Wave Geometry No. 47

Cosmology and Conformally Flat Space. II.[(1)]

By

Hyôitirô TAKENO.

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§ 1. Introduction.

As we have seen in the previous paper, in the ordinary relativistic cosmologies the line element of the form:

$$L_1:\quad ds^2=-F(r,t)(dr^2+r^2\,d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2\,,\quad \left(F=e^{2g(t)}\Big/\left[1+\frac{r^2}{4R^2}\right]^2\right)$$

is usually adopted, as having the properties:

(i) the existence of a non-null vector v_i satisfying the equation

$$\nabla_i v_j=\beta g_{ij}\,,\qquad \left(\beta=\frac{1}{4}\nabla_s v^s,\ v_i=\partial_i v\right) \tag{1.1}$$

(ii) the conformal flatness of the V_4 defined by L_1.[(2)]

The purpose of this paper is to find various simple forms of the line elements obtained from L_1, from the point of view of transformations of coordinates.

Besides L_1, we shall also consider the following three kinds of line element

$$L_2:\quad ds^2=e^{2\sigma(r,t)}(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2)$$

$$L_3:\quad ds^2=-A(r,t)\,dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)\,dt^2$$

$$L_4:\quad ds^2=-A(r,t)\,dr^2-B(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)\,dt^2\,,\quad (B=\text{const.})$$

which include almost all the forms of line element discussed in the Relativity theories and cosmologies, in which the spherical symmetricity of the line elements is fundamental. Thus, the transformations of coordinates (assumed as non-singular) which we are going to discuss will be restricted to the form of

$$G:\quad r=r(\bar{r},\bar{t})\,,\quad t=t(\bar{r},\bar{t})\,,\quad [\text{i. e. } G^{-1}:\ \bar{r}=\bar{r}(r,t),\ \bar{t}=\bar{t}(r,t)]$$

which preserves the spherical symmetry of any quantity.

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(1) This paper is continued from H. Takeno, this Journal, **10** (1940), 173: (W.G. No. 39).

(2) W.G. No. 39, 200, Theorem 23.

§ 2. Conditions of G in the general case.

Any one of L_i $(i=1,2,3,4)$ is a special form of the most general spherically symmetric line element

$$L: ds^2=-A(r,t)dr^2-B(r,t)(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)dt^2. \qquad (2.1)$$

Now, we shall assume that (2.1) is transformed into another spherically symmetric line element

$$\bar{L}: d\bar{s}^2=-\bar{A}(\bar{r},\bar{t})d\bar{r}^2-\bar{B}(\bar{r},\bar{t})(d\theta^2+\sin^2\theta d\phi^2)+\bar{C}(\bar{r},\bar{t})d\bar{t}^2. \qquad (2.2)$$

Then, from the transformation law of g_{ij},

$$\bar{g}_{ij}=\frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}g_{lm}, \qquad (2.3)$$

we have, as the condition to be satisfied by G,

$$\left.\begin{aligned} -\bar{A}&=-\left(\frac{\partial r}{\partial \bar{r}}\right)^2A+\left(\frac{\partial t}{\partial \bar{r}}\right)^2C, \quad \bar{C}=-\left(\frac{dr}{\partial \bar{t}}\right)^2A+\left(\frac{\partial t}{\partial \bar{t}}\right)^2C \\ 0&=-\frac{\partial r}{\partial \bar{r}}\frac{\partial r}{\partial \bar{t}}A+\frac{\partial t}{\partial \bar{r}}\frac{\partial t}{\partial \bar{t}}C, \quad B=\bar{B}. \end{aligned}\right\} \qquad (2.4)$$

But it is obvious that (2.4) is equivalent to

$$\bar{B}=B \quad \text{i.e.} \quad \sqrt{\bar{B}}=\eta\sqrt{B}, \qquad (\eta^2=1) \qquad (2.5)$$

and

$$\left.\begin{aligned} &\sqrt{A}\sqrt{\bar{C}}\frac{\partial r}{\partial \bar{r}}=\epsilon\sqrt{C}\sqrt{\bar{A}}\frac{\partial t}{\partial \bar{t}}, \quad \sqrt{A}\sqrt{\bar{A}}\frac{\partial r}{\partial \bar{t}}=\epsilon\sqrt{C}\sqrt{\bar{C}}\frac{\partial t}{\partial \bar{r}} \\ &-\bar{A}=-\left(\frac{\partial r}{\partial \bar{r}}\right)^2A+\left(\frac{\partial t}{\partial \bar{r}}\right)^2C, \quad (\epsilon^2=1) \end{aligned}\right\} \qquad (2.6)$$

Hence we have

Theorem 1. *A necessary and sufficient condition for L to be transformed into $\bar{L}$ by a transformation G is given by (2.5) and (2.6).*

Accordingly, so long as r and t which satisfy (2.5) and (2.6) are not found as functions of $\bar{r}$ and $\bar{t}$, the two line-elements L and $\bar{L}$ are not transformable into each other by G. It is generally rather difficult to solve (2.5) and (2.6) directly, but when a solution v_i of (1.1) is known, we can solve the problem in a simple way by virtue of the transformation law of v.

§ 3. On the line element L_1. I. General form of the line element of S_4 and E_4.

We shall denote a four-dimensional space of constant curvature and a four-dimensional flat space by S_4 and E_4 respectively. S_4 and E_4 are characterized by

$$K_{ijlm}=k^2(g_{im}g_{jl}-g_{il}g_{jm})\,,\quad (3.1\text{a}) \qquad K_{ijlm}=0\,, \quad (3.1\text{b})$$

respectively. In this section, first, we shall show that *when ds^2 takes the form*

$$ds^2=-A(r,t)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2\,, \tag{3.2}$$

the general solution of (3.1) must be of the form L_1; and then we shall find the actual form of F.[1]

Since S_4 and E_4 are both conformally flat, it follows that[2]

$$A=[a(t)]^2[1+b(t)r^2]^{-2}\,, \qquad (a, b \text{ are any functions})\,. \tag{3.3}$$

Next, in (3.1a), if we put $(i, j, l, m)=(2, 3, 2, 3)$, we have

$$(1+br^2)^2(k^2a^2-4b-\dot{a}^2)+2a\dot{a}\dot{b}r^2(1+br^2)-a^2\dot{b}^2r^4=0\,, \tag{3.4}$$

so

$$k^2a^2-4b-\dot{a}^2=0\,, \qquad \dot{b}=0\,. \tag{3.5}$$

Hence we know that b must be constant and (3.2) coincides with L_1 if we take $b=\dfrac{1}{4R^2}$.

When $b=0$, from (3.5), we have

$$A=c^2e^{\pm 2kt} \quad \text{or} \quad A=c^2\,, \qquad (c \text{ is constant}) \tag{3.6}$$

according as $k\neq 0$ or $k=0$. Next, when $b\neq 0$, from (3.5), we have $\dot{a}(k^2a-\ddot{a})=0$; but if $\dot{a}=0$, ds^2 becomes coincident with that of Einstein-type space; which, as is easily seen, does not satisfy (3.1). So we abandon this case. Therefore, according as $k\neq 0$ or $k=0$, we have

$$a=c_1e^{kt}+c_2e^{-kt}\,, \qquad \left(4c_1c_2k^2=\frac{1}{R^2}\right) \tag{3.7}$$

or

$$a=c_1t+c_2\,, \qquad \left(c_1^2=-\frac{1}{R^2}\right). \tag{3.8}$$

Conversely, by direct substitution we can readily show that (3.6), (3.7), and (3.8) satisfy (3.1). So we have

Theorem 2. *The most general forms of the line elements of S_4 and E_4 which are of the form (3.2) are given by*

$$S_4:\quad A=c^2e^{\pm 2kt}\,; \quad A=\frac{(c_1e^{kt}+c_2e^{-kt})^2}{\left[1+\dfrac{r^2}{4R^2}\right]^2}\,, \quad \left(4c_1c_2k^2=\frac{1}{R^2}\right) \tag{3.9}$$

$$E_4:\quad A=c^2\,; \qquad A=\frac{(c_1t+c_2)^2}{\left[1+\dfrac{r^2}{4R^2}\right]^2}\,, \quad \left(c_1^2=-\frac{1}{R^2}\right). \tag{3.10}$$

(1) The solutions of (3.1a) have already been obtained by the present writer; this Journal, **7** (1937), 44 (W. G. No. 11).

(2) W. G. No. 39, 189, Theorem 14.

These four kinds of line elements are transformed into

$$S_4 \begin{cases} ds^2 = -e^{2kt}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2 & (3.11) \\ ds^2 = -\dfrac{(e^{kt}+e^{-kt})^2}{4k^2R^2\left[1+\dfrac{r^2}{4R^2}\right]^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2 & (3.12) \end{cases}$$

$$E_4 \begin{cases} ds^2 = -(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2 & (3.13) \\ ds^2 = \dfrac{t^2}{R^2\left[1+\dfrac{r^2}{4R^2}\right]^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2 & (3.14) \end{cases}$$

by the transformations $cr=\epsilon\bar{r}$, $\eta t=\bar{t}$; $r=\epsilon\bar{r}$, $c_1e^{kt}=\eta e^{k\bar{t}}/2kR$; $cr=\epsilon\bar{r}$, $t=\eta\bar{t}$; $r=\epsilon\bar{r}$, $c_1t+c_2=\eta c_1\bar{t}$, respectively.(1) $(\epsilon^2=\eta^2=1)$.

Moreover, it is easily seen that by the transformation $r^2=4R^2\bar{r}^2$, two line-elements (3.12) and (3.14) are transformable into the form in which $4R^2=1$.

§ 4. On the line element L_1. II. Solution of (1.1).

In this section we intend to find the general spherically symmetric solution v of the equation (1.1) with respect to the line element L_1.

Since v is spherically symmetric, we have $v=v(r, t)$. Hence $v_2=v_3=0$. Consequently, if we calculate $\{^{k}_{ij}\}$, and substitute it into (1.1), we have

$$\text{(a)}\quad \partial_{11}v-\frac{F'}{2F}\partial_1 v-F\dot{g}\partial_4 v=-F\partial_{44}v\,,\qquad \text{(b)}\quad \partial_{14}v-\dot{g}\partial_1 v=0\,,$$

$$\text{(c)}\quad (r^2F)'\frac{\partial_1 v}{2F}-r^2F\dot{g}\partial_4 v=-r^2F\partial_{44}v\,,\qquad \text{(d)}\quad \beta=\partial_{44}v\,.$$

First, in the case: $\partial_1 v=0$ and $\partial_4 v\neq 0$, we have at one

$$v_1=0\,,\quad v_4=c_1e^g\,;\quad v=c_1\int e^g dt\,. \tag{4.1}$$

Secondly, in the case: $\partial_1 v\neq 0$ and $\partial_4 v=0$, from (c) we have $(r^2F)'=0$; but obviously this equation cannot be satisfied by F. Hence such a case does not occur.

Lastly, when $\partial_1 v\neq 0$ and $\partial_4 v\neq 0$, from (a), (b), and (c) we have

$$\partial_1 v=\varphi(r)e^g \quad\text{and}\quad \varphi(r)=pr\left(1+\frac{r^2}{4R^2}\right)^{-2} \tag{4.2}$$

where p is an arbitrary constant. If we assume that $\dfrac{1}{R^2}=0$, from (4.2) and (a) we have

(1) These transformations are all denoted by G_1, further on; (see § 5). Moreover, we can make use of $r=\epsilon\bar{r}$, $ce^{\pm kt}=\eta e^{k\bar{t}}$ instead of the first transformation.

$$e^g = ce^{c't}\,; \quad v = \frac{p}{2} r^2 e^g + \phi(t)\,, \quad p = e^g(\dot{g}\dot{\phi} - \ddot{\phi})\,. \tag{4.3}$$

If $c' \neq 0$, e^g in (4.3) gives S_4 in which $c'^2 = k^2$ (Theorem 2), and the corresponding $\phi(t)$ is obtained from

$$\dot{\phi} = e_1 e^{c't} + e_2 e^{-c't} \tag{4.4}$$

where e_1 and e_2 are constants satisfying $2cc'e_2 = p$. If $c' = 0$, e^g in (4.3) gives E_4, and the corresponding ϕ is obtained from $c\ddot{\phi} = -p_1$.

