Wave Geometry No. 46

Spin Transformations. I.

By

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§ 1. Introduction.

Any 4-4 matrices γ_i which satisfy the relations

$$\gamma_{(i}\gamma_{j)}=g_{ij}I \qquad i, j=1, 2, 3, 4 \tag{1.1}$$

for any given fundamental tensor g_{ij} in a 4-dimensional Riemannian space are given as follows[1]:

$$\gamma_i=S^{-1}h_i^r\overset{\circ}{\gamma}_r S, \tag{1.2}$$

where S is any 4-4 matrix, $\overset{\circ}{\gamma}_i$ are any 4-4 matrices satisfying $\overset{\circ}{\gamma}_{(i}\overset{\circ}{\gamma}_{j)}=\delta_{ij}I$, and h_j^i satisfy the following relations:

$$\sum_{r=1}^{4} h_i^r h_j^r=g_{ij}, \tag{1.3}$$

i. e. arbitrary γ_i are given by $H=\| h_{ij}\|$ (i indicate the rows and j the columns) and a spin matrix S. Now let us consider the space Γ_4 consisting of all γ_i $(=h_i^r\overset{\circ}{\gamma}_r)$ where $\overset{\circ}{\gamma}_r$ are fixed and h_j^i may take all the values satisfying relations (1.3). An element γ_i of Γ_4 evidently satisfies (1.1). Further, let us consider the spin transformation S of the elements γ_i's of Γ_4, such that $'\gamma_i=S^{-1}\gamma_i S$. The set of all such S we write $\mathfrak{S}$; then, clearly, $\mathfrak{S}$ makes a group. So for any γ_i and $'\gamma_i$ satisfying (1.1), there exists S such that transitive group leaving Γ_4 invariant. There now arises the problem of determining group $\mathfrak{S}$.

If any two elements γ_i and $'\gamma_i$ of Γ_4 be written as follows:

$$\gamma_i=h_i^r\overset{\circ}{\gamma}_r \quad \text{and} \quad '\gamma_i=k_i^r\overset{\circ}{\gamma}_r,$$

then, from (1.3), we have $H^*H=K^*K=G$, where $H=\| h_j^i\|$, $K=\| k_j^i\|$, and $G=\| g_{ij}\|$, and the asterisk denotes the transposed matrix. If we put $HK^{-1}\equiv A$, we have $A^*A=I$, i. e. A is an orthogonal matrix. Then we say that γ_i and $'\gamma_i$ have the same or opposite orientations, according as the orthogonal matrix A is proper or improper. Specially, if we take δ_{ij} for

(1) Pauli, Ann. d. Physik. **18** (1933).
Newman, Jour. London Math. Soc. **7** (1932), p. 93.

g_{ij}, then $H=\|h^i_j\|$ satisfying (1.3) becomes an orthogonal matrix, and then the space Γ_4 of γ_i can be considered as a vector space whose basis is $\{\mathring{\gamma}_i\}$. If we say that γ_i and γ_j $(i \neq j)$ are perpendicular when, and only when, $\gamma_{(i}\gamma_{j)}=0$, then $\{\gamma_i\}$ forms an orthogonal ennuple in the space Γ_4. In this case the orientation of the system γ_i is equivalent to the orientation of an orthogonal ennuple h^i_j.

In this paper we assume that A is a real matrix, i. e. the elements of A are all real numbers. Now if, for any two elements $\gamma_i=h^j_i\mathring{\gamma}_j$, and $'\gamma_i=k^j_i\mathring{\gamma}_j$ of Γ_4, HK^{-1} is real, we say that the systems $\{\gamma_i\}$ and $\{'\gamma_i\}$ are equivalent; and this equivalence is obviously reflective and transitive. Then all the elements of Γ_4 are classified into certain sets $R_4, S_4, \ldots.$ of elements such that elements of the same set are equivalent to one another. Thus, in this paper, instead of Γ_4 we shall consider any sub-space of Γ_4, say R_4, in which any two elements are related to each other by real orthogonal matrices. By $\bar{\mathfrak{S}}$ we denote the set of S such that $'\gamma_i=S^{-1}\gamma_i S$ for any two elements γ_i and $'\gamma_i$ of R_4. $\bar{\mathfrak{S}}$ is clearly a sub-group of $\mathfrak{S}$.

Brauer and Weyl(1) have algebraically classified the spin matrices S of $\mathfrak{S}$, and therefore of $\bar{\mathfrak{S}}$, into two classes; but in order to determine the concrete forms, the infinitesimal method has been adopted there. In this paper, however, where consideration is purely abstract, regarding $\mathring{\gamma}_i$ as any operator satisfying $\mathring{\gamma}_{(i}\mathring{\gamma}_{j)}=\delta_{ij}I$ we shall algebraically evaluate S and then classify the elements of $\bar{\mathfrak{S}}$. That is to say, in § 2–4 we actually evaluate S for any given γ_i and $'\gamma_i$ in a sub-space R_4; and as its corollary, we prove the existence of operator S such that $'\gamma_i=S^{-1}\gamma_i S$ for any given γ_i and $'\gamma_i$ in R_4(2). In § 5 we classify the elements of $\bar{\mathfrak{S}}$ into two classes, one preserving the orientations and the other changing them; and in § 6 we determine the concrete forms of spin operators S of $\bar{\mathfrak{S}}$. In § 7 we describe the infinitesimal method; and in § 8 we discuss the relations of our method to the infinitesimal method, and then obtain the simple relation between Cayley's parametrization of an orthogonal matrix and the spin operator. In § 9–§ 11 we extend the result above-obtained for 8-8 matrix, and show that the same procedure can be extended to 2^n-2^n matrix.

The result of this paper holds good also for the general case in which the condition of reality is removed.(3) Therefore we can apply this result to 4-dimensional space-time. Lastly we shall suggest that the case of 8-8 matrix seems to be applicable to the atomic nucleus.

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(1) R. Brauer and H. Weyl, Amer. Jour. of Math. **57** (1935), pp. 425–449.

(2) The proof of the existence of operator S for any operator γ_i and $'\gamma_i$ satisfying $\gamma_{(i}\gamma_{j)}={}'\gamma_{(i}{}'\gamma_{j)}=\delta_{ij}I$ has been given by Eddington. Our result is a special case of that result. Cf. A. S. Edington; Jour. Lond. Math. Soc. **7** (1932), pp. 58–68.

