right)$$

for any $\xi(E)$ belonging to $\mathfrak{L}_2(\beta)$.

From (1) we have $\|\mathfrak{K}_\theta(_{(E)},E')\|^2=\beta(E)$; hence, from (*d*), $\{\mathfrak{K}_\theta(_{(E)},E')\}$ converges strongly to $\mathfrak{K}_{\theta_0}(_{(E)},E')$.(2) Therefore we have (2).

Thus we have proved the first part of the theorem; the second part follows easily by putting $\mathfrak{K}_\theta(E,E')=U_\theta\beta(EE')$.(3)

2. Take Ω for the Euclidean space of one dimension. Then E is, as usually done, a finite open interval; say, (a,b) or a point.(4) According to which, we put $\beta(E)=b-a$ or 0. Then L_2 and $\mathfrak{L}_2(\beta)$ are isomorphic with the correspondence $f(x)\longleftrightarrow\int_E f(x)dx$. From this the kernel of the Fourier transform must be

(1) F. Maeda, loc. cit., 32.
(2) F. Maeda, loc. cit., 29.
(3) F. Maeda, this Journal, **5** (1935), 113–115.
(4) F. Maeda, this Journal, **6** (1936), 42.

$$\frac{1}{\sqrt{2\pi}}\int_E\int_{E'} e^{ixx'}dxdx',$$

which we denote by $F(E, E')$.

Let $[\theta]\equiv\theta \pmod{2\pi}$ and $-\pi<[\theta]\leqq\pi$. Let $s, s', \ldots.$ and $c, c', \ldots.$ stand for $\sin\theta, \sin\theta', \ldots.$ and $\cos\theta, \cos\theta', \ldots..$ Denote by E^0 the set which is symmetric to E with respect to the origin of Ω. Modifying the kernel given by Condon, we define

$$\mathfrak{K}_\theta(E, E')=\begin{cases} \dfrac{e^{i\left\{\frac{\pi}{4}-\frac{[\theta]}{2}\right\}}}{\sqrt{2\pi s}}\displaystyle\int_E\int_{E'} e^{i\left(-\frac{c}{2s}x^2+\frac{1}{s}xx'-\frac{c}{2s}x'^2\right)}dxdx' & \text{for } [\theta]\neq 0, \pi, \\ \beta(EE') & \text{for } [\theta]=0, \\ \beta(E^0E') & \text{for } [\theta]=\pi. \end{cases}$$

We shall now show that U_θ are a family of unitary operators which is continuously isometric with $\mathfrak{D}$ and immerses the Fourier transform.

From the foregoing definition of $\mathfrak{K}_\theta(E, E')$ we see that $\mathfrak{K}_{\frac{\pi}{2}}(E, E')=F(E, E')$ and $\mathfrak{K}_\theta(E, E')=\mathfrak{K}_{\theta'}(E, E')$ when $\theta\equiv\theta' \pmod{2\pi}$. Hence it will suffice to show that (*a*), (*b*), (*c*), and (*d*) of sec. I hold good; for then we shall see that $\mathfrak{K}_\theta(E, E')=\mathfrak{K}_{\theta'}(E, E')$ when, and only when, $\theta\equiv\theta' \pmod{2\pi}$; thus U_θ will be continuously isomorphic with $\mathfrak{D}$. That (*b*), (*c*), and (*d*) hold good follows at once from the form of $\mathfrak{K}_\theta(E, E')$. As for (*a*), we may assume that $0<[\theta]<\pi$, $[\theta']\neq 0, \pi$; for, in fact, $\mathfrak{K}_\theta(E, E')=\mathfrak{K}_\theta(E', E)$ implies $\mathfrak{K}_\theta\mathfrak{K}_{\theta'}(E, E')=\mathfrak{K}_{\theta'}\mathfrak{K}_\theta(E, E')$; so (*a*) and $\mathfrak{K}_{-\theta}\mathfrak{K}_{-\theta'}=\mathfrak{K}_{-\theta-\theta'}$ are equivalent. Put $\theta''=\theta+\theta'$.

Case 1. $[\theta'']\neq 0, \pi$.

$$\mathfrak{K}_\theta\mathfrak{K}_{\theta'}(E, E')=\frac{e^{i\left\{\frac{\pi}{2}-\frac{[\theta]}{2}-\frac{[\theta']}{2}\right\}}}{\sqrt{2\pi s}\ \sqrt{2\pi s'}}\int_{-\infty}^{+\infty}\left(\int_E e^{i\left(-\frac{c}{2s}x^2+\frac{1}{s}xx''-\frac{c}{2s}x''^2\right)}dx\right)\left(\int_{E'} e^{i\left(-\frac{c'}{2s'}x''^2+\frac{1}{s'}x''x'-\frac{c'}{2s'}x'^2\right)}dx'\right)dx''$$

$$=\frac{e^{i\left\{\frac{\pi}{2}-\frac{[\theta]}{2}-\frac{[\theta']}{2}\right\}}}{\sqrt{2\pi s}\ \sqrt{2\pi s'}}\lim_{n\to\infty}\int_E\int_{E'} e^{i\left(-\frac{c''}{2s''}x^2+\frac{1}{s''}xx'-\frac{c''}{2s''}x'^2\right)}dxdx' \int_{-r}^{n} e^{-i\frac{s''}{2ss'}\left(x''-\frac{s'}{s''}x-\frac{s}{s''}x'\right)^2}dx''$$

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(1) F. Maeda, ibid., 43.