When $\frac{1}{R^2} \neq 0$ in the equation above, from (4.2) and (a) we can easily obtain

$$v = -\frac{2R^2 p e^g}{1 + \frac{r^2}{4R^2}} + \phi(t)\,, \quad p = e^g(\dot{g}\dot{\phi} - \ddot{\phi})\,, \quad e^{2g}\ddot{g} = \frac{1}{R^2}\,; \tag{4.5}$$

and from the last of these equations we have for e^g

$$\text{(i) when } e^g\dot{g} \neq \text{const., } e^g = c_1 e^{kt} + c_2 e^{-kt} \quad \left(\begin{array}{l} k,\, c_1,\, c_2 \text{ are constants} \\ \text{satisfying } 4c_1c_2k^2 = \frac{1}{R^2} \end{array}\right) \tag{4.6}$$

$$\text{(ii) when } e^g\dot{g} = \text{const., } e^g = c_1 t + c_2 \quad \left(\begin{array}{l} c_1,\, c_2 \text{ are constants} \\ \text{satisfying } c_1^2 = -\frac{1}{R^2} \end{array}\right) \tag{4.7}$$

These two e^g's give S_4 and E_4 respectively (Theorem 2, § 3), and the corresponding ϕ's are given by

$$\dot{\phi} = e_1 e^{kt} + e_2 e^{-kt} \tag{4.8}$$

and

$$\dot{\phi} = e_1 t + e_2 \tag{4.9}$$

where e_1 and e_2 are constants satisfying $2k(c_2e_1 - c_1e_2) = p$ and $c_2e_1 - c_1e_2 = p$ respectively. Hence we have

Theorem 3. *When ds^2 is of the form L_1, the general spherically symmetric solution v of the eqaution (1.1) is given by*

(i) *when V_4 is neither S_4 nor E_4, for arbitrary e^g*

$$v = c\int e^{g(t)}\,dt \tag{4.10}$$

(ii) *when V_4 is S_4 in which $K = \frac{k^2}{12}$, according as F takes the first or the second form of (3.9),*

$$v = \frac{p}{2} r^2 e^g + \phi(t)\,, \quad \dot{\phi} = e_1 e^{\pm kt} + e_2 e^{\mp kt}\,, \quad (e^g = ce^{\pm kt},\ 2cke_2 = p) \tag{4.11}$$

$$\left.\begin{array}{l} v=-2R^2p\left(1+\dfrac{r^2}{4R^2}\right)^{-1}e^g+\phi(t)\,, \quad \dot{\phi}=e_1e^{kt}+e_2e^{-kt} \\ \left(e^g=c_1e^{kt}+c_2e^{-kt},\ 2k(c_2e_1-c_1e_2)=p\right), \end{array}\right\} \tag{4.12}$$

or

(iii) *when* V_4 *is* E_4, *according as* F *takes the first or the second form of* (*3.10*),

$$v=\frac{c}{2}pr^2+\phi(t)\,, \quad \ddot{\phi}=-\frac{p}{c} \tag{4.13}$$

or

$$\left.\begin{array}{l} v=-2R^2p\left(1+\dfrac{r^2}{4R^2}\right)^{-1}(c_1t+c_2)+\phi(t)\,, \quad \dot{\phi}=e_1t+e_2\,. \\ (c_2e_1-c_1e_2=p) \end{array}\right\} \tag{4.14}$$

Thus we have, corresponding to (3.11), (3.12), (3.13), and (3.14),

$$v=m\left(r^2e^{kt}-\frac{1}{k^2}e^{-kt}\right)+ne^{kt}+q \tag{4.15}$$

$$v=m\left(e^{kt}-\frac{r^2}{4R^2}e^{-kt}\right)\left(1+\frac{r^2}{4R^2}\right)^{-1}+n(e^{kt}-e^{-kt})+q \tag{4.16}$$

$$v=m(t^2-r^2)+nt+q^{(1)} \tag{4.17}$$

$$v=m\left(1-\frac{r^2}{4R^2}\right)t\left(1+\frac{r^2}{4R^2}\right)^{-1}+nt^2+q \tag{4.18}$$

where m, n, and q are arbitrary constants.

§ 5. On the line element L_1. III. Transformations which make the form of L_1 invariant.

In this section we shall obtain the general form of the transformations which keep the form of L_1 invariant. First, we assume that our V_4 is neither S_4 nor E_4; the transformations in the case of S_4 and E_4 will be considered in the later sections.

Suppose that, by a transformation G, L_1 is transformed into

$$\bar{L}_1:\quad d\bar{s}^2=-\bar{F}(d\bar{r}^2+\bar{r}^2d\theta^2+\bar{r}^2\sin^2\theta d\phi^2)+d\bar{t}^2\,, \quad \left(\bar{F}=e^{2\bar{g}(\bar{t})}\Big/\left[1+\frac{\bar{r}^2}{4\bar{R}^2}\right]^2\right)$$

then, in consequence of Theorem 1, the condition is

$$Fr^2=\bar{F}\bar{r}^2 \quad \text{i. e.} \quad re^g\left(1+\frac{r^2}{4R^2}\right)^{-1}=\eta\bar{r}e^{\bar{g}}\left(1+\frac{\bar{r}^2}{4\bar{R}^2}\right)^{-1} \tag{5.1}$$

and

$$\frac{r}{\bar{r}}\frac{\partial t}{\partial\bar{t}}=\epsilon\frac{\partial r}{\partial\bar{r}}\,, \quad \frac{r}{\bar{r}}F\frac{\partial r}{\partial\bar{t}}=\epsilon\frac{\partial t}{\partial\bar{r}}\,, \quad -\frac{r^2}{\bar{r}^2}F=-\left(\frac{\partial r}{\partial\bar{r}}\right)^2F+\left(\frac{\partial t}{\partial\bar{t}}\right)^2. \tag{5.2}$$

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(1) This result coincides with that of (11.20) in W. G. No. 39.

But since v, corresponding to L_1 and $\bar{L}_1$, are $c\int e^g dt$ and $\bar{c}\int e^{\bar{g}} d\bar{t}$ respectively, from (5.2) and the transformation law of v we have at once

$$t=t(\bar{t}), \quad r=r(\bar{r}), \quad \frac{dt}{d\bar{t}}=\epsilon\frac{\bar{r}}{r}\frac{dr}{d\bar{r}}=\epsilon', \quad (\epsilon'^2=1). \tag{5.3}$$

Therefore, as the general solutions of (5.1) and (5.2), we have the following two kinds of transformations:

$$G_1: t=\eta\bar{t}+a, \quad r=b\bar{r}; \quad G_2: t=\eta\bar{t}+a, \quad r=\frac{b}{\bar{r}} \tag{5.4}$$

where a and b are arbitrary constants, and by G_1 and G_2, R and $g(t)$ undergo the following transformations,

$$\text{by } G_1: \quad e^{2\bar{g}}=b^2e^{2g(\eta\bar{t}+a)}, \quad \bar{R}^2=R^2/b^2 \tag{5.5}$$

and

$$\text{by } G_2: \quad e^{2\bar{g}}=\frac{16R^4}{b^2}e^{2g(\eta\bar{t}+a)}, \quad \bar{R}^2=\frac{b^2}{16R^2}. \tag{5.6}$$

If we classify L_1 as follows:

L_{1a}: L_1 in which $\frac{1}{R^2}=0$ i.e. $F=e^{2g}$

L_{1b}: L_1 in which $\frac{1}{R^2}\neq 0$,

then, as is seen from (5.5) and (5.6), G connecting L_{1a} with L_{1b} does not exist, and G_2 is a transformation connecting two ds^2's of the form L_{1b}. Hence we have

Theorem 4. *When V_4 is neither S_4 nor E_4, the general form of G which keeps the form of L_1 invariant is given by G_1 and G_2, and L_{1a} and L_{1b} are not transformable to each other. Furthermore, G which keeps the form of L_{1a} invariant is G_1 alone, whereas G which keeps that of L_{1b} invariant is given by both G_1 and G_2.*

§ 6. On the line element L_2. I. General form of the line element of S_4 and E_4.

The general form of the line element of S_4 and E_4 which are of the form L_2 is given by

$$e^{-\sigma}=p(r^2-t^2)+c_2t+c_3 \qquad (c_2^2=k^2-4pc_3), \tag{6.1}$$

where p, c_2, and c_3 are constants.(1) And by transformations of the form $a\bar{r}=\epsilon r$, $a\bar{t}+b=t$,(2) ds^2's become:

(1) W. G. No. 39, 185. Theorem 10.

(2) These transformations are all to be denoted by G_3. See § 8.

(i) when V_4 is S_4

$$ds^2=\frac{1}{\left(1-\frac{k^2}{4}X\right)^2}(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2)\,,\quad (X\equiv t^2-r^2) \tag{6.2}$$

or

$$ds^2=\frac{1}{k^2t^2}(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2)\,, \tag{6.3}$$

(ii) when V_4 is E_4

$$ds^2=\frac{1}{(t^2-r^2)^2}(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2)\,, \tag{6.4}$$

or

$$ds^2=-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2\,. \tag{6.5}$$

Here (6.5) is the same as (3.13), and v corresponding to (6.2), (6.3), and (6.4) are given by

$$v=m\frac{1}{1-\frac{k^2}{4}X}+n\frac{t}{1-\frac{k^2}{4}X}+q\,, \tag{6.6}$$

$$v=m\frac{X}{t}+n\frac{1}{t}+q\,, \tag{6.7}$$

and

$$v=m\frac{1}{X}+n\frac{t}{X}+q\,. \tag{6.8}$$

§ 7. On the line element L_2. II. Solution of (1.1).

When ds^2 is of the form L_2, the spherically symmetric solutions $e^{2\sigma}$ and v of (1.1) are given (excluding the solutions obtained by the transformation $t=\bar{t}+\text{const.}$) by[1]

$$L_{2\text{I}}:\quad \left.\begin{aligned} &ds^2=f(X)(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2) &(7.1)\\ &v=c_1\int f(X)\,dX \qquad (X\equiv t^2-r^2) &(7.2)\end{aligned}\right\}$$

$$L_{2\text{II}}:\quad \left.\begin{aligned} &ds^2=\frac{1}{t^2}\Phi(Y)(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2) &(7.3)\\ &v=c_2\int\Phi(Y)\,dY \qquad \big(Y\equiv(X-a)/t\big) &(7.4)\end{aligned}\right\}$$

$$L_{2\text{III}}:\quad \left.\begin{aligned} &ds^2=h(t)(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2) &(7.5)\\ &v=c_3\int h(t)\,dt &(7.6)\end{aligned}\right\}$$

(1) W. G. No. 39, 206, Theorem 28.

where c_i $(i=1, 2, 3)$ are arbitrary constants. Specially, when V_4 is either S_4 or E_4, as stated in the preceding section, two independent v's correspond to one line element, the additional constants being excluded.

§ 8. On the line element L_2. III. Transformations which make the form of L_2 invariant.

Suppose that the line element L_2 is transformed into

$$\bar{L}_2:\quad d\bar{s}^2=e^{2\bar{\sigma}(\bar{r},\bar{t})}(-d\bar{r}^2-\bar{r}^2d\theta^2-\bar{r}^2\sin^2\theta d\phi^2+d\bar{t}^2)$$

by a transformation G; then, in consequence of Theorem 1, the conditions to be satisfied by G are

$$re^{\sigma}=\eta\bar{r}e^{\bar{\sigma}} \tag{8.1}$$

and
$$\frac{\partial r}{\partial \bar{r}}=\epsilon\frac{\partial t}{\partial \bar{t}},\quad \frac{\partial r}{\partial \bar{t}}=\epsilon\frac{\partial t}{\partial \bar{r}},\quad \left(\frac{\partial t}{\partial \bar{r}}\right)^2-\left(\frac{\partial r}{\partial \bar{r}}\right)^2=-\frac{r^2}{\bar{r}^2}. \tag{8.2}$$

From the first two equations of (8.2) we have

$$\epsilon r=g_1(\xi)+g_2(\bar{\xi}),\qquad t=g_1(\xi)-g_2(\bar{\xi})+c, \tag{8.3}$$

where g_1 and g_2 are arbitrary functions, $\xi\equiv\bar{t}+\bar{r}$, $\bar{\xi}\equiv\bar{t}-\bar{r}$, and c is a constant. Substituting (8.3) into the third equation of (8.2), after some calculation we have two systems of solutions of (8.2), as follows:

$$\text{(i)}\quad g_1=p\xi+q,\qquad g_2=-p\bar{\xi}-q \tag{8.4}$$

i. e.,
$$G_3:\quad \epsilon r=a\bar{r},\qquad t=a\bar{t}+b$$

$$\text{(ii)}\quad g_1=\frac{1}{p(p\xi+q)}+c,\quad g_2=\frac{-1}{p(p\bar{\xi}+q)}-c \tag{8.5}$$

i. e.,
$$G_5:\quad \epsilon r=\frac{-2\bar{r}}{(p\bar{t}+q)^2-p^2\bar{r}^2},\quad t=\frac{2(\bar{t}+q/p)}{(p\bar{t}+q)^2-p^2\bar{r}^2}+l,$$

where $p, q, a, b,$ and l are arbitrary constants. But G_3 is a special form of G_5 in which $l=\frac{b}{2}-\frac{1}{pq}$, $\frac{a}{2}=-\frac{1}{q^2}$, $p\to 0$. Hence we have

Theorem 5. *The general transformation G which makes the form of the line element L_2 invariant is G_5, the transformation law of $e^{2\sigma}$ being given by (8.1).*

Since G_5 is equivalent to the product of the following three transformations

$$\text{(a)}\begin{cases}\epsilon r=\dfrac{2}{p}r_1\\[2ex] t=\dfrac{2}{p}t_1+l\end{cases}\qquad \text{(b)}\begin{cases}r_1=\dfrac{r_2}{(t_2)^2-(r_2)^2}\\[2ex] t_1=\dfrac{t_2}{(t_2)^2-(r_2)^2}\end{cases}\qquad \text{(c)}\begin{cases}-r_2=p\bar{r}\\[2ex] t_2=p\bar{t}+q,\end{cases}$$

we have

Theorem 6. *The transformation G's which keep the form of L_2 invariant are given by G_3, G_4, and their combinations, where*

$$G_4:\ r=\frac{\bar r}{\bar X},\quad t=\frac{\bar t}{\bar X}.\quad (\bar X\equiv \bar t^2-\bar r^2)$$

As a matter of course G_5's make a group and G_3's its subgroup.