(3) We shall give the proof in the next paper.

§ 2. Determination of S.

Take any two elements $\gamma_i = h_i^r \mathring{\gamma}_r$ and $'\gamma_i = k_i^r \mathring{\gamma}_r$ of Γ_4 where $i, r = 1, 2, 3, 4$, and consider the matrices $H \equiv \| h_j^i \|$ and $K \equiv \| k_j^i \|$; then HK^{-1} becomes an orthogonal matrix. Now we shall determine S satisfying the relation

$$'\gamma_i = k_i^r \mathring{\gamma}_r = S^{-1} \gamma_i S = S^{-1} h_i^r \mathring{\gamma}_r S. \tag{2.1}$$

If we put $HK^{-1} \equiv A = \| a_j^i \|$, from (2.1) we have

$$\mathring{\gamma}_i = S^{-1} a_i^j \mathring{\gamma}_j S. \tag{2.2}$$

Expanding S by basic elements,

$$S = \Lambda I + \Lambda^i \mathring{\gamma}_i + \Lambda^5 \mathring{\gamma}_5 + \Lambda^{i5} \mathring{\gamma}_i \mathring{\gamma}_5 + \Lambda^{ij} \mathring{\gamma}_i \mathring{\gamma}_j, \tag{2.3}$$

where

$$\left.\begin{aligned} &\mathring{\gamma}_{(i} \mathring{\gamma}_{j)} = \delta_{ij} I \\ &\Lambda^{ij} = -\Lambda^{ji}, \quad \text{and} \quad \mathring{\gamma}_5 = \mathring{\gamma}_1 \mathring{\gamma}_2 \mathring{\gamma}_3 \mathring{\gamma}_4, \end{aligned}\right\} \tag{2.4}$$

and taking into account the relations:

$$\left.\begin{aligned} &\mathring{\gamma}_i \mathring{\gamma}_j \mathring{\gamma}_k = -\mathring{\epsilon}_{ijk}{}^{l} \mathring{\gamma}_l \mathring{\gamma}_5 \qquad (i, j \neq) \\ &\mathring{\gamma}_i \mathring{\gamma}_j \mathring{\gamma}_5 = -\frac{1}{2} \mathring{\epsilon}_{ij}{}^{kl} \mathring{\gamma}_k \mathring{\gamma}_l \quad (i \neq j) \end{aligned}\right\} \tag{2.5}$$

where $\mathring{\epsilon}_{ijkl} = 0$ when any two of i, j, k, l are equal,

$= 1$ when (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$,

$= -1$ when (i, j, k, l) is an odd permutation of $(1, 2, 3, 4)$,

from (2.2) and (2.3) we have

$$\left.\begin{aligned} &\text{(i)} \quad (a_i^k - \delta_i^k) \Lambda_k = 0, \\ &\text{(ii)} \quad (a_i^j - \delta_i^j) \Lambda + 2(a_i^k + \delta_i^k) \Lambda_k{}^{j} = 0, \\ &\text{(iii)} \quad (a_i^k + \delta_i^k) \Lambda_k{}^{5} = 0, \\ &\text{(iv)} \quad (a_i^j + \delta_i^j) \Lambda^5 = (a_i^k - \delta_i^k) \Lambda^{lp} \mathring{\epsilon}_{klp}{}^{j}, \\ &\text{(v)} \quad (a_i^p + \delta_i^p)(\delta_p^j \Lambda^k - \delta_p^k \Lambda^j) = (a_i^l - \delta_i^l) \Lambda^{p5} \mathring{\epsilon}_{lp}{}^{jk}. \end{aligned}\right\} \tag{2.6}$$

Thus the problem of determining S for given H and K becomes that of solving Λ's from (2.6) for given a_j^i.

§ 3. Determination of S for same orientations.

In order to solve (2.6), we use the following well-known theorems:

Lemma 1. *If F is a real skew-symmetric matrix whose degree is even, say $2n$, there exists a real orthogonal matrix P such that*

$$P^{-1}FP=\tilde{F}=\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}\dotplus\begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix}\dotplus\cdots\cdots\dotplus\begin{pmatrix} 0 & a_n \\ -a_n & 0 \end{pmatrix}.$$

Lemma 2. *If A is a real proper orthogonal matrix whose degree is even, say $2n$, there exists a real orthogonal matrix T such that*

$$T^{-1}AT=\tilde{A}=\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}\dotplus\begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}\dotplus\cdots\cdots\dotplus\begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix}.$$

When $\| a^i_j \|$ is real and proper, by Lemma 2 we can choose a real orthogonal matrix T such that

$$T^{-1}AT=\tilde{A}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\dotplus\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{3.1}$$

Put $T=\| t^i_j \|$ and $t^r_i\mathring{\gamma}_r={}'\mathring{\gamma}_i$; then ${}'\mathring{\gamma}_{(i}{}'\mathring{\gamma}_{j)}=\delta_{ij}I$, because T is an orthogonal matrix. If we write ${}'\mathring{\gamma}_5$ for ${}'\mathring{\gamma}_1{}'\mathring{\gamma}_2{}'\mathring{\gamma}_3{}'\mathring{\gamma}_4$, then ${}'\mathring{\gamma}_5=\text{det.}|T|\cdot\mathring{\gamma}_5$ and $\text{det.}|T|=+1$ or -1, according as the matrix T is proper or improper. If we put $T^{-1}=\| T^i_j \|$, then $\mathring{\gamma}_i=T^r_i{}'\mathring{\gamma}_r$. Substituting this into (2.2), we get:

$${}'\mathring{\gamma}_i=S^{-1}t^r_i a^s_r T^j_s{}'\mathring{\gamma}_j S$$

or

$${}'\mathring{\gamma}_i=S^{-1}\tilde{a}^j_i{}'\mathring{\gamma}_j S, \tag{3.2}$$

where $\| \tilde{a}^i_j \|=\tilde{A}=T^{-1}AT$ and

$$S={}'\Lambda I+{}'\Lambda^5{}'\mathring{\gamma}_5+{}'\Lambda^i{}'\mathring{\gamma}_i+{}'\Lambda^{i5}{}'\mathring{\gamma}_i{}'\mathring{\gamma}_5+{}'\Lambda^{ij}{}'\mathring{\gamma}_i{}'\mathring{\gamma}_j, \tag{3.3}$$