But $\int_{-n}^{n} e^{-i\frac{s''}{2ss'}\left(x''-\frac{s'}{s''}x-\frac{s'}{s''}x'\right)^2}dx''$ converges uniformly to

$$e^{-\frac{\pi}{4}i+ai}\frac{\sqrt{2\pi s}\ \sqrt{s'}}{\sqrt{s''}},$$

when $n\to+\infty$, where $a=\pi$ for $s''<0, s'>0$, and otherwise $a=0$.

Since $[\theta]+[\theta']-2a=[\theta'']$, hence

$$\begin{aligned}\mathfrak{R}_\theta\mathfrak{R}_{\theta'}(E,E')&=\frac{e^{i\left\{\frac{\pi}{4}-\frac{[\theta'']}{2}\right\}}}{\sqrt{2\pi s''}}\int_E\int_{E'} e^{i\left(-\frac{c''}{2s''}x^2+\frac{1}{s''}xx'-\frac{c''}{2s''}x'^2\right)}dx\,dx'\\ &=\mathfrak{R}_{\theta+\theta'}(E,E').\end{aligned}$$

Case 2. $[\theta'']=0$.

Since $0<[\theta]<\pi$, therefore $[\theta']=-[\theta]$. Denote by $\phi_E(\lambda)$ the characteristic function of E.

$$\begin{aligned}\mathfrak{R}_\theta\mathfrak{R}_{\theta'}(E,E')&=\frac{1}{2\pi s}\int_{-\infty}^{+\infty}\left(\int_E e^{i\left(-\frac{c}{2s}x^2+\frac{1}{s}xx''-\frac{c}{2s}x''^2\right)}dx\right)\\ &\qquad\left(\int_{E'} e^{i\left(\frac{c}{2s}x''^2-\frac{1}{s}x''x'+\frac{c}{2s}x'^2\right)}dx'\right)dx''\\ &=\frac{1}{2\pi s}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} e^{\frac{i}{s}xx''}\left(e^{-\frac{c}{2s}ix^2}\phi_E(x)\right)dx\right\}\\ &\qquad\overline{\left\{\int_{-\infty}^{+\infty} e^{\frac{i}{s}x'x''}\left(e^{-\frac{c}{2s}ix'^2}\phi_{E'}(x')\right)dx'\right\}}dx''\\ &=\frac{1}{s}\int_{-\infty}^{+\infty}\phi_E\left(\frac{x}{s}\right)\phi_{E'}\left(\frac{x}{s}\right)dx\\ &=\beta(EE')\\ &=\mathfrak{R}_{\theta+\theta'}(E,E').\end{aligned}$$

Case 3. $[\theta'']=\pi$.

Since $0<[\theta]<\pi$, therefore $[\theta']=\pi-[\theta]$.

$$\begin{aligned}\mathfrak{R}_\theta\mathfrak{R}_{\theta'}(E,E')&=\frac{1}{2\pi s}\int_{-\infty}^{+\infty}\left(\int_E e^{i\left(-\frac{c}{2s}x^2+\frac{1}{s}xx''-\frac{c}{2s}x''^2\right)}dx\right)\\ &\qquad\left(\int_{E'} e^{i\left(\frac{c}{2s}x''^2+\frac{1}{s}x''x'+\frac{c}{2s}x'^2\right)}dx'\right)dx''\end{aligned}$$

$$=\frac{1}{2\pi s}\int_{-\infty}^{+\infty}\overline{\left\{\int_{-\infty}^{+\infty}e^{\frac{i}{s}xx''}\left(e^{\frac{c}{2s}ix^2}\phi_{E^0}(x)\right)dx\right\}}$$

$$\left\{\int_{-\infty}^{+\infty}e^{\frac{ix'x''}{s}}\left(e^{\frac{c}{2s}ix'^2}\phi_{E'}(x')\right)dx'\right\}dx''$$

$$=\frac{1}{s}\int_{-\infty}^{+\infty}\phi_{E^0}\left(\frac{x}{s}\right)\phi_{E'}\left(\frac{x}{s}\right)dx$$

$$=\beta(E^0E')$$

$$=\mathfrak{K}_{\theta+\theta'}(E,E').$$

Thus we have reached the desired conclusion.

3. Since L_2 and $\mathfrak{L}_2(\beta)$ are isomorphic with the correspondence $f(x)\longleftrightarrow\int_E f(x)dx$, there corresponds to U_θ in $\mathfrak{L}_2(\beta)$ the operator in L_2 which transforms $f(x)$ into

$$\begin{cases}\dfrac{e^{i\left\{\frac{\pi}{4}-\frac{[\theta]}{2}\right\}}}{\sqrt{2\pi s}}\,[\lim]_{A\to+\infty}\displaystyle\int_{-A}^{+A}e^{i\left(-\frac{c}{2s}x^2+\frac{1}{s}xx'-\frac{c}{2s}x'^2\right)}f(x')\,dx' & \text{for } [\theta]\neq 0,\ \pi, \\ f(x) & \text{for } [\theta]=0, \\ f(-x) & \text{for } [\theta]=\pi. \end{cases}$$

These are unitary and continuously isomorphic with $\mathfrak{D}$ and immerse the classical Plancherel-Fourier transform for $[\theta]=\frac{\pi}{2}$.

In conclusion, the writer wishes to express his hearty thanks to Prof. F. Maeda for his kind guidance.