§ 9. Transformation G connecting $L_{2\mathrm{I}}$, $L_{2\mathrm{II}}$, and $L_{2\mathrm{III}}$.

In this section we shall study whether $L_{2\alpha}$ ($\alpha=$ I, II, III) in § 7 are transformable into one another, and consider, in the transformable case, how to find the actual forms of transformation. From Theorems 5 and 6 it is clear that we can solve these problems by studying whether the line elements obtained from $L_{2\mathrm{I}}$, $L_{2\mathrm{II}}$, and $L_{2\mathrm{III}}$ by suitable G_3 are transformable by G_4 or not.(1)

Now, if we denote by $L'_{2\alpha}$ the line elements transformed from $L_{2\alpha}$ ($\alpha=$ I, II, III) by the transformation G_3's, the $e^{2\sigma}$'s and v's corresponding to $L'_{2\alpha}$ are given by

$$L'_{2\mathrm{I}}:\ e^{2\sigma_1}=a_1^2 f(X'),\quad v=c_1\int f(X')\,dX',\quad \left(X'\equiv(a_1t_1+b_1)^2-a_1^2r_1^2\right)$$

$$L'_{2\mathrm{II}}:\ e^{2\sigma_2}=\frac{a_2^2}{(a_2t_2+b_2)^2}\,\Phi(Y'),\quad v=c_2\int\Phi(Y')\,dY',\quad \left(Y'\equiv\frac{(a_2t_2+b_2)^2-a_2^2r_2^2-a}{a_2t_2+b_2}\right)$$

$$L'_{2\mathrm{III}}:\ e^{2\sigma_3}=a_3^2h(t'),\quad v=c_3\int h(t')dt',\quad (t'\equiv a_3t_3+b_3)$$

where a_i and b_i ($i=1,2,3$) are the respective parameters of G_3's and the suffices to σ, r, t are put in order to distinguish the cases.

(i) *Transformations connecting $L'_{2\mathrm{III}}$ and $L'_{2\mathrm{II}}$.* If we take the transformation G of the form: $r_3=r_3(r_2,t_2)$ and $t_3=t_3(r_2,t_2)$, from the transformation law of v, t_3 must be a function of Y', hence we have

$$\frac{\partial t_3}{\partial r_2}\{(a_2t_2+b_2)^2+a_2^2r_2^2+a\}+\frac{\partial t_3}{\partial t_2}\,2a_2r_2(a_2t_2+b_2)=0\,. \tag{9.1}$$

But from the form of G_4 it follow that $t_3=t_2/(t_2^2-r_2^2)$; so that, from (9.1), we have

$$b_2=a=0\,;\quad \text{so}\quad Y'=a_2(t_2^2-r_2^2)/t_2\,. \tag{9.2}$$

Conversely, when (9.2) holds good, we can easily prove that $L'_{2\mathrm{III}}$ is transformable into $L'_{2\mathrm{II}}$ by G_4. Further, in consequence of (8.1), in this transformation $e^{2\sigma_2}$ and $\Phi(Y')$ are given by

$$e^{2\sigma_2}=\frac{r_3^2}{r_2^2}\,e^{2\sigma_3}=\frac{a_3^2}{(t_2^2-r_2^2)^2}\,h\!\left(a_3\frac{t_2}{t_2^2-r_2^2}+b_3\right), \tag{9.3}$$

(1) Since G_3^{-1} is a transformation of the same kind as G_3.

and $$\Phi(Y')=t_2^2 e^{2\sigma_2}=\frac{a_2^2a_3^2}{Y'^2}h\left(\frac{a_2a_3}{Y'}+b_3\right). \tag{9.4}$$

Hence we have

Theorem 7_1. *If V_4 is neither S_4 nor E_4, $L'_{2\,\mathrm{III}}$ and $L'_{2\,\mathrm{II}}$ are transformable by G_4 when, and only when, $b_2=a=0$. Consequently $L_{2\,\mathrm{III}}$ and $L_{2\,\mathrm{II}}$ are transformable by G when, and only when, $a=0$.*[1]

The actual forms of G's which transform an $L_{2\,\mathrm{III}}$ into an $L_{2\,\mathrm{II}}$ will easily be obtained by using (9.2). Among these transformations the simplest is obtained by putting $a_2=1=a_3$ and $b_2=0=b_3$; this is nothing but G_4.

(ii) *Transformations connecting $L'_{2\,\mathrm{III}}$ and $L'_{2\,\mathrm{I}}$.* If the equation of transformation is $r_3=r_3(r_1, t_1)$ and $t_3=t_3(r_1, t_1)$, from the transformation law of v, t_3 must be a function of X', so t_3, i. e. $t_1/(t_1^2-r_1^2)$, must satisfy

$$\frac{\partial t_3}{\partial r_1}(a_1t_1+b_1)+\frac{\partial t_3}{\partial t_1}a_1r_1=0\,; \tag{9.5}$$

but this is obviously impossible. Hence we have

Theorem 7_2. *$L'_{2\,\mathrm{III}}$ and $L'_{2\,\mathrm{I}}$ are not transformable by G_4, and consequently $L_{2\,\mathrm{III}}$ and $L_{2\,\mathrm{I}}$ are also not transformable by G, provided that V_4 is neither S_4 nor E_4.*

(iii) *Transformations connecting $L'_{2\,\mathrm{I}}$ and $L'_{2\,\mathrm{II}}$.* If the equation of G is $r_1=r_1(r_2, t_2)$ and $t_1=t_1(r_2, t_2)$, from the transformation law of v, X' becomes a function of Y', so that the equation obtained from (9.1) must hold good if we substitute X' for t_3. Then, by using the expressions for r_1 and t_1 in G_4, we have the following relation as a necessary and sufficient condition for the transformability of $L'_{2\,\mathrm{I}}$ and $L'_{2\,\mathrm{III}}$,

$$a_1a_2=2b_1b_2\,, \qquad b_2^2=-a\,. \tag{9.6}$$

But since a_1 and a_2 must not be zero, it is impossible for b_1, b_2, and a to be zero. Then, as is readily calculated from (9.6), the relation

$$Y'=\frac{2a_2b_1(t_2^2-r_2^2)}{2b_1t_2+a_1}+2b_2\,; \qquad X'=\frac{2a_1a_2b_1}{Y'-2b_2}+b_1^2 \tag{9.7}$$

holds good, so that $e^{2\sigma_2}$ and $\Phi(Y')$ obtained by the transformation are given by

$$e^{2\sigma_2}=\frac{r_1^2}{r_2^2}e^{2\sigma_1}=\frac{a_1^2}{(t_2^2-r_2^2)}f\left(\frac{2a_1a_2b_1}{Y'-2b_2}+b_1^2\right) \tag{9.8}$$

and $$\Phi(Y')=\frac{(a_2t_2+b_2)^2}{a_2^2}e^{2\sigma_2}=\frac{a_1^2a_2^2}{(Y'-2b_2)^2}f\,. \tag{9.9}$$

---

(1) The expression "$L_{2\,\mathrm{III}}$ and $L_{2\,\mathrm{II}}$ are transformable by G when, and only when, $a=0$" means that an arbitrary $L_{2\,\mathrm{III}}$ is transformed into a certain one of $L_{2\,\mathrm{II}}$ in which $a=0$; also, an arbitrary $L_{2\,\mathrm{II}}$ in which $a=0$ is transformed into a certain one of $L_{2\,\mathrm{III}}$; and none of $L_{2\,\mathrm{III}}$ is ever transformable into the form of $L_{2\,\mathrm{II}}$ in which $a\neq 0$ (and vice versa). Hereafter we shall often use such an expression, whenever it can easiiy be understood.

Hence we obtain

Theorem 7_3. *$L'_{2\mathrm{I}}$ and $L'_{2\mathrm{II}}$ are transformable by G_4 when, and only when, (9.6) holds good; and consequently $L_{2\mathrm{I}}$ and $L_{2\mathrm{II}}$ are transformable by G when, and only when, $a \neq 0$, provided that V_4 is neither S_4 nor E_4.*

The actual form of G which transforms an $L_{2\mathrm{I}}$ into a certain $L_{2\mathrm{II}}$ is obtained as a product of an arbitrary G_3 whose parameters are $a_1(\neq 0)$ and $b_1(\neq 0)$, G_4, and G_3^{-1} whose parameters are a_2 and b_2 determined by (9.6). In this case the transformation is impossible by G_4 alone.

Summarizing theorems 7_1, 7_2, and 7_3, we have

Theorem 8. *Provided that V_4 is neither S_4 nor E_4, the line elements $L_{2\alpha}$ ($\alpha=$ I, II, III) are classified into two categories as follows:*

$$L_{2\mathrm{a}}:\ [L_{2\mathrm{III}}, L_{2\mathrm{II}}(a=0)], \qquad L_{2\mathrm{b}}:\ [L_{2\mathrm{I}}, L_{2\mathrm{II}}(a \neq 0)].$$

And line elements in the same category are intertransformable by G, but line elements in one category are not transformable into those of the other.

§ 10. Relations between L_1 and L_2.

In § 5 we have studied the relations between two kinds of L_1, and in § 9 the relations between any two of the three kinds of L_2. In this section we shall consider the transformations which connect L_1 and L_2. Let us suppose that the V_4 is neither S_4 nor E_4, and that L_1 is transformed into $\bar{L}_2$ by G. From Theorem 1, the conditions to be satisfied by $r(\bar{r}, \bar{t})$ and $t(\bar{r}, \bar{t})$ are

$$Fr^2 = e^{2\bar{\sigma}}\bar{r}^2 \quad \text{i. e.} \quad re^g\left(1+\frac{r^2}{4R^2}\right)^{-1} = \eta e^{\bar{\sigma}}\bar{r} \tag{10.1}$$

and
$$\frac{\partial t}{\partial \bar{t}} = \epsilon\sqrt{F}\frac{\partial r}{\partial \bar{r}}, \quad \frac{\partial t}{\partial \bar{r}} = \epsilon\sqrt{F}\frac{\partial r}{\partial \bar{t}}, \quad \left(\frac{\partial t}{\partial \bar{r}}\right)^2 - \left(\frac{\partial t}{\partial \bar{t}}\right)^2 = -e^{2\bar{\sigma}}. \tag{10.2}$$

Now, we shall first show that by G an $L_{1\mathrm{a}}$ is transformable into the form of $L_{2\mathrm{a}}$. For this purpose it will be sufficient if we prove that $L_{1\mathrm{a}}$ is transformable into $L_{2\mathrm{III}}$. Hence if we put

$$\sqrt{F} = e^{g(t)}, \qquad e^{2\bar{\sigma}} = e^{2\bar{\sigma}(\bar{t})} \tag{10.3}$$

in (10.1) and (10.2), we have, from (10.1) and the transformation law of v,

$$e^g r = \eta e^{\bar{\sigma}}\bar{r}, \qquad c_1\int e^g dt = c_2\int e^{2\bar{\sigma}} d\bar{t}; \tag{10.4}$$

and from (10.4) and (10.2),

$$c_2 e^{\bar{\sigma}} = \epsilon\eta c_1 e^g, \qquad c_2 r = c_1 \epsilon \bar{r} \tag{10.5}$$

and
$$d_{\bar{t}}\bar{\sigma} = \epsilon\eta e^{\bar{\sigma}} d_t g; \tag{10.6}$$

here (10.5) determines a transformation, and (10.6) gives the condition to be satisfied by $\bar{\sigma}$. Now, if we consider $\dot{g}$ as a function of e^g and put $\dot{g}=H(e^g)$ (the actual form of H is determined if the concrete form of e^g is given as a function of t); then, from (10.6), we have

$$d_{\bar{t}}\bar{\sigma}=\epsilon\eta e^{\bar{\sigma}}H\left(\epsilon\eta\frac{c_2}{c_1}e^{\bar{\sigma}}\right). \tag{10.7}$$

This is the equation which determines $\bar{\sigma}$ when e^g is given. Conversely, it is evident that when $\bar{\sigma}$ satisfies (10.7), L_1 is transformed into $\bar{L}_2$ by (10.5). Hence, by putting $\epsilon\eta c_2/c_1=c$, we have

Theorem 9_1. *The line element L_{1a} is transformable into $L_{2\text{III}}$ by G whose general form is given by*

$$\bar{r}=\eta c r\,,\qquad e^g=ce^{\bar{\sigma}}\,, \tag{10.8}$$

provided that V_4 is neither S_4 nor E_4.