where

$${}'\Lambda=\Lambda,\quad {}'\Lambda^5=\epsilon\Lambda^5,\quad {}'\Lambda^i=T^i_k\Lambda^k,\quad {}'\Lambda^{i5}=\epsilon T^i_k\Lambda^{k5},\quad {}'\Lambda^{ij}=T^i_kT^j_l\Lambda^{kl}, \tag{3.4}$$

or

$$\Lambda={}'\Lambda,\quad \Lambda^5=\epsilon{}'\Lambda^5,\quad \Lambda^i=t^i_r{}'\Lambda^r,\quad \Lambda^{i5}=\epsilon t^i_r{}'\Lambda^{r5},\quad \Lambda^{ij}=t^i_kt^j_l{}'\Lambda^{kl}, \tag{3.5}$$

and $\epsilon\equiv\text{det.}|T|$. To simplify description, we write $\Lambda, \Lambda^5, \ldots$ etc., and a^j_i, instead of ${}'\Lambda, {}'\Lambda^i, \ldots$ etc., and $\tilde{a}^j_i$. Then we have, from (3.2), the same relation as (2.6), in which

$$A=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\dotplus\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

To solve (2.6), we consider the problem in the following cases:

Case I. $\quad |I+A|\neq 0, \quad |I-A|\neq 0;$

Case II. $\quad |I+A|=0, \quad |I-A|\neq 0;$

Case III. $\quad |I+A|\neq 0, \quad |I-A|=0;$

Case IV. $\quad |I+A|=0, \quad |I-A|=0.$

In what follows we use the letters of indices as follows:

$$i, j, k, l, \ldots\ldots=1, 2, 3, 4$$

$$a, b, \ldots\ldots\ldots=1, 2$$

$$x, y, \ldots\ldots\ldots=3, 4$$

Case I. From (2.6) (i) and (iii) we have $\Lambda_k = \Lambda_k^{\;5} = 0$, and from (ii), we have $\Lambda^{ax}=0$, $\Lambda^{12}=\frac{1}{2}\tan\frac{\theta}{2}\Lambda$, $\Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda$. From (iv), $\Lambda^5=\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\Lambda$. Then (v) becomes an identity. Thus we have

$$\left.\begin{aligned} &\Lambda^k=\Lambda^{k5}=0\,, \quad \Lambda^{ax}=0\,, \quad \Lambda^{12}=\frac{1}{2}\tan\frac{\theta}{2}\Lambda\,, \quad \Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda\,, \\ &\Lambda^5=\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\Lambda\,, \qquad \text{and } \Lambda \text{ is arbitrary.} \end{aligned}\right\} \tag{3.6}$$

Case II. From $|I+A|=0$, $\theta\equiv\pi$ or $\varphi\equiv\pi$ (mod. 2π). First we assume that $\theta\equiv\pi$ (mod. 2π) and $\varphi\not\equiv\pi$ (mod. 2π). Then $a_a^i=-\delta_a^i$. From (i) $\Lambda_k=0$, and from (ii) $\Lambda=0$, $\Lambda^{ax}=0$, $\Lambda^{xy}=0$; from (iii) $\Lambda^{x5}=0$, from (iv) $\Lambda^{12}=\frac{1}{2}\cot\frac{\varphi}{2}\Lambda^5$, and from (v) $\Lambda^{x5}=0$. Rewriting (3.6) as follows:

$$\Lambda=2\Lambda^{12}\cot\frac{\theta}{2}\,, \quad \Lambda^{12}=\frac{1}{2}\cot\frac{\varphi}{2}\Lambda^5\,, \quad \Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda\,,$$

and putting $\theta\equiv\pi$ (mod. 2π), we see that $\Lambda=0$; consequently $\Lambda^{34}=0$, and $\Lambda^{12}=\frac{1}{2}\cot\frac{\varphi}{2}\Lambda^5$. Thus, from (3.6), we can obtain the result for when $\theta\equiv\pi$ (mod. 2π). When $\theta\equiv\pi$ and $\varphi\equiv\pi$ (mod. 2π), we have again the same result.

Case III. Here $\theta\equiv 0$ or $\varphi\equiv 0$ (mod. 2π). By the same treatment as above we have (3.6), in which θ or φ is congruent to 0 mod. 2π.

Case IV. Here, we can take $a_a^i=\delta_a^i$ and $a_x^i=-\delta_x^i$. Then we have: $\Lambda^k=\Lambda^{k5}=0$, $\Lambda=\Lambda^5=\Lambda^{aj}=0$; and Λ^{34} is arbitrary. So in the same way as in Case II, we have (3.6), in which $\theta\equiv 0$ and $\varphi\equiv\pi$ (mod. 2π).

Thus, as the general solution of (3.2) we have

$$\left.\begin{aligned} &\Lambda^k=\Lambda^{k5}=0\,, \quad \Lambda^{ax}=0\,, \quad \Lambda^{12}=\frac{1}{2}\tan\frac{\theta}{2}\Lambda\,, \quad \Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda\,, \\ &\qquad\qquad \Lambda^5=\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\Lambda\,. \end{aligned}\right\} \tag{3.7}$$

As is easily seen from this, for given γ_i and $'\gamma_i$ of R_4 S is determined uniquely except for a numerical factor, and Λ, Λ^5, Λ^{ij} are real except for a common factor. Returning to $\Lambda, \Lambda^5, \ldots$ in (3.5), from these in (3.7) $\Lambda^k=\Lambda^{k4}=0$. Thus S defined by (2.2) for the real proper orthogonal matrix $A=\|a_j^i\|$, i. e. for γ_i and $'\gamma_i$ of R_4 of the same orientations, must be of the form

$$S=\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j\,,$$

where Λ, Λ^5, Λ^{ij} are real function except for a common factor.

§ 4. Determination of S for opposite orientations.