Next we shall prove that $L_{1\text{b}}$ is transformable into the form of L_{2b}. For this purpose it is sufficient if we prove that $L_{1\text{b}}$ is transformable into $L_{2\text{I}}$. As in the foregoing case, from the transformation law of v, we have

$$c_1\int e^g dt=-\frac{c_2}{2}\int f(\bar{X})d\bar{X}\equiv\varphi(\bar{X})\,,\qquad (\bar{X}\equiv\bar{t}^2-\bar{r}^2)\,, \tag{10.9}$$

and, after some calculation, from (10.1), (10.2), and (10.3)[1] we have

$$c_2^2\bar{X}f=e^{2g(t)}c_1^2 \tag{10.10}$$

and

$$\frac{1+\dfrac{r^2}{4R^2}}{r}=\epsilon'\frac{c_2}{c_1}\frac{\sqrt{\bar{X}}}{\bar{r}}\,,\qquad \frac{1-\dfrac{r^2}{4R^2}}{r}=-\epsilon\frac{c_2}{c_1}\frac{\bar{t}}{\bar{r}}\,,\quad (\epsilon'^2=1)\,. \tag{10.11}$$

Hence, in consequence of (10.11),

$$r=-\frac{c_1}{c_2}\frac{2}{\bar{r}}(\epsilon'\sqrt{\bar{X}}+\epsilon\bar{t})\,,\qquad \left(\frac{c_2}{c_1}\right)^2=-\frac{1}{R^2}\,. \tag{10.12}$$

Conversely, we can readily show that when f satisfies (10.10), G defined by (10.9) and (10.12) transforms $L_{1\text{b}}$ into $L_{2\text{I}}$. Hence, putting $c_1/c_2=c$, we have

Theorem 9_2. *The line element $L_{1\text{b}}$ is transformable into $L_{2\text{I}}$ by G whose general form is given by*

$$r=-\frac{2c}{\bar{r}}(\epsilon'\sqrt{\bar{X}}+\epsilon\bar{t})\,,\quad -c\int e^g dt=\int f(\bar{X})d\bar{X}\,,\quad (c^2=-R^2) \tag{10.13}$$

(1) In this calculation the following equations are used:

$$\frac{\partial r}{\partial\bar{r}}=-\epsilon\frac{c_2}{c_1}\bar{t}f\cdot\left(1+\frac{r^2}{4R^2}\right)e^{-2g}=-\epsilon\epsilon'\frac{r}{\bar{r}}\frac{\bar{t}}{\sqrt{\bar{X}}}\,,\quad \text{similarly}\quad \frac{\partial r}{\partial\bar{t}}=\epsilon\epsilon'\frac{c_1^2r}{\sqrt{\bar{X}}}\,.$$

where c is a constant, provided that V_4 is neither S_4 nor E_4.[1]

To obtain the actual form of the function $f(\bar{X})$ when the concrete form of e^g is given, we have only to express t as a function of $(f, \bar{r}, \bar{t})$ from (10.10), and solve $f(\bar{X})$ from (10.9). And we can easily prove that the transformation obtained by using (10.1) in place of the second equation of (10.13) is equivalent to the transformation defined by (10.13).

As a special case of Theorem 9_2, if we put $e^g=1$, we have $f=-\dfrac{R^2}{\bar{X}}$ in consequence of (10.10). So that we know that the line element of Einstein type

$$ds^2=-\frac{1}{\left[1+\dfrac{r^2}{4R^2}\right]^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+dt^2 \qquad (10.14)$$

is transformable into

$$d\bar{s}^2=-\frac{R^2}{\bar{X}}(-d\bar{r}^2-\bar{r}^2d\theta^2-\bar{r}^2\sin^2\theta d\phi^2+d\bar{t}^2) \qquad (\overline{10.15})$$

by the transformation

$$r=\frac{2c}{\bar{r}}(\epsilon'\sqrt{\bar{X}}+\epsilon\bar{t}),\quad ct=R^2\log\bar{X}+\text{const.},\quad (c^2=-R^2). \qquad (10.16)$$

When R is real, (10.16) becomes an imaginary transformation.

Summarizing Theorems 9_1 and 9_2, we have

Theorem 10. *L_{1a} and L_{2a}, and L_{1b} and L_{2b}, are transformable by G, but L_{1a} and L_{2b}, and L_{1b} and L_{2a}, are not transformable.*

§ 11. On the line element L_3. I. General form of the line element of S_4 and E_4.

In this section we shall assume that in S_4 and E_4, ds^2 is of the form L_3. It is well-known that when ds^2 is static (i. e., both A and C are functions of r alone), the general forms of the line elements of S_4 and E_4 are given by

$$ds^2=-\frac{dr^2}{1-k^2r^2}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+(1-k^2r^2)dt^2 \qquad (11.1)$$

and (6.5) respectively, where the constant factor of C is excluded.

Now let us find the general form of ds^2, assuming that $A=A(r,t)$ and $C=C(r,t)$. Substituting g_{ij} of L_3 into (3.1), and putting $(i,j,l,m)=(1,2,2,4)$, we see that A must be static, i. e. a function of r alone. Next, putting $(i,j,l,m)=(2,3,2,3)$ and $(2,4,2,4)$, we have

(1) From Theorems 8, 9_1, and 9_2 we know that L_{1a} is not transformable into L_{2I}. This can easily be proved directly as follows: In the proof of Theorem 9_2, if we consider the case in which $1/R^2=0$, the equation obtained from (10.11) by putting $1/R^2=0$ should hold good; but it is obviously impossible. Therefore L_{1a} is not transformable into L_{2I}.

in S_4: $\quad A=(1-k^2r^2)^{-1}, \quad C=\varphi(t)(1-k^2r^2)$ (11.2)

in E_4: $\quad A=1, \quad C=\varphi(t),$ (11.3)

where $\varphi(t)$ is an arbitrary function of t. Conversely, it is evident that A and C thus obtained satisfy (3.1). Hence we have

Theorem 11. *In S_4 and E_4, the general forms of the line elements of the form L_3 are given by (11.2) and (11.3) respectively.*

Clearly, the line elements (11.2) and (11.3) are transformable into (11.1) and (6.5) respectively by the transformation $r=\eta\bar{r}$, $d\bar{t}=\epsilon\sqrt{\varphi}\,dt$.(1) v's corresponding to (11.2) and (11.3) will be obtained in the next section.

§ 12. On the line element L_3. II. Solution of (1.1).

A necessary and sufficient condition for L_3 to be conformally flat is given by(2)

$$1-A+\frac{r}{2}\left(\frac{A'}{A}-\frac{C'}{C}\right)-\frac{r^2}{C}\left(\frac{2\ddot{A}C-\dot{A}\dot{C}}{4C}-\frac{2C''C-C'^2}{4C}+\frac{A'C'-\dot{A}^2}{4A}\right)=0. \tag{12.1}$$

Hence, to obtain the forms of L_3 which is transformed from L_1 by G, we have only to solve (1.1) under the additional condition (12.1). (1.1) corresponding to L_3 becomes

(a) $\partial_{11}v-\dfrac{A'}{2A}\partial_1 v-\dfrac{\dot{A}}{2C}\partial_4 v=\dfrac{1}{r}\partial_1 v$ $\quad$ (b) $\partial_{44}v-\dfrac{C'}{2A}\partial_1 v-\dfrac{\dot{C}}{2C}\partial_4 v=-\dfrac{C}{rA}\partial_1 v$

(c) $\partial_{14}v-\dfrac{\dot{A}}{2A}\partial_1 v-\dfrac{C'}{2C}\partial_4 v=0$ $\quad$ (d) $\beta=-\dfrac{1}{rA}\partial_1 v.$

From these equations and (12.1), $A, C,$ and v can be determined. But it is rather difficult to solve the equations as they are, on account of the complexity of the form of (12.1); so we shall, as a first step, solve the equations on the assumption that A is static, i.e. $A=A(r)$.

(i) When $\partial_1 v=0$ and $\partial_4 v\neq 0$. From (c), we have $C'=0$. Hence, as the general solution we have

$$ds^2=-\frac{dr^2}{1-\dfrac{r^2}{R^2}}-r^2d\theta^2-r^2\sin^2\theta d\phi^2+C(t)dt^2 \tag{12.2}$$

and

$$v=c_1\int\sqrt{C(t)}\,dt. \qquad (\beta=0) \tag{12.3}$$

Obviously (12.2) is transformed by the transformation $r=\eta\bar{r}$, $\epsilon\sqrt{C}\,dt=d\bar{t}$ into the ordinary form of line element of Einstein type:

(1) This transformation is denoted by G_6 later on. See § 13.

(2) W.G. No. 39. 187, Theorem 12.

$$ds^2 = -\frac{dr^2}{1-\frac{r^2}{R^2}} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2, \tag{12.4}$$

and v corresponding to (12.4) becomes

$$v = mt + q, \tag{12.5}$$

where m and q are arbitrary constants.
(ii) When $\partial_1 v \neq 0$ and $\partial_4 v = 0$. From (b), we have $C = a(t)r^2$; and then, from (12.1), $A = 0$; this is a trivial case.
(iii) When $\partial_1 v \neq 0$ and $\partial_4 v \neq 0$. From (a), (b), and (c), we have

$$\partial_1 v = p_1(t) r\sqrt{A}, \quad \partial_4 v = p_2(t)\sqrt{C} \quad (p_1 \neq 0,\ p_2 \neq 0) \tag{12.6}$$

where p_1 and p_2 are functions of t satisfying

$$\dot{p}_1 r\sqrt{A} = p_2 \frac{C'}{2\sqrt{C}}, \qquad \dot{p}_2 - \frac{C'}{2\sqrt{AC}} p_1 r = -p_1\sqrt{\frac{C}{A}}. \tag{12.7}$$

When $\dot{p}_1 = 0$ here, we have (11.3), and

$$v = -\frac{p_1}{2}\left\{\left(\int\sqrt{C}\,dt\right)^2 - r^2\right\} + p_3\int\sqrt{C}\,dt + p_4, \tag{12.8}$$

where p_1, p_3, and p_4 are arbitrary constants. As a matter of course, if we transform (11.3) into (6.5) by the transformation $\sqrt{C}\,dt = d\bar{t}$, (12.8) is transformed into (4.17).

When $p_1 \neq 0$, from (12.7) we have

$$C = \left(\dot{p}_2 - \frac{p_1\dot{p}_1 r^2}{p_2}\right) b(t), \qquad \frac{1}{A(r)} = \frac{1}{b p_1^2}\left(\dot{p}_2 - \frac{p_1\dot{p}_1 r^2}{p_2}\right). \tag{12.9}$$

Hence it must follow that $\dot{p}_2/(bp_1^2) \equiv h = \text{const.}$ and $\dot{p}_1/(bp_1p_2) \equiv k^2 = \text{const.}$. Substituting these results into (12.1), we have $h = 1$ and $\frac{p_1\dot{p}_1}{p_2\dot{p}_2} = k^2$. Hence, as the general solutions for A and C, we have (11.2) in which $\varphi = b\dot{p}_2$. To obtain v corresponding to this ds^2 changing the variables: $\eta\sqrt{\varphi}\,dt = d\bar{t}$, we have (11.1) for ds^2 and (12.6) becomes $\dot{p}_1 = -k^2 p_2$ and $\dot{p}_2 = -p_1$, from which we have

$$v = me^{kt}\sqrt{1-k^2r^2} + ne^{-kt}\sqrt{1-k^2r^2} + q, \tag{12.10}$$

where m, n, and q are arbitrary constants. Hence we have
Theorem 12. *When ds^2 is of the form L_3 in which $A = A(r)$, the general solution of (1.1) and (12.1) are given by the following three forms:*

(11.1), (12.10); (6.5), (4.17); (12.4), (12.5);

excluding those connected by G_6, i. e. $\left(r = \eta\bar{r},\ t = f(\bar{t})\right)$.

Accordingly, if we assume that A is static, there exists no V_4 which gives the solution of (1.1) and (12.1) except for S_4, E_4, and Einstein-type V_4.

§ 13. On the line element L_3. III. Transformations which make the form of L_3 invariant.

In the same way as in § 5 and § 8, we have the theorem concerning the transformations which transform L_3 into

$$\bar{L}_3:\quad d\bar{s}^2 = -\bar{A}(\bar{r}, \bar{t})d\bar{r}^2 - \bar{r}^2 d\theta^2 - \bar{r}^2 \sin^2\theta d\phi^2 + \bar{C}(\bar{r}, \bar{t})d\bar{t}^2\,.$$

Theorem 13. *The general form of the transformation which keeps the form L_3 invariant is given by*

$$G_6:\quad r = \eta\bar{r}\,,\quad t = f(\bar{t})\,.$$

And by G_6, A and C undergo the transformation

$$A = \bar{A}\,,\qquad C\left(\frac{dt}{d\bar{t}}\right)^2 = \bar{C}\,.$$

The proof is omitted here as it is just the same as for Theorems 4 and 5.

§ 14. Transformations connecting L_3 and L_1.

In consequence of Theorem 1, the conditions to be satisfied by G which transforms L_3 into $\bar{L}_1$ are

$$r = \eta\sqrt{\bar{F}}\,\bar{r}\,,\qquad \left(\sqrt{\bar{F}} = e^{\bar{g}}\left(1 + \frac{\bar{r}^2}{4R^2}\right)^{-1}\right) \tag{14.1}$$

$$\sqrt{A}\,\frac{\partial r}{\partial \bar{r}} = \epsilon\sqrt{C}\,\sqrt{\bar{F}}\,\frac{\partial t}{\partial \bar{t}}\,,\quad \sqrt{A}\,\sqrt{\bar{F}}\,\frac{\partial r}{\partial \bar{t}} = \epsilon\sqrt{C}\,\frac{\partial t}{\partial \bar{r}}\,,\quad \bar{F} = \left(\frac{\partial r}{\partial \bar{r}}\right)^2 A - \left(\frac{\partial t}{\partial \bar{r}}\right)^2 C. \tag{14.2}$$

Therefore, when $\bar{F}$ is given, from the equations above we have[(1)]

(i) when $\bar{L}_1$ is $\bar{L}_{1b}$ $\left(\text{i. e. } \frac{1}{R^2} \neq 0\right)$, excluding the case of Einstein-type ds^2 (in which $\dot{\bar{g}} = 0$)

$$G:\qquad r = \eta\sqrt{\bar{F}}\,\bar{r}\,,\quad t = \varphi(\xi)\,,\quad \left(\xi \equiv 2R^2\log\frac{\alpha}{\beta} + 2\int\frac{d\bar{t}}{e^{2\bar{g}}\dot{\bar{g}}}\right) \tag{14.3}$$

$$\frac{1}{A} = \frac{\beta^2}{\alpha^2} - r^2\dot{\bar{g}}^2\,,\qquad C = A\frac{\beta^2 e^{4\bar{g}}\dot{\bar{g}}^2}{4\alpha^2(d_\xi\varphi)^2} \tag{14.4}$$

(1) From (14.1) and (14.2), we have $\frac{\partial t}{\partial \bar{r}} = \epsilon\eta\sqrt{\frac{A}{C}}\,e^{2\bar{g}}\,\bar{r}\,\frac{\dot{\bar{g}}}{\alpha^2}$, $\frac{\partial t}{\partial \bar{t}} = \epsilon\eta\sqrt{\frac{A}{C}}\,\frac{\beta}{\alpha}$; so we have $\frac{\partial t}{\partial \bar{r}}\,\frac{\alpha\beta}{e^{2\bar{g}}\dot{\bar{g}}\bar{r}} - \frac{\partial t}{\partial \bar{t}} = 0$, from which $t = \varphi(\xi)$ is obtained.

where $\alpha\equiv 1+\frac{\bar{r}^2}{4R^2}$, $\beta\equiv 1-\frac{\bar{r}^2}{4R^2}$, and φ is arbitrary,

(ii) when $\bar{L}_1$ is $\bar{L}_{1a}\left(\text{i. e. } \frac{1}{R^2}=0\right)$

$$G: \quad r=\eta e^{\bar{g}}\bar{r}, \quad t=\varphi(\xi), \quad \left(\xi\equiv\bar{r}^2+2\int\frac{d\bar{t}}{e^{2\bar{g}}\dot{\bar{g}}}\right) \tag{14.5}$$

$$\frac{1}{A}=1-r^2\dot{\bar{g}}^2, \qquad C=A\frac{e^{4\bar{g}}\dot{\bar{g}}^2}{4(d_\xi\varphi)^2} \tag{14.6}$$

where φ is arbitrary.

The fact that the transformation G contains an arbitrary function corresponds to the invariance of the form L_3 under G_6.

In the case of Einstein-type ds^2, the transformation which transforms (10.14) into $(\overline{12.4})$ is obtained as follows. From (4.10) and (12.5), v and $\bar{v}$ are given by

$$v=mt+q, \qquad \bar{v}=\bar{m}\bar{t}+\bar{q}. \tag{14.7}$$

Accordingly t must be a linear function of $\bar{t}$. By applying this relation to (14.1) and (14.2), we can easily prove the following theorem:

Theorem 14. *The general form of G which transforms (10.14) into $(\overline{12.4})$ is given by*

$$r=\eta\bar{r}\left(1+\frac{\bar{r}^2}{4R^2}\right)^{-1}, \quad t=\epsilon\bar{t}+c, \quad (c \text{ is a constant}). \tag{14.8}$$

By this transformation the coefficients m and q in v undergo the transformation

$$\bar{m}=m\epsilon, \qquad \bar{q}=mc+q. \tag{14.9}$$

And, from Theorem 14, we know that both (10.14) and (12.4) are of the same category as L_{1b} and L_{2b}.

§ 15. Transformations connecting L_3 and $\bar{L}_2$.

As in the preceding section, the conditions to be satisfied by G which transforms L_3 into $\bar{L}_2$ are

$$r=\eta e^{\bar{\sigma}}\bar{r} \tag{15.1}$$

$$\sqrt{A}\frac{\partial r}{\partial\bar{r}}=\epsilon\sqrt{C}\frac{\partial t}{\partial\bar{t}}, \quad \sqrt{A}\frac{\partial r}{\partial\bar{t}}=\epsilon\sqrt{C}\frac{\partial t}{\partial\bar{r}}, \quad e^{2\bar{\sigma}}=\left(\frac{\partial r}{\partial\bar{r}}\right)^2A-\left(\frac{\partial t}{\partial\bar{r}}\right)^2C. \tag{15.2}$$

Therefore, when $e^{\bar{\sigma}}$ is given, from the equations above we have[1]

(i) when $\bar{L}_2$ is $\bar{L}_{2\mathrm{I}}$, excluding the case of Einstein-type ds^2 (in which $1+2\bar{X}(d_{\bar{X}}\bar{\sigma})=0$)

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(1) As in § 14, $t=\varphi(\xi)$ is obtained from the relation $\frac{\partial t}{\partial\bar{t}}\bar{r}\dot{\bar{\sigma}}=\frac{\partial t}{\partial\bar{r}}(1+\bar{r}\bar{\sigma}')$.

$$G: \qquad r=\eta e^{\bar{\sigma}}\bar{r}\,, \quad t=\varphi(\xi)\,, \quad \left(\xi\equiv\log\bar{t}-\int\frac{(d_{\bar{X}}\bar{\sigma})d\bar{X}}{1+2\bar{X}(d_{\bar{X}}\bar{\sigma})}\right) \tag{15.3}$$

$$\frac{1}{A}=1-4\bar{r}^2(d_{\bar{X}}\bar{\sigma})\{1+\bar{X}(d_{\bar{X}}\bar{\sigma})\}\,, \qquad C=Ae^{2\bar{\sigma}}\bar{t}^2(1+2\bar{X}d_{\bar{X}}\bar{\sigma})^2/(d_{\xi}\varphi)^2 \tag{15.4}$$

(ii) when $\bar{L}_2$ is $\bar{L}_{2\,\mathrm{III}}$,

$$G: \qquad r=\eta e^{\bar{\sigma}}\bar{r}\,, \quad t=\varphi(\xi)\,, \quad \left(\xi\equiv\frac{1}{2}\bar{r}^2+\int\frac{d\bar{t}}{\dot{\bar{\sigma}}}\right) \tag{15.5}$$

$$\frac{1}{A}=1-\bar{r}^2\dot{\bar{\sigma}}^2\,, \qquad C=Ae^{2\bar{\sigma}}\dot{\bar{\sigma}}^2/(d_{\xi}\varphi)^2\,, \tag{15.6}$$

where φ is an arbitrary function.

In the case of Einstein-type space, there are two kinds of line element of the form L_2, namely (i) that belonging to the type $L_{2\,\mathrm{I}}$, i. e. (10.15), and (ii) that belonging to the type $L_{2\,\mathrm{II}}$, in which $a\neq 0$,[1] i. e.

$$ds^2=\frac{4aR^2}{(X-a)^2+4at^2}(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2)\,. \tag{15.7}$$

But we have seen that (10.15) and (15.7) are intertransformable by G, the equation of transformation being given in Theorem 7_3. Accordingly, we have only to consider the transformation which transforms (12.4) into $(\overline{10.15})$. But in consequence of (4.10) and (7.2), we have, respectively,

$$v=mt+q \quad \text{and} \quad \bar{v}=\bar{m}\log\bar{X}+\bar{q}\,.$$

Hence, from (15.1), (15.2), and (15.7), we can readily prove the following theorem.

Theorem 15. *The general form of the transformation G which transforms (12.4) into $(\overline{10.15})$ is given by*[2]

$$r=\frac{i\eta R}{\sqrt{\bar{X}}}\bar{r}\,, \quad t=\frac{i\epsilon' R}{2}\log\bar{X}+c\,. \quad (c \text{ is a const. and } \epsilon'=\epsilon\eta) \tag{15.8}$$

Furthermore, we see that by (15.8) the coefficients m and q in v undergo the transformation

$$\bar{m}=\frac{i\epsilon' R}{2}m\,, \qquad \bar{q}=q+cm\,, \tag{15.9}$$

And the transformation (15.8) becomes real only when R is purely imaginary, $\bar{X}>0$ $\left(\text{corresponding to } 1-\frac{r^2}{R^2}>0\right)$, and c is real.

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(1) W. G. No. 39, 210.

(2) A special form of this transformation is given in W. G. No. 39, 186.

§ 16. Transformations connecting the line elements of S_4.