When $\gamma_i\ (=h_i^r\mathring{\gamma}_r)$ and $'\gamma_i\ (=k_i^r\mathring{\gamma}_r)$ have opposite orientations, taking $\bar{\gamma}_i\ (=\bar{h}_i^r\mathring{\gamma}_r)$ of R_4 with opposite orientations compared with γ_j, and considering T such that

$$T^{-1}\gamma_i T=\bar{\gamma}_i\,, \tag{4.1}$$

we have, from (2.1),

$$S^{-1}T\bar{\gamma}_i T^{-1}S={}'\gamma_i\,. \tag{4.2}$$

If we put $T^{-1}S=U$, we see from (4.2) that U is a transformation of $\bar{\gamma}_i$ into $'\gamma_i$ preserving the orientation; therefore, when the reality $\bar{H}K^{-1}$ is taken into account, U has the following form:

$$U=\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j\,. \tag{4.3}$$

Therefore, if T is determined by (4.1), S is determined by $S=TU$. Specially, if we take $\bar{H}=|\bar{h}_j^i|$ as follows:

$$H\bar{H}^{-1}\equiv A\equiv\begin{pmatrix}-1&&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}\quad\text{i. e.}\quad \begin{matrix}\bar{h}_a^i=-h_a^i\\ \bar{h}_4^i=h_4^i\end{matrix}\ \begin{pmatrix}a=1,2,3\\ i=1,2,3,4\end{pmatrix} \tag{4.4}$$

T is obtained from (2.6) by putting $a_a^i=-\delta_a^i$, $a_4^i=\delta_4^i$ $(a=1,2,3,\ i=1,2,3,4)$; that is to say, $\Lambda_a=0$ (from (i)), $\Lambda=0$ (from (ii)), $\Lambda^{ij}=0$ and $\Lambda^5=0$ (from (iv)), $\Lambda^{i5}=0$ (from (v)), i. e. $T=\lambda\mathring{\gamma}_4$. Therefore S transforming any two of γ_i's of opposite orientations is obtained as follows:

$$S=\lambda\mathring{\gamma}_4(\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j)={}'\Lambda^i\mathring{\gamma}_i+{}'\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5\,,$$

i. e. S has the following form:

$$S=\Lambda^i\mathring{\gamma}_i+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5\,,$$

where Λ^i, Λ^{i5} are real except for a common factor.

§ 5. Classification of $\mathfrak{S}$.

Putting together the results obtained in § 3 and § 4, we have:

Theorem 1. *For any two given elements $\gamma_i=h_i^r\mathring{\gamma}_r$ and $'\gamma_i=k_i^r\mathring{\gamma}_r$ of the space R_4, there exists one, and only one, (except for a numerical factor) S such that $'\gamma_i=S^{-1}\gamma_i S$. When γ_i and $'\gamma_i$ have the same orientations, S has the following form:*

$$S=\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j\,;$$

and when γ_i and $'\gamma_i$ have opposite orientations, S' has the following form:

$$S=\Lambda^i\mathring{\gamma}_i+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5\,.$$

And in both cases, the coefficients of $I, \mathring{\gamma}_5, \mathring{\gamma}_i, \mathring{\gamma}_{[i}\mathring{\gamma}_{j]}, \mathring{\gamma}_i\mathring{\gamma}_5$, *are real except for a common factor.*

We denote the operator of the form $\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j$ by S_1, and the operator of the form $\Lambda^i\mathring{\gamma}_i+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5$ by S_2. Then, from the identity:

$$\mathring{\gamma}_5(\Lambda I+\Lambda^i\mathring{\gamma}_i+\Lambda^5\mathring{\gamma}_5+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j)\mathring{\gamma}_5$$
$$=\Lambda I-\Lambda^i\mathring{\gamma}_i+\Lambda^5\mathring{\gamma}_5-\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j,$$

we know that S_1 is characterized by the relation $\mathring{\gamma}_5 S_1\mathring{\gamma}_5=S_1$, and S_2 by $\mathring{\gamma}_5 S_2\mathring{\gamma}_5=-S_2$. So that γ_i and $'\gamma_i$ have the same, or opposite, orientations, according as S mediating γ_i and $'\gamma_i$ satisfies $\mathring{\gamma}_5 S\mathring{\gamma}_5=S$ or $\mathring{\gamma}_5 S\mathring{\gamma}_5=-S$ and conversely.(1)

Next we consider the case when γ_i, and consequently S, are 4-4 matrices. If we take Dirac's matrices as $\mathring{\gamma}_i$,(2) then

$$S_1=\left(\begin{array}{cc|cc}\times&\times&0&\\\times&\times&&\\\hline &0&\times&\times\\&&\times&\times\end{array}\right)\quad\text{and}\quad S_2=\left(\begin{array}{cc|cc}0&&\times&\times\\&&\times&\times\\\hline\times&\times&0&\\\times&\times&&\end{array}\right).$$

Next, instead of $\mathring{\gamma}_i$ satisfying $\mathring{\gamma}_{(i}\mathring{\gamma}_{j)}=\delta_{ij}I$, we take any operator $\tilde{\gamma}_i$ satisfying $\tilde{\gamma}_{(i}\tilde{\gamma}_{j)}=g_{ij}I$, and consider the analogous problem. That is, we consider the space $\tilde{\Gamma}_4$ consisting of all $\gamma_i\ (=h_i^r\gamma_r)$ where γ_i are fixed and $\tilde{h}_j^i$ may take all the values satisfying $\tilde{h}_j^r\tilde{h}_j^s g_{rs}=g_{ij}$, and we shall investigate the form of spin operator S such that $'\gamma_i=S^{-1}\gamma_i S$ for any two $\gamma_i\ (=\tilde{h}_i^r\tilde{\gamma}_r)$ and $\tilde{\gamma}_i\ (=k_i^r\tilde{\gamma}_r)$ of $\tilde{\Gamma}_4$. Here the condition that HK^{-1} with respect to $\mathring{\gamma}_i$ is real becomes that $\mathring{H}\tilde{H}\tilde{K}^{-1}\mathring{H}^{-1}$ is real where $\tilde{\gamma}_i=\mathring{h}_i^k\mathring{\gamma}_k$ and $\mathring{H}=\|\mathring{h}_j^i\|$. Thus, as in Γ_4, in $\tilde{\Gamma}_4$ we shall restrict ourselves to one of the sub-spaces of $\tilde{\Gamma}_4$, say $\tilde{R}_4$.(3) Then we have the same result as in Theorem 1,(4) $\mathring{\gamma}_i$ and $\mathring{\gamma}_5$ being replaced by $\tilde{\gamma}_i$ and $\tilde{\gamma}_5=\dfrac{1}{\sqrt{\det.|\mathring{H}|}}\tilde{\gamma}_{[1}\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_{4]}$.