The line elements of S_4, and the corresponding v's, are given by the following five systems.

$$S_{\text{I}}\left\{\begin{aligned} &ds_1^2=-e^{2kt_1}(dr_1^2+r_1^2d\theta^2+r_1^2\sin^2\theta d\phi^2)+dt_1^2\,,\\ &\overset{1}{v}=m_1\left(r_1^2e^{kt_1}-\frac{1}{k^2}e^{-kt_1}\right)+n_1e^{kt_1}+q_1\,,\end{aligned}\right.$$

$$S_{\text{II}}\left\{\begin{aligned} &ds_2^2=-\frac{(e^{kt_2}+e^{-kt_2})^2}{4k^2R^2\left(1+\dfrac{r_2^2}{4R^2}\right)^2}(dr_2^2+r_2^2d\theta^2+r_2^2\sin^2\theta d\phi^2)+dt_2^2\,,\\ &\overset{2}{v}=m_2\left(e^{kt_2}-\frac{r_2^2}{4R^2}e^{-kt_2}\right)\left(1+\frac{r_2^2}{4R^2}\right)^{-1}+n_2(e^{kt_2}-e^{-kt_2})+q_2\,,\end{aligned}\right.$$

$$S_{\text{III}}\left\{\begin{aligned} &ds_3^2=\frac{1}{\left(1-\dfrac{k^2}{4}X_3\right)^2}(-dr_3^2-r_3^2d\theta^2-r_3^2\sin^2\theta d\phi^2+dt_3^2)\,,\\ &\overset{3}{v}=m_3\frac{1}{1-\dfrac{k^2}{4}X_3}+n_3\frac{t_3}{1-\dfrac{k^2}{4}X_3}+q_3\,,\qquad (X_3\equiv t_3^2-r_3^2)\,,\end{aligned}\right.$$

$$S_{\text{IV}}\left\{\begin{aligned} &ds_4^2=\frac{1}{k^2t_4^2}(-dr_4^2-r_4^2d\theta^2-r_4^2\sin^2\theta d\phi^2+dt_4^2)\,,\\ &\overset{4}{v}=m_4\frac{X_4}{t_4}+n_4\frac{1}{t_4}+q_4\,,\qquad (X_4\equiv t_4^2-r_4^2)\,,\end{aligned}\right.$$

$$S_{\text{V}}\left\{\begin{aligned} &ds_5^2=-\frac{dr_5^2}{1-k^2r_5^2}-r_5^2d\theta^2-r_5^2\sin^2\theta d\phi^2+(1-k^2r_5^2)dt_5^2\,,\\ &\overset{5}{v}=m_5e^{kt_5}\sqrt{1-k^2r_5^2}+n_5e^{-kt_5}\sqrt{1-k^2r_5^2}+q_5\,.\end{aligned}\right.$$

The most general forms of ds^2's and corresponding v's are obtained by operating on S_ρ ($\rho=\text{I, II},\ldots,\text{V}$) the transformations G_1, G_1, G_3, G_3, G_6. Of the five S_ρ, ds_1^2 and ds_2^2 belong to L_1, ds_3^2 and ds_4^2 to L_2, and ds_5^2 to L_3. If k and R are both real, the signatures of the five ds^2's are the same. In this section we shall determine the forms of G's connecting S_ρ ($\rho=\text{I},\ldots,\text{V}$) with one another.

(i) *Transformations connecting* S_{I} *and* S_{II}. To simplify the calculation, operating a transformation ($r_2^2=4R^2\bar{r}_2^2$, $t_2=\bar{t}_2$) to S_{II}, we have the resulting system $\bar{S}_{\text{II}}$:

$$\bar{S}_{\text{II}}\left\{\begin{aligned} &d\bar{s}_2^2=-\frac{(e^{kt_2}+e^{-kt_2})^2}{k^2(1+\bar{r}_2^2)^2}(d\bar{r}_2^2+\bar{r}_2^2d\theta^2+\bar{r}^2\sin^2\theta d\phi^2)+dt_2^2\\ &\bar{v}_2=\bar{m}_2(e^{kt_2}-\bar{r}_2^2e^{-kt_2})(1+\bar{r}_2^2)^{-1}+\bar{n}_2(e^{kt_2}-e^{-kt_2})+\bar{q}_2\,.\end{aligned}\right.$$

From (2.5) and the transformation law of v, we have

$$r_1 e^{kt_1} = \eta \bar{r}_2 \frac{e^{kt_2}+e^{-kt_2}}{k(1+\bar{r}_2^2)} . \tag{16.1}$$

$$e^{kt_1} = \alpha(e^{kt_2} - \bar{r}_2^2 e^{-kt_2})(1+\bar{r}_2^2)^{-1} + \beta(e^{kt_2} - e^{-kt_2}) + \gamma \tag{16.2}$$

where α, β, and γ are any constants to be determined. From (16.1), (16.2), and (2,6), i. e.

$$\frac{r_1}{\bar{r}_2}\frac{\partial t_1}{\partial t_2} = \epsilon \frac{\partial r_1}{\partial \bar{r}_2}, \quad \frac{r_1}{\bar{r}_2} e^{2kt_1} \frac{\partial r_1}{\partial t_2} = \epsilon \frac{\partial t_1}{\partial \bar{r}_2}, \quad \left\{\frac{r_1^2}{\bar{r}_2^2} - \left(\frac{\partial r_1}{\partial \bar{r}_2}\right)^2\right\} e^{2kt_1} = \left(\frac{\partial t_1}{\partial \bar{r}_2}\right)^2, \tag{16.3}$$

we have

$$\alpha(\epsilon - 1) = 2\beta, \qquad \gamma = 0 . \tag{16.4}$$

Therefore we obtain two transformations connecting $S_{\rm I}$ and $\bar{S}_{\rm II}$, corresponding to $\epsilon = +1$ and $\epsilon = -1$. Hence we have

Theorem 16. *The general forms of G's connecting $S_{\rm I}$ and $S_{\rm II}$ are given by*

$$T_{\rm IIIa}: \quad r_1 e^{kt_1} = \eta \frac{r_2(e^{kt_2}+e^{-kt_2})}{2Rk\left(1+\frac{r_2^2}{4R^2}\right)}, \qquad e^{kt_1} = c\frac{e^{kt_2} - \frac{r_2^2}{4R^2}e^{-kt_2}}{\left(1+\frac{r_2^2}{4R^2}\right)}$$

$$T_{\rm IIIb}: \quad r_1 e^{kt_1} = \eta \frac{r_2(e^{kt_2}+e^{-kt_2})}{2Rk\left(1+\frac{r_2^2}{4R^2}\right)}, \qquad e^{kt_1} = c\frac{e^{-kt_2} - \frac{r_2^2}{4R^2}e^{kt_2}}{\left(1+\frac{r_2^2}{4R^2}\right)},$$

where c is an arbitrary constants.

Further, in consequence of the equation

$$r_1^2 e^{kt_1} - \frac{1}{k^2}e^{-kt_1} = -\frac{1}{ck^2}\left(e^{kt_2} - e^{-kt_2}\frac{r_2^2}{4R^2}\right)\left(1+\frac{r_2^2}{4R^2}\right)^{-1} + \frac{1}{ck^2}(e^{kt_2} - e^{-kt_2}), \tag{16.5}$$

the equation of transformation of the coefficients m, n, q in v becomes

$$\text{in } T_{\rm IIIa}: \quad m_2 = -\frac{m_1}{ck^2} + cn_1, \quad n_2 = \frac{m_1}{ck^2}, \quad q_2 = q_1 . \tag{16.6}$$

Similarly

$$\text{in } T_{\rm IIIb}: \quad m_2 = -\frac{m_1}{ck^2} + cn_1, \quad n_2 = -cn_1, \quad q_2 = q_1 . \tag{16.7}$$

(ii) *Transformations connecting $S_{\rm III}$ and $S_{\rm IV}$.* Such transformations can be obtained by a method analogous to that used in (i).[1] In this case,

(1) In this case, as the equation corresponding to (16.2) it is convenient to use

$$\frac{1}{t_4} = \left(1 - \frac{k^2}{4}X_3\right)^{-1}(\alpha + \beta t_3) + \gamma .$$

In this way, as the equations defining $T_{\rm III\,IV}$ we obtain (16.8) and (16.11), which are equivalent to those mentioned in Theorem 17.

however, we can obtain the transformations in a simpler way by using the result attained in § 8, as follows. By Theorem 5, the transformation must be of the form G_5 satisfying (8.1), i. e.

$$\frac{r_4}{kt_4}=\frac{\eta r_3}{\left(1-\frac{k^2}{4}X_3\right)}, \tag{16.8}$$

from which we have

$$2\left(1-\frac{k^2}{4}X_3\right)=-\epsilon\eta k\left[2\left(t_3+\frac{q}{p}\right)+l\{(pt_3+q)^2-p^2r_3^2\}\right]. \tag{16.9}$$

Therefore, as a necessary and sufficient condition to be satisfied by G_5, we have

$$l=-\epsilon'\frac{k}{2p^2}, \quad q=\frac{2\epsilon'}{k}p. \quad (\epsilon'=-\epsilon\eta) \tag{16.10}$$

Hence we have

Theorem 17. *The general form of the transformation G connecting S_{III} and S_{IV} is given by*

$$T_{\mathrm{III\,IV}}: \quad \epsilon r_4=\frac{-2r_3}{p^2[(t_3^2+2\epsilon'/k)^2-r_3^2]}, \quad t_4=\frac{2\epsilon'\left(1-\frac{k^2}{4}X_3\right)}{kp^2[(t_3+2\epsilon'/k)^2-r_3^2]}, \quad (\epsilon'^2=1)$$

where p^2 is an arbitrary constant. Here we may use (16.8) in place of the first equation.

From the form of $T_{\mathrm{III\,IV}}$ given above, we have

$$\frac{1}{t_4}=\left(1-\frac{k^2}{4}X_3\right)^{-1}\left(\frac{4p^2}{k}\epsilon'+2p^2t_3\right)-\frac{2p^2}{k}\epsilon' \tag{16.11}$$

$$\frac{X}{t_4}=\left(1-\frac{k^2}{4}X_3\right)^{-1}\left(\frac{k}{p^2}\epsilon'-\frac{k^2}{2p^2}t_3\right)-\frac{k}{2p^2}\epsilon'. \tag{16.12}$$

So we have

$$\text{in } T_{\mathrm{III\,IV}}: \quad m_3=\epsilon'\left(\frac{4p^2}{k}m_4+\frac{k}{p^2}n_4\right), \quad n_3=2p^2m_4-\frac{k^2}{2p^2}n_4,$$

$$q_3=q_4-\epsilon'\left(\frac{2p^2}{k}m_4+\frac{k}{2p^2}n_4\right). \tag{16.13}$$

(iii) *Transformations connecting S_{I} and S_{IV}.* Substituting

$$r_1e^{kt_1}=\eta\frac{r_4}{kt_4}, \qquad e^{kt_1}=\alpha\frac{X_4}{t_4}+\beta\frac{1}{t_4}+\gamma \tag{16.14}$$

into (2.6), in the same way as previously in (i), we have two systems of solutions α, β, and γ:

(a) when $\epsilon=\eta$, $\beta=0=\gamma$, α is arbitrary

and (b) when $\epsilon=-\eta$, $\alpha=0=\gamma$, β is arbitrary.

Hence we have

Theorem 18. *The general forms of the transformations connecting S_{I} and S_{IV} are given by*

$$T_{\mathrm{IIV\,a}}: \qquad r_1 e^{kt_1} = \eta \frac{r_4}{kt_4}, \qquad e^{kt_1} = \alpha \frac{t_4^2 - r_4^2}{t_4},$$

$$T_{\mathrm{IIV\,b}}: \qquad r_1 e^{kt_1} = \eta \frac{r_4}{kt_4}, \qquad e^{kt_1} = \beta \frac{1}{t_4},$$

where α, β are arbitrary constants.

It is to be noticed that in place of the first equations in $T_{\mathrm{IIV\,a}}$ and $T_{\mathrm{IIV\,b}}$, we may use

$$T_{\mathrm{IIV\,a}}: \ r_1 = \frac{\eta}{k\alpha} \frac{r_4}{X_4}; \qquad T_{\mathrm{IIV\,b}}: \ r_1 = \frac{\eta}{k\beta} r_4. \tag{16.15}$$

And by these transformations the coefficients in v undergo the transformation

$$\text{in } T_{\mathrm{IIV\,a}}: \ m_4 = n_1\alpha, \quad n_4 = -m_1 \frac{1}{k^2\alpha}, \quad q_4 = q_1, \tag{16.16}$$

$$\text{in } T_{\mathrm{IIV\,b}}: \ m_4 = -m_1 \frac{1}{k^2\beta}, \quad n_4 = n_1\beta, \quad q_4 = q_1. \tag{16.17}$$

Thus there are two kinds of transformations connecting S_{I} and S_{IV}, the forms of which are entirely different from each other. From this result, so far as the fundamental tensor g_{ij} is concerned no ambiguities arise owing to the different kinds of transformation; but when we are considering vectors and tensors other than g_{ij}, it is necessary to make clear which transformation should be adopted. This remark may be applied to other similar cases.