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(1) This is the same result as that obtained by Brauer and Weyl. Brauer and Weyl, loc. cit.

(2)
$$\mathring{\gamma}_1=\left(\begin{array}{cc|cc}0&&1&0\\&&0&1\\\hline 1&0&0&\\0&1&&\end{array}\right),\quad \mathring{\gamma}_2=\left(\begin{array}{cc|cc}0&&0&-i\\&&-i&0\\\hline 0&i&0&\\i&0&&\end{array}\right),\quad \mathring{\gamma}_3=\left(\begin{array}{cc|cc}0&&0&1\\&&-1&0\\\hline 0&-1&0&\\1&0&&\end{array}\right),$$
$$\mathring{\gamma}_4=\left(\begin{array}{cc|cc}0&&-i&0\\&&0&i\\\hline i&0&0&\\0&-i&&\end{array}\right)\quad\text{and}\quad \mathring{\gamma}_5=\left(\begin{array}{cc|cc}1&0&0&\\0&1&&\\\hline 0&&-1&0\\&&0&-1\end{array}\right).$$

(3) The meaning of $\tilde{R}_4$ is analogous to that of R_4 in Γ_4.

(4) By making use of $\mathring{H}$, the problem in $\tilde{R}_4$ can be reduced to the case of R_4. Returning to $\tilde{R}_4$, we have the same result as in Theorem 1.

§ 6. Determination of $\bar{\mathfrak{S}}$.

In this section we shall determine the set $\bar{\mathfrak{S}}$ of the operators S which leave the space R_4 invariant, that is $S^{-1}R_4S=R_4$. But $'\gamma_i=S^{-1}{}'\gamma_iS$, in which $\gamma_i\,(=h_i^r\mathring{\gamma}_r)$ is any element of the space R_4, and we shall find the condition that $'\gamma_i$ can be written as $'\gamma_i=k_i^r\mathring{\gamma}_r$, and HK^{-1} is real where $H=\|h_j^i\|$, $K=\|k_j^i\|$, and

$$\sum_{r=1}^{4}h_i^rh_j^r=g_{ij}\,,\qquad \mathring{\gamma}_{(i}\mathring{\gamma}_{j)}=\delta_{ij}I\,. \tag{6.1}$$

Since, necessarily, $'\gamma_{(i}{}'\gamma_{j)}=g_{ij}I$, we have $\sum_{r=1}^{4}k_i^rk_j^r=g_{ij}$. By Theorem 1, if γ_i and $'\gamma_i$ have the same orientations, and HK^{-1} is real, S must be of the form:

$$S=S_1\equiv\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j\,, \tag{6.2}$$

where Λ, Λ^5, Λ^{ij} are real except for a common factor; and if they have opposite orientations, S must be of the form:

$$S=S_2\equiv\Lambda^i\mathring{\gamma}_i+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5\,, \tag{6.3}$$

where Λ^i, Λ^{i5} are real except for a common factor. Thus, in order that $S^{-1}{}'\gamma_iS\equiv{}'\gamma_i$ may belong to the space R_4, S must be either S_1 or S_2.

When S is of the form S_1, if we put $R\equiv\|\Lambda^{ij}\|$ (i indicates the rows and j the columns), from the fact that $R^*+R=0$ and Λ^{ij} are real except for a common factor, we know by Lemma 1 that, after a suitable real orthogonal transformation $T=\|t_j^i\|$, R is transformed as follows:

$$T^{-1}RT={}'R=\begin{pmatrix}0 & '\Lambda^{12}\\ -'\Lambda^{12} & 0\end{pmatrix}\dot{+}\begin{pmatrix}0 & '\Lambda^{34}\\ -'\Lambda^{34} & 0\end{pmatrix}.$$

If we put $'\mathring{\gamma}_i=t_i^r\mathring{\gamma}_r$, or $\mathring{\gamma}_i=T_i^r{}'\mathring{\gamma}_r$ where $T^{-1}=\|T_j^i\|$, S can be written as follows:

$$S=S_1={}'\Lambda I+{}'\Lambda^5{}'\mathring{\gamma}_5+{}'\Lambda^{ij}{}'\mathring{\gamma}_i{}'\mathring{\gamma}_j\,, \tag{6.4}$$

where

$$'\Lambda=\Lambda\,,\quad '\Lambda^5=\epsilon\Lambda^5\,,\quad '\Lambda^{ij}=T_k^iT_l^j\Lambda^{kl}=\sum_l T_k^i\Lambda^{kl}t_j^l\,,\quad \epsilon=\text{det.}|T| \tag{6.4}$$

(Cf. (3.4)). Now, if we assume that there exists S^{-1} for S of the form S_1, the conditions that, for S given by (6.4), $S^{-1}\gamma_iS\equiv{}'\gamma_i$ belong to the space R_4 are, from § 3, as follows:

$$'\Lambda^{12}=\frac{1}{2}\tan\frac{\theta}{2}{}'\Lambda\,,\quad '\Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}{}'\Lambda\,,\quad '\Lambda^5=\tan\frac{\theta}{2}\tan\frac{\varphi}{2}{}'\Lambda\,. \tag{6.6}$$

Eliminating θ and φ above, we obtain the condition:

$$'\Lambda^{12}{}'\Lambda^{34}=\frac{1}{4}{}'\Lambda^5{}'\Lambda\,, \tag{6.7}$$

or, in the original Λ's,

$$\frac{1}{2}\mathring{\epsilon}_{ijkj}\Lambda^{ij}\Lambda^{kl}=\Lambda\Lambda^{5}\text{[1]}; \tag{6.8}$$

which is the required condition.