(iv) In the same way as (i) we can prove

Theorem 19. *The general forms of the transformations connecting S_{I} and S_{V} and the corresponding transformations of the coefficients in v are given by*

$$T_{\mathrm{IV\,a}}: \ r_1 e^{kt_1} = \eta r_5, \quad e^{kt_1} = \alpha e^{kt_5}\sqrt{1 - k^2 r_5^2}, \quad (\alpha \text{ is arbitrary})$$

$$m_5 = \alpha n_1, \quad n_5 = -\frac{m_1}{\alpha k^2}, \quad q_5 = q_1; \tag{16.18}$$

$$T_{\mathrm{IV\,b}}: \ r_1 e^{kt_1} = \eta r_5, \quad e^{kt_1} = \beta e^{-kt_5}\sqrt{1 - k^2 r_5^2}, \quad (\beta \text{ is arbitrary})$$

$$m_5 = -\frac{m_1}{\beta k^2}, \quad n_5 = \beta n_1, \quad q_5 = q_1. \tag{16.19}$$

(v) The other transformations connecting S_ρ ($\rho = \mathrm{I}, \ldots, \mathrm{V}$) with one another can be obtained by combining the above-mentioned $T_{\mathrm{I\,II}}$, $T_{\mathrm{III\,IV}}$, $T_{\mathrm{I\,IV}}$, and $T_{\mathrm{I\,V}}$. But if we adhere to these four transformations only, we have the trouble of combining some of them (perhaps three) in order to obtain another (say $T_{\mathrm{I\,III}}$, $T_{\mathrm{III\,V}}$, etc.). But by adding $T_{\mathrm{I\,III}}$ to the above-given four, we can simplify the combination of transformations. Actually,

$$T_{\mathrm{I\,III}}: \quad r_1 e^{kt_1} = \frac{\eta r_3}{1-\frac{k^2}{4}X_3}, \qquad e^{kt_1} = a\left\{\frac{2+k\epsilon t_3}{1-\frac{k^2}{4}X_3} - 1\right\},$$

$$m_3 = -\frac{2}{ak^2}m_1 + 2an_1, \quad n_3 = \frac{\epsilon}{ak}m_1 + ak\epsilon n_1, \quad q_3 = q_1 + \frac{1}{ak^2}m_1 - n_1 a \qquad (16.20)$$

where a is an arbitrary constant.

§ 17. Transformations connecting the line elements of E_4.

The line elements and the corresponding v's obtained in our discussion are as follows:

$$E_1 \begin{cases} ds_1^2 = -dr_1^2 - r_1^2 d\theta^2 - r_1^2 \sin^2\theta d\phi^2 + dt_1^2, \\ \overset{1}{v} = m_1(t_1^2 - r_1^2) + n_1 t_1 + q_1, \end{cases}$$

$$E_2 \begin{cases} ds_2^2 = \dfrac{t_2^2}{R^2\left(1+\dfrac{r_2^2}{4R^2}\right)^2}(dr_2^2 + r_2^2 d\theta^2 + r_2^2 \sin^2\theta d\phi^2) + dt_2^2, \\ \overset{2}{v} = m_2\left(1-\dfrac{r_2^2}{4R^2}\right)t_2\left(1+\dfrac{r_2^2}{4R^2}\right)^{-1} + n_2 t_2^2 + q_2, \end{cases}$$

$$E_3 \begin{cases} ds_3^2 = \dfrac{1}{(t_3^2 - r_3^2)^2}(-dr_3^2 - r_3^2 d\theta^2 - r_3^2 \sin^2\theta d\phi^2 + dt_3^2) \\ \overset{3}{v} = m_3 \dfrac{1}{X_3} + n_3 \dfrac{t_3}{X_3} + q_3. \end{cases}$$

The most general forms of ds^2's and v's are obtained from E_α (α=I, II, III) by operating the transformations (G_1, G_3, G_6), G_1, G_3. ds_1^2 belongs to any of L_1, L_2, and L_3, and ds_2^2 and ds_3^2 belong to L_1 and L_2 respectively. When R is real, the signature of ds_2^2 differs from those of other ds^2's. If we find the general form of the transformations connecting E_α with one another and the transformations of the corresponding m_i, n_i, and q_i ($i=1, 2, 3$), we have

$$U_{\mathrm{I\,II}}: \quad r_1 = \eta \frac{ir_2 t_2}{R\left(1+\frac{r_2^2}{4R^2}\right)}, \qquad t_1 = \epsilon \frac{t_2\left(1-\frac{r_2^2}{4R^2}\right)}{1+\frac{r_2^2}{4R^2}} + c,$$

$$m_2 = \epsilon(2cm_1 + n_1), \quad n_2 = m_1, \quad q_2 = q_1 + c^2 m_1 + cn_1, \qquad (17.1)$$

$$U_{\mathrm{I\,III}}:\quad r_1=\eta\frac{r_3}{t_3^2-r_3^2}\,,\qquad t_1=\epsilon\frac{t_3}{t_3^2-r_3^2}+c\,,$$

$$m_3=m_1\,,\quad n_3=\epsilon(2cm_1+n_1)\,,\quad q_3=q_1+c^2m_1+cn_1\,,\tag{17.2}$$

$$U_{\mathrm{II\,III}}:\quad \frac{ir_2t_2}{R\left(1+\frac{r_2^2}{4R^2}\right)}=\eta\frac{r_3}{t_3^2-r_3^2}\,,\qquad \frac{t_2\left(1-\frac{r_2^2}{4R^2}\right)}{1+\frac{r_2^2}{4R^2}}=\epsilon\frac{t_3}{t_3^2-r_3^2}+c$$

$$m_3=n_2\,,\quad n_3=\epsilon(m_2+2cn_2)\,,\quad q_3=q_2+cm_2+c^2n_2\tag{17.3}$$

where c's are arbitrary constants and $U_{\mathrm{II\,III}}$ is the combination of $U_{\mathrm{I\,II}}$ and $U_{\mathrm{I\,III}}$. Here $U_{\mathrm{I\,II}}$ and $U_{\mathrm{II\,III}}$ are real transformations when R is purely imaginary.

§ 18. Solution of (1.1) when $v=v(r,\theta,\phi,t)$.

Above (§ 4, § 7, § 12) we have solved equation (1.1) on the assumption that v is spherically symmetric. In this section we shall try to solve (1.1) when v contains not only r, t but also θ, ϕ.

When ds^2 is of the form L_1, (1.1) becomes

$$\partial_{12}v-\frac{B'}{2B}\partial_2 v=\partial_{13}v-\frac{B'}{2B}\partial_3 v=\partial_{23}v-\cot\theta\,\partial_3 v=0\tag{18.1}$$

$$\partial_{a4}v-\dot g\partial_a v=0\,,\qquad (a=1,2,3)\tag{18.2}$$

$$\partial_{11}v-\frac{F'}{2F}\partial_1 v-F\dot g\partial_4 v=-F\partial_{44}v\tag{18.3}$$

$$\partial_{22}v+\frac{B'}{2F}\partial_1 v-r^2F\dot g\partial_4 v=-r^2F\partial_{44}v\tag{18.4}$$

$$\partial_{33}v+\frac{B'}{2F}\sin^2\theta\,\partial_1 v+\sin\theta\cos\theta\,\partial_2 v-r^2F\sin^2\theta(\dot g\partial_4 v-\partial_{44}v)=0\tag{18.5}$$

$$\beta=\partial_{44}v\,.\qquad (B\equiv r^2F)\tag{18.6}$$

If $\partial_3 v\neq 0$, from (18.1) and (18.2) we have

$$v=\left(\sin\theta\int b(\phi)d\phi+\int a(\theta)d\theta\right)r\sqrt{F}+c(r,t)\,,\tag{18.7}$$

where $a(\theta)$, $b(\phi)$, and $c(r,t)$ are arbitrary functions. Substituting (18.7) into (18.4), after some calculation we have

$$-r\sqrt{F}+\frac{B'^2}{4rF^{\frac{3}{2}}}-r^3F\dot g\partial_4\sqrt{F}+r^3F\partial_{44}\sqrt{F}=0\,;\tag{18.8}$$

and further substituting the actual form of F into this equation, we have

$$e^{2g}\ddot{g}=\frac{1}{R^2}, \tag{18.9}$$

which coincides with the third equation of (4.5). Hence, by the same calculation as in § 4, we see that F coincides with A given by either (3.9) or (3.10); which shows that V_4 must be either S_4 or E_4. So that when V_4 is neither S_4 nor E_4, we must have $\partial_3 v=0$.

Similarly, we can prove that $\partial_2 v=0$ when V_4 is neither S_4 nor E_4; so v must be spherically symmetric. Therefore, we know that (4.10) gives the general solution v of (1.1) even when we do not assume that v is spherically symmetric.

Similarly, we obtain the same result when ds^2 is of the form L_2, provided V_4 is neither S_4 nor E_4; the proof being omitted.[1] Hence we have

Theorem 20. *When V_4 is neither S_4 nor E_4, if we assume that ds^2 is of the form L_1 or L_2, the solution v of (1.1) becomes spherically symmetric. Accordingly, when ds^2 is of the form L_1, (4.10) gives the general solution of v; and when ds^2 is of the form L_2, the general solution ds^2 and the corresponding v are given by $L_{2\text{I}}$, (7.2); $L_{2\text{II}}$, (7.3); $L_{2\text{III}}$, (7.4).*

When V_4 is either S_4 or E_4, (1.1) becomes completely integrable for v_i, so that the general solution of v must contain four arbitrary constants at least excluding the additive constant. As the simplest example in an E_4 we shall find v corresponding to ds^2 of E_1 (cf. (3.13)). If we put $\sqrt{F}=1$ and $\dot{g}=0$ in (18.1), ..., (18.7), from (18.7) and (18.4) we have

$$\partial_\theta a+\int a d\theta=p, \quad p+\partial_r c+r\partial_{tt}c=0, \quad (p \text{ is const.}) \tag{18.10}$$

On the other hand, from (18.2) and (18.3) we have $\partial_{rr}c+\partial_{tt}c=0=\partial_{rt}c$. Making use of these results, as the general solution of v we have

$$v=r\sin\theta(c_1\sin\phi+c_2\cos\phi)+c_3 r\cos\theta+c_4 t+c_5(t^2-r^2)+c_6, \tag{18.11}$$

where c_i ($i=1, \ldots, 6$) are arbitrary constants.[2] Hence, we know that there are five kinds of v_i which are linearly independent with constant coefficients.

In the same way, we can obtain the solution v's corresponding to the

(1) In this proof, as the equations corresponding to (18.9) we have

$$f'+r\ddot{f}=0, \quad f''+\ddot{f}=0, \quad \dot{f}'=0, \quad (f\equiv e^{-\sigma})$$

from which we can easily obtain (6.1), i.e. the equation defining S_4 and E_4.

(1.1) has only one independent solution of v in the coordinate system in which ds^2 is of the form L_1. From this, we may conclude that when ds^2 is of the form L_{2a} (a=I, II, III), (7.2), (7.4), and (7.6) give the general solution of v even when we do not assume that v is spherically symmetric.

(2) β corresponding to (18.11) is given by c_5, which coincides with the result of the corollary in W. G. No. 39, 201.

other forms of ds^2 of S_4 and E_4. For example, the general solution v corresponding to ds^2 of Robertson's form (i. e. ds^2 of S_I) is obtained as follows:

$$v=e^{kt}\{r\sin\theta(c_1\sin\phi+c_2\cos\phi)+c_3r\cos\theta+c_4\}$$
$$+c_5\left(r^2e^{kt}-\frac{1}{k^2}e^{-kt}\right)+c_6. \qquad (18.12)$$

N. B. In the ordinary (x, y, z, t)-coordinate system in which ds^2 is given by

$$(E_4): \qquad ds^2=-dx^2-dy^2-dz^2+dt^2 \qquad (18.13)$$

or

$$(S_4): \qquad ds^2=-e^{2kt}(dx^2+dy^2+dz^2)+dt^2, \qquad (18.14)$$

(18.11) and (18.12) become

$$v=c_1x+c_2y+c_3z+c_4t+c_5(t^2-r^2)+c_6,$$
$$v=e^{kt}(c_1x+c_2y+c_3z+c_4)+c_5\left(r^2e^{kt}-\frac{1}{k^2}e^{-kt}\right)+c_6.$$

In this coordinate system, the calculation is simpler than in (r, θ, ϕ, t)-system.

§ 19. On the line element L_4.

The line element L_4 is obtained by putting $B=$const. in the most general spherically symmetric line-element (2.1), and cannot be transformed into the line element in which B is not constant by the transformation G. Line elements of the form L_4 are not in general adopted in the ordinary relativities and cosmologies. In wave geometry, however, we have treated them several times; therefore it is not purposeless to study such L_4.

In L_1, the coefficient of $d\theta^2$ cannot be constant, so it is evident that by any G we cannot transform L_1 into L_4. Further, we can prove that not only not by G, but also by no transformation of (r, θ, ϕ, t), can we transform L_1 into L_4.

To prove this, we have only to obtain ds^2 which gives a conformally flat space and is of the form L_4, and then to prove that the equation (1.1) relating to this ds^2 has no solution v. But such ds^2 is always transformable into the form

$$ds^2=\frac{B}{r^2}(-dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+dt^2), \qquad (B=\text{const.}) \quad (19.1)$$

by a suitable transformation,[1] while (19.1) is of the form L_2, and V_4 defined by (19.1) is neither S_4 nor E_4. Hence, by Theorem 20, v must be spherically symmetric; moreover, (19.1) does not belong to any of L_{2a} ($a=$I, II, III); so it is evident that (1.1) has no solution v.
Hence we have

(1) W. G. No. 39, 186, Theorem 11.