When $S=S_2=\Lambda^i\mathring{\gamma}_i+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5$, $\mathring{\gamma}_4 S$ is of the form S_1; therefore, putting $\mathring{\gamma}_4 S\equiv T$, we have $S^{-1}\gamma_i S=T^{-1}\mathring{\gamma}_4\mathring{\gamma}_i\mathring{\gamma}_4 T$. But if we put $\mathring{\gamma}_4\mathring{\gamma}_i\mathring{\gamma}_4=\bar{\gamma}_i$, then $\bar{\gamma}_i=\bar{h}_i^r\mathring{\gamma}_r$ and $\bar{h}_i^a=-h_i^a$, $\bar{h}_i^4=h_i^4$ $(a=1,2,3)$; therefore $\bar{\gamma}_i$ belongs to R_4. Thus $S^{-1}\gamma_i S=T^{-1}\bar{h}_i^r\mathring{\gamma}_r T={}'\gamma_i$; therefore, if we assume that there exists S^{-1} for S, the necessary and sufficient condition for $'\gamma_i$ to belong to the space R_4 is that the coefficient of expansion of T satisfy (6.8). Now T can be written as follows:

$$T=\mathring{\gamma}_4 S={}'\Lambda I+{}'\Lambda^5\mathring{\gamma}_5+{}'\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j$$

where

$$'\Lambda=\Lambda^4,\quad '\Lambda^5=\Lambda^{45},\quad '\Lambda^{a4}=-\frac{1}{2}\Lambda^a,\quad '\Lambda^{ab}=\frac{1}{2}\Lambda^{c5}\mathring{\epsilon}_{c4}^{\cdot\cdot ab}\quad (a,b,c=1,2,3),$$

and $'\Lambda, '\Lambda^5, '\Lambda^{ij}$ are real except for a common factor, because of the reality of Λ^i, Λ^{i5}. Substituting these $'\Lambda, '\Lambda^s, '\Lambda^{ij}$ into (6.8), we have:

$$'\Lambda'\Lambda^5=\frac{1}{2}\mathring{\epsilon}_{ijkl}{}'\Lambda^{ij}{}'\Lambda^{kl}$$

i. e.

$$\Lambda_4\Lambda^{45}=\mathring{\epsilon}_{a4bc}{}'\Lambda^{a4}{}'\Lambda^{bc}+\mathring{\epsilon}_{abc4}{}'\Lambda^{ab}{}'\Lambda^{c4}=-\Lambda_a\Lambda^{a5},$$

so that

$$\Lambda_i\Lambda^{i5}=0 \tag{6.9}$$

So the condition for S of the form S_2 is $\Lambda_i\Lambda^{i5}=0$.

From the identities:

$$(\Lambda I+\Lambda^5\mathring{\gamma}_5+\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j)(\Lambda I+\Lambda^5\mathring{\gamma}_5-\Lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j)$$
$$=[C\Lambda)^2+(\Lambda^5)^2+2\Lambda^{ij}\Lambda_{ij}]I+(2\Lambda\Lambda^5-\mathring{\epsilon}_{ijkl}\Lambda^{ij}\Lambda^{kl})\mathring{\gamma}_5,$$

and

$$(\Lambda^i\mathring{\gamma}_i+\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5)(\Lambda^i\mathring{\gamma}_i-\Lambda^{i5}\mathring{\gamma}_i\mathring{\gamma}_5)=(\Lambda^i\Lambda_i+\Lambda^{i5}\Lambda_i^{\cdot 5})I-2\Lambda^i\Lambda_i^{\cdot 5}\mathring{\gamma}_5,$$

we see that for S of the form S_1 satisfying (6.8), or of the form S_2 satisfying (6.9), there exists S^{-1}[2]

Thus we have the following theorem.

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(1) (6.7) can be written as follows:

$$\frac{1}{2}\mathring{\epsilon}_{ijkl}{}'\Lambda^{ij}{}'\Lambda^{kl}=\epsilon\Lambda\Lambda^5$$

$$\therefore\ \frac{1}{2}\mathring{\epsilon}_{ijkl}T_p^iT_q^jT_r^kT_s^l\Lambda^{pq}\Lambda^{rs}=\epsilon\Lambda\Lambda^5,$$

$$\text{i. e.}\quad \frac{1}{2}\mathring{\epsilon}_{pqrs}\Lambda^{pq}\Lambda^{rs}=\Lambda\Lambda^5,$$

i. e. we have (6.8).

(2) The trivial case when $S=0$ is excluded.

Theorem 2. *The set* $\bar{\mathfrak{S}}$ *of S's which leave the space* R_4 *invariant consists of two parts* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$; $\mathfrak{S}_1$, *consists of S's which have the form* $S_1 = \Lambda I + \Lambda^5 \mathring{\gamma}_5 + \Lambda^{ij} \mathring{\gamma}_i \mathring{\gamma}_j$, $\Lambda, \Lambda^5, \Lambda^{ij}$ *being real except for a common factor, and whose coefficients* Λ, Λ^5, *and* Λ^{ij} *satisfy the following relation*

$$\frac{1}{2} \mathring{\epsilon}_{ijkl} \Lambda^{ij} \Lambda^{kl} = \Lambda \Lambda^5 ; \tag{6.10}$$

$\mathfrak{S}_2$ *consists of S's which have the form* $S_2 = \Lambda^i \mathring{\gamma}_i + \Lambda^{i5} \mathring{\gamma}_i \mathring{\gamma}_5$, Λ^i, Λ^{i5} *being real except for a common factor, and whose coefficients* Λ^i *and* Λ^{i5} *satisfy the following relation*

$$\Lambda_i \Lambda^{i5} = 0 . \tag{6.11}$$

The element S of $\mathfrak{S}_1$ *reserves the orientation of* γ_i *in* R_4, *and S of* γ_2 *changes the orientation of* γ_i *of* R_4.

In the case of $\tilde{R}_4$, if we make use of $\mathring{H}$ as in § 5, the problem of determining $\bar{\mathfrak{S}}$ can be reduced to the case of R_4. Then, returning to $\tilde{R}_4$, we have the same result as in Theorem 2,[1] (6.10) and (6.11) being replaced by

$$\frac{1}{2} \mathring{\epsilon}_{ijkl} \tilde{\Lambda}^{ij} \tilde{\Lambda}^{kl} = \tilde{\Lambda} \tilde{\Lambda}^s , \tag{6.12}$$

and

$$\tilde{\Lambda}^i \tilde{\Lambda}_i^{\cdot 5} = 0 ,^{(2)} \tag{6.13}$$

respectively, where $\tilde{\Lambda}, \tilde{\Lambda}^i, \ldots .$ etc. are coefficients of expansion of S with respect to $\tilde{\gamma}_i$, and $\tilde{\epsilon}_{ijkl} = \text{det.} |\mathring{H}| \cdot \mathring{\epsilon}_{ijkl} = \pm \sqrt{\text{det.} | g_{ij} |} \cdot \mathring{\epsilon}_{ijkl}$.

§ 7. Infinitesimal Description of S.

The problems discussed in the previous sections have been, in short, to determine S satisfying the relations

$$a_i^j \mathring{\gamma}_j = S \mathring{\gamma}_i S^{-1} , \qquad (i, j = 1, 2, 3, 4) \tag{7.1}$$

for any real orthogonal matrix $A = \| a_j^i \|$, where $\mathring{\gamma}_{(i} \mathring{\gamma}_{j)} = \delta_{ij} I$. If we denote the vector space whose basis is $\mathring{\gamma}_i$ by Γ, the transformations in the space Γ consist of two kinds, one being a coordinate transformation: $'\gamma_i = a_i^j \gamma_j$, and the other a spin transformation: $'\gamma_i = S^{-1} \gamma_i S$. The problem discussed in this paper, in other words, is to investigate the relations of these two

(1) In the case of $\tilde{R}_4$, the condition of reality of coefficients of S does not hold good, but there exists S^{-1} for $S \neq 0$, because $(\tilde{\Lambda})^2 + (\tilde{\Lambda}^5)^2 + 2\tilde{\Lambda}^{ij} \tilde{\Lambda}_{ij} = (\lambda)^2 + (\lambda^5)^2 + 2\lambda^{ij}\lambda_{ij}$ and $\tilde{\Lambda}^i \tilde{\Lambda}_i + \tilde{\Lambda}^{i5} \tilde{\Lambda}_i^{\cdot 5} = \lambda^i \lambda_i + \lambda^{i5} \lambda_i^{\cdot 5}$ where $\lambda_i, \lambda^i, \ldots .$ are coefficients of expansion of S with respect to $\mathring{\gamma}_i$.

(2) The up-and-down of indices is carried on for the tensors witn ripple-marks with respect to g_{ij}, and for the tensors with no ripple-mark with respect to δ_{ij}.

transformations. When A is an improper orthogonal matrix, the corresponding S is obtained as a product of two S's, say S_1, S_2, in which S_1, corresponds to a special improper orthogonal matrix and S_2 is any corresponding to the general proper orthogonal matrix. Thus the determination of S is reduced to the case when A is proper. Now, by Lemma 2, in suitably chosen orthogonal coordinates, any real proper orthogonal matrix A can be reduced to the direct sum of matrices of the form

$$\begin{pmatrix} \cos\theta \sin\theta \\ -\sin\theta \cos\theta \end{pmatrix}.$$

But the linear transformation A of the form above in the space Γ can be regarded as a finite form generated by the infinitesimal transformation of type $I+\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\delta\theta$, where θ is a parameter. Thus, any real proper orthogonal transformation A is an integral of an infinitesimal transformation,[1] so that S corresponding to any real proper orthogonal transformation A can be regarded as an integral of an infinitesimal S corresponding to a real infinitesimal orthogonal transformation. If we put $S=I+\sigma$ (σ is infinitesimal), and $A=I+\tau$ (τ is infinitesimal satisfying $\tau^*+\tau=0$), then, by (7.1), we have

$$t^i_{\cdot j}\mathring{\gamma}_j=\sigma\mathring{\gamma}_i-\mathring{\gamma}_i\sigma\,, \tag{7.2}$$

where $\| t^i_{\cdot j}\|=\tau$. Expanding σ in $\mathring{\gamma}_i$'s, from (7.2) we have

$$\sigma=\lambda I+\frac{1}{4}t^{ij}\mathring{\gamma}_i\mathring{\gamma}_j\,, \tag{7.3}$$

so that the infinite form of S can be obtained as follows:

$$S=e^\sigma=e^{\lambda I}\cdot e^{\frac{1}{4}t^{ij}\mathring{\gamma}_i\mathring{\gamma}_j}. \tag{7.4}$$

But by Lemma 1, by chosing suitable orthogonal coordinates in Γ, t^{ij} except t^{12} and t^{34} can be equated to zero. Put $t^{12}=\theta$ and $t^{34}=\varphi$; then, by making use of the formulae: $(\mathring{\gamma}_1\mathring{\gamma}_2)^2=(\mathring{\gamma}_3\mathring{\gamma}_4)^2=-I$, we have

$$\left.\begin{aligned} S&=e^{\lambda I}e^{\frac{1}{2}\theta\mathring{\gamma}_1\mathring{\gamma}_2}e^{\frac{1}{2}\varphi\mathring{\gamma}_3\mathring{\gamma}_4} \\ &=e^{\lambda I}\Big(I\cos\frac{\theta}{2}+\mathring{\gamma}_1\mathring{\gamma}_2\sin\frac{\theta}{2}\Big)\Big(I\cos\theta\frac{\varphi}{2}+\mathring{\gamma}_3\mathring{\gamma}_4\sin\frac{\varphi}{2}\Big) \\ &=a\,\Big(I+\tan\frac{\theta}{2}\mathring{\gamma}_1\mathring{\gamma}_2\Big)\Big(I+\tan\frac{\varphi}{2}\mathring{\gamma}_3\mathring{\gamma}_4\Big) \\ &=a\,\Big(I+\tan\frac{\theta}{2}\mathring{\gamma}_1\mathring{\gamma}_2+\tan\frac{\varphi}{2}\mathring{\gamma}_3\mathring{\gamma}_4+\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\mathring{\gamma}_5\Big), \end{aligned}\right\} \tag{7.5}$$

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(1) The complex proper orthogonal transformation is not necessarily generated by the repetition of an infinitesimal orthogonal transformation.
H. Taber, Bull. New York Math. Soc. **3** (1894), pp. 251-259. or
H. Taber, Proc. Lond. Math. Soc. **21** (1895), pp. 364-376.

where a is an arbitrary numerical factor and $\mathring{\gamma}_5=\mathring{\gamma}_1\mathring{\gamma}_2\mathring{\gamma}_3\mathring{\gamma}_4$. Since the coefficient of $\mathring{\gamma}_{[i}\mathring{\gamma}_{j]}$ in (7.5) are the same as those expressed by (6.6), they satisfy condition (6.8). Conversely, from the discussion in § 6, in a suitable coordinate system in Γ, S of the form S_1 (cf. Theorem 2) satisfying condition (6.8) can be written as (7.5), and consequently as (7.4). Therefore condition (6.8) is equivalent to the condition that S of the form S_1 is generated by the repetition of an infinitesimal spin transformation corresponding to an infinitesimal orthogonal transformation.