Theorem 21. *The line element of the form* L_1 *cannot be transformed into the form* L_4 *by any transformation of* (r, θ, ϕ, t). *Accordingly, the space defined by* L_4 *is entirely different from that defined by the line element of the form* L_1.

§ 20. Group of motions.

In this section we shall consider the group of motions in the space V_4 defined by L_1. Let the infinitesimal motion be

$$'x^i = x^i + \xi^i \partial\tau, \qquad (i=1, 2, 3, 4); \tag{20.1}$$

then the operator of the motion is given by $\xi^j \partial_j$. The vector ξ^i is obtained as the solution of Killing's equation

$$\nabla_{(i}\xi_{j)} = 0 \tag{20.2}$$

or, in contravariant form,

$$\xi^h \frac{\partial g_{lm}}{\partial x^h} + \frac{\partial \xi^i}{\partial x^l} g_{im} + \frac{\partial \xi^j}{\partial x^m} g_{lj} = 0 \tag{20.3}$$

To solve (20.3), if we put $x = r\sin\theta\cos\phi (\equiv x^1)$, $y = r\sin\theta\sin\phi(\equiv x^2)$, $z = r\cos\theta(\equiv x^3)$, and $t = t(\equiv x^4)$, L_1 can be written as

$$L_1' : \ ds^2 = -F(r, t)(dx^2 + dy^2 + dz^2) + dt^2, \quad \left(F = e^{2g(t)} \Big/ \left[1 + \frac{r^2}{4R^2}\right]^2\right) \tag{20.4}$$

and (20.3) becomes

$$\left.\begin{aligned} &\frac{F'}{r}(\xi^1 x + \xi^2 y + \xi^3 z) + \xi^4 \dot{F} + 2F\frac{\partial \xi^1}{\partial x} = 0; \quad \frac{\partial \xi^2}{\partial z} + \frac{\partial \xi^3}{\partial y} = 0, \\ &-F\frac{\partial \xi^1}{\partial t} + \frac{\partial \xi^4}{\partial x} = 0, \quad \text{and cyclic}; \quad \frac{\partial \xi^4}{\partial t} = 0. \end{aligned}\right\} \tag{20.5}$$

By solving (20.5), we get the following result (as the calculation is easy and somewhat long, we omit it):

Theorem 22. *The operators of motions in the space defined by* L_1', *are given by*

I. *In the case:* $\dfrac{1}{R^2} = 0$,

Ia. *when* $\dot{g} = const. = k (\neq 0)$, *taking* $e^{2g} = e^{2kt}$ *by* G_1 (*i.e.* ds^2 *is* S_I')

$\overset{1}{T}, \overset{2}{T}, \overset{3}{T};\ \overset{1}{S}, \overset{2}{S}, \overset{3}{S};\ \overset{1}{R}, \overset{2}{R}, \overset{3}{R};\ U,$ (*group of 10 parameters*)

Ia′. *when* $\dot{g} = const. = 0$, *taking* $e^{2g} = 1$ *by* G_1 (*i.e.* ds^2 *is* E_I')

$\overset{1}{T}, \overset{2}{T}, \overset{3}{T};\ \overset{1}{\bar{S}}, \overset{2}{\bar{S}}, \overset{3}{\bar{S}};\ \overset{1}{R}, \overset{2}{R}, \overset{3}{R};\ \bar{U},$ (*group of 10 parameters*)

Ib. *when* $\dot{g} \neq const.$; $\overset{1}{T}, \overset{2}{T}, \overset{3}{T};\ \overset{1}{R}, \overset{2}{R}, R,$ (*group of 6 parameters*).

II. *In the case:* $\dfrac{1}{R^2} \neq 0$, IIa. *when* $e^{2g}\ddot{g}=\dfrac{1}{R^2}$ *and* $\dot{g}e^g \neq const.$, *taking* $e^{2g}=(e^{kt}+e^{-kt})^2/4k^2R^2$ *by* G_1 (*i. e.* ds^2 *is* S'_{II}).

$$\overset{1}{P},\overset{2}{P},\overset{3}{P};\ \overset{1}{Q},\overset{2}{Q},\overset{3}{Q};\ \overset{1}{R},\overset{2}{R},\overset{3}{R};\ U', \qquad (\textit{group of 10 parameters})$$

IIa′. *when* $e^{2g}\ddot{g}=\dfrac{1}{R^2}$ *and* $\dot{g}e^g = const.$, *taking* $e^{2g}=-\dfrac{t^2}{R^2}$ *by* G_1 (*i. e.* ds^2 *is* E'_{II})

$$\overset{1}{\bar{P}},\overset{2}{\bar{P}},\overset{3}{\bar{P}};\ \overset{1}{V},\overset{2}{V},\overset{3}{V};\ \overset{1}{R},\overset{2}{R},\overset{3}{R};\ \bar{U}', \qquad (\textit{group of 10 parameters})$$

IIb. *when* $\dot{g}=0$, (*i. e.* ds^2 *is of Einstein type* (*10.14*))

$$\overset{1}{V},\overset{2}{V},\overset{3}{V};\ \overset{1}{R},\overset{2}{R},\overset{3}{R};\ \bar{U}, \qquad (\textit{group of 7 parameters})$$

IIc. *when* $e^{2g}\ddot{g}\neq\dfrac{1}{R^2}$ *and* $\dot{g}\neq 0$,

$$\overset{1}{V},\overset{2}{V},\overset{3}{V};\ \overset{1}{R},\overset{2}{R},\overset{3}{R}, \qquad (\textit{group of 6 parameters})$$

where $\overset{a}{T},\overset{a}{S},U,\ldots\ldots$, *etc.* ($a=1,2,3$) *are operators of motions, in which* ξ^i *are given by*

	ξ^1	ξ^2	ξ^3	ξ^4
$\overset{1}{T}$	1	0	0	0
$\overset{1}{S}$	$e^{-2kt}-4R^2k_2\gamma$	$2k^2xy$	$2k^2xz$	$-2kx$
$\overset{1}{R}$	0	$-z$	y	0
U	kx	ky	kz	-1
$\overset{1}{\bar{S}}$	t	0	0	x
$\bar{U}$	0	0	0	-1
$\overset{1}{P}$	$\dfrac{2kR}{\alpha}(e^{kt}-e^{-kt}\gamma)$	$\dfrac{kxye^{-kt}}{R\alpha}$	$\dfrac{kxze^{-kt}}{R\alpha}$	$\dfrac{x}{R}\delta$
$\overset{1}{Q}$	$\dfrac{2kR}{\alpha}(e^{-kt}-e^{kt}\gamma)$	$\dfrac{kxye^{kt}}{R\alpha}$	$\dfrac{kxze^{kt}}{R\alpha}$	$-\dfrac{x}{R}\delta$
U'	$kx\beta$	$ky\beta$	$kz\beta$	$-\left(1-\dfrac{r^2}{4R^2}\right)\delta$
$\overset{1}{\bar{P}}$	$-\dfrac{r}{t}(1+\gamma)$	$\dfrac{xy}{2Rt}$	$\dfrac{xz}{2Rt}$	$-\dfrac{x}{R}\delta$
$\overset{1}{V}$[1]	$1-\gamma$	$\dfrac{xy}{2R^2}$	$\dfrac{xz}{2R^2}$	0
$\bar{U}'$.	$\dfrac{x}{t}$	$\dfrac{y}{t}$	$\dfrac{z}{t}$	$-\left(1-\dfrac{r^2}{4R^2}\right)\delta$

(1) As is easily seen, the following relation exists:

$$\overset{a}{V}=\frac{1}{2kR}(\overset{a}{P}+\overset{a}{Q}).$$

where $\alpha\equiv e^{kt}+e^{-kt}$, $\beta\equiv\frac{1}{\alpha}(e^{kt}-e^{-kt})$, $\gamma\equiv\frac{r^2-2x^2}{4R^2}$ and $\delta\equiv\left(1+\frac{r^2}{4R^2}\right)^{-1}$, *and* $\overset{2}{T}, \overset{3}{T}$; $\overset{2}{P}, \overset{3}{P}$;.... *are obtained form* $\overset{1}{T}, \overset{1}{P}$,.... *by the cyclic change of* x, y, z.

Among these operators only 7 operators

$$\overset{a}{R},\ U,\ U',\ \bar{U},\ \bar{U}'$$

make $r=0$ invariant.

Further, we can prove that $\overset{a}{R}$ is invariant by G, and that in S_4 and E_4 the following transformation laws hold good:

$$\text{when } S_{\rm I}\rightarrow S_{\rm II} \text{ by } T_{\rm IIa}\ (\eta=c=1),\qquad \overset{a}{T}, \overset{a}{S}, U\rightarrow \overset{a}{P}, \overset{a}{Q}, U',$$

$$\text{when } S_{\rm I}\rightarrow S_{\rm II} \text{ by } T_{\rm IIb}\ (\eta=c=1),\qquad \overset{a}{T}, \overset{a}{S}, U\rightarrow \overset{a}{Q}, \overset{a}{P}, -U',$$

$$\text{when } E_{\rm I}\rightarrow E_{\rm II} \text{ by } U_{\rm II}\ (\epsilon=\eta=1,\ c=0),\qquad \overset{a}{T}, \overset{a}{\bar{S}}, \bar{U}\rightarrow i\overset{a}{\bar{P}}, -iR\overset{a}{V}, \bar{U}'.$$

Lastly, in preparation for some future applications, we add the form of the alternants between the operators of each of the groups obtained above. (The proof is easy, so we omit it.)

$$(\overset{a}{T}, \overset{b}{T})=(\overset{a}{S}, \overset{b}{S})=(\overset{a}{R}, U)=0,\quad (\overset{a}{R}, \overset{b}{R})=-\epsilon_{abc4}\overset{c}{R},\quad (\overset{a}{R}, \overset{b}{T})=-\epsilon_{abc4}\overset{c}{T},$$

$$(\overset{a}{R}, \overset{b}{S})=-\epsilon_{abc4}\overset{c}{S},\quad (\overset{a}{T}, \overset{b}{S})=-2k^2\epsilon_{abc4}\overset{c}{R}+2k\delta_{ab}U,\quad (U, \overset{a}{T})=-\overset{a}{T},$$

$$(U, \overset{a}{S})=\overset{a}{S};$$

$$(\overset{a}{\bar{S}}, \overset{b}{\bar{S}})=\epsilon_{abc4}\overset{c}{R},\quad (\overset{a}{R}, \overset{b}{\bar{S}})=-\epsilon_{abc4}\overset{c}{\bar{S}},\quad (\overset{a}{T}, \overset{b}{\bar{S}})=-\delta_{ab}\bar{U},\quad (\bar{U}, \overset{a}{\bar{S}})=-\overset{a}{T},$$

$$(\bar{U}, \overset{a}{T})=(\overset{a}{R}, \bar{U})=0;$$

$$(\overset{a}{P}, \overset{b}{P})=(\overset{a}{Q}, \overset{b}{Q})=(\overset{a}{R}, U')=0,\quad (\overset{a}{R}, \overset{b}{P})=-\epsilon_{abc4}\overset{c}{P},\quad (\overset{a}{R}, \overset{b}{Q})=-\epsilon_{abc4}\overset{c}{Q},$$

$$(\overset{a}{P}, \overset{b}{Q})=-2k^2\epsilon_{abc4}\overset{c}{R}+2k\delta_{ab}U',\quad (U', \overset{a}{P})=-\overset{a}{P},\quad (U', \overset{a}{Q})=Q_a;$$

$$(\overset{a}{\bar{P}}, \overset{b}{\bar{P}})=(\bar{U}', \overset{a}{\bar{P}})=(\overset{a}{R}, \bar{U}')=0,\quad (\overset{a}{\bar{P}}, \overset{b}{V})=-\frac{1}{R}\delta_{ab}\bar{U}',\quad (\overset{a}{R}, \overset{b}{\bar{P}})=-\epsilon_{abc4}\overset{c}{\bar{P}},$$

$$(\bar{U}', \overset{a}{V})=\frac{1}{R}\overset{a}{\bar{P}},\quad (\overset{a}{V}, \overset{b}{V})=-\frac{1}{R^2}\epsilon_{abc4}\overset{c}{R},\quad (\overset{a}{R}, \overset{b}{V})=-\epsilon_{abc4}\overset{c}{V};$$

$$(\overset{a}{V}, \bar{U})=0.\qquad (a, b, c=1, 2, 3)$$

This problem was discussed at a special Seminar of Geometry and Theoretical Physics in the Hirosima University, and research into it has been carried on under the Scientific-Research Expenditure of the Department of Education.

Hirosima Rikugun Yônen Gakkô, Hirosima.