Thus S can be determined also by the infinitesimal method. But the method used in § 2–§ 6 is purely algebraic, whereas the infinitesimal method is analytic. In both methodes, however, $\mathring{\gamma}_i$ are taken as mere operators; therefore in this sense both methods are abstract.

§ 8. The relation between Cayley's parametrization of an orthogonal matrix and the spin matrix.

If an orthogonal matrix A is given, provided that det. $|I+A|\neq 0$, we can construct a matrix T such that $T=(I-A)(I+A)^{-1}$. Here T becomes skew-symmetric. This construction of T from A is called Cayley's parametrization, and the orthogonal matrix A in which det. $|I+A|\neq 0$ is called non-exceptional, and A in which det. $|I+A|=0$ is called exceptional.

The spin matrix S leaving the space Γ_4 invariant and reserving the orientation of γ_i is solved for a proper orthogonal matrix $A=\|a^i_j\|$ in (2.6). (ii), (iv) in (2.6) can be rewritten in matrix form as follows:

$$(A-I)\Lambda=2R(A+I) \tag{8.1}$$

$$(A+I)\Lambda^5=R'(A-I) \tag{8.2}$$

where $R=\|\Lambda^{ij}\|$, $R'=\|\Lambda^{lp}\mathring{\varepsilon}_{jlp}{}^{\cdot\cdot\cdot i}\|$ (i denotes the rows and j the columns). Therefore, when A is non-exceptional, $R=\dfrac{\Lambda}{2}(I-A)(I+A)^{-1}$; this shows that $R=-\dfrac{\Lambda}{2}T$, where T is a Cayley's skew-symmetric matrix of A. Likewise, when det. $|I-A|\neq 0$, from (8.2) we can determine R', and consequently R, in matrix form. Therefore, except when det. $|I+A|=$ det. $|I-A|=0$, R is determined in matrix form from (8.1) or (8.2) without our assuming the reality of A. The result of § 3 shows that, for this last exceptional case also, where exists R, provided that A is real.(1)

Now, an infinitesimal real orthogonal transformation A can be written as $A=I+\tau$, where infinitesimal matrix τ is real and skew-symmetric. Therefore $A=\left(I+\frac{1}{2}\tau\right)\left(I-\frac{1}{2}\tau\right)^{-1}=\left(I-\left(-\frac{1}{2}\tau\right)\right)\left(I+\left(-\frac{1}{2}\tau\right)\right)^{-1}$. Compar-

(1) In the next paper we shall show that R or R' always exists without the assumption of reality of A, and then determine condition (6.10) for complex A.

ing this with Cayley's parametrization $A=(I-T)(I+T)^{-1}$, we have $T=\frac{1}{2}\tau$. So that, from (7.3), $R=\left\|\frac{1}{4}t^{ij}\right\|=\frac{1}{4}\tau=-\frac{1}{2}T$, i. e. $A=(I+2R)(I-2R)^{-1}$ i. e. $(A-I)=2R(A+I)$. This is simply (8.1). Thus relation (8.1), i. e. the relation between Cayley's parametrization and the spin matrix S, is the finite algebraic form of the relation between the infinitesimal rotation and the corresponding S.

§ 9. Extension of the problem to 8-8 matrices.

We shall extend the problem discussed above to 8-8 matrices. The actual form of 8-8 matrices E_λ satisfying $E_{(\lambda}E_{\mu)}=\delta_{\lambda\mu}I$ has been given by Newman(1) as follows:

$$E_\lambda=S^{-1}\mathring{E}_\lambda S, \qquad (\lambda=1,2,\ldots\ldots,7) \tag{9.1}$$

where

$$\mathring{E}^a=\begin{pmatrix} & i\mathring{\gamma}_a \\ -i\mathring{\gamma}_a & \end{pmatrix}, \quad (a=1,2,\ldots\ldots,5) \quad \mathring{E}_6=\begin{pmatrix} & I \\ I & \end{pmatrix}, \quad \mathring{E}_7=\begin{pmatrix} I & \\ & -I \end{pmatrix}, \tag{9.2}$$

and $\mathring{\gamma}_{(a}\mathring{\gamma}_{\beta)}=\delta_{a\beta}I$ $(a, \beta=1,2,\ldots\ldots,5)$. Since $\mathring{\gamma}_5=\epsilon\mathring{\gamma}_1\mathring{\gamma}_2\mathring{\gamma}_3\mathring{\gamma}_4$ $(\epsilon=\pm1)$, E_λ given by (9.1) satisfies the following relations:

$$E_1E_2\ldots\ldots E_5E_6=i\epsilon E_7. \tag{9.3}$$

But if we are to aim at application to physics, it is desirable that $\mathring{E}_a$ $(a=1,2,\ldots\ldots,5)$ should have the form $\mathring{E}_a=\begin{pmatrix} A_a & 0 \\ 0 & B_a\end{pmatrix}$, where A_a and B_a are 4-4 matrices. To find such $\mathring{E}_a$, from $\mathring{E}_{(a}\mathring{E}_{\beta)}=\delta_{a\beta}I$, $A_{(a}A_{\beta)}=B_{(a}B_{\beta)}=\delta_{a\beta}I$, so that of necessity $A_a=\mathring{\gamma}_a$ and $B_a=\bar{\mathring{\gamma}}_a$.(2) If we put

$$\mathring{E}_\lambda(\lambda=6.7)\equiv\begin{pmatrix} X_\lambda & Y_\lambda \\ Z_\lambda & U_\lambda\end{pmatrix},$$

X_λ. Y_λ, Z_λ, and U_λ being 4-4 matrices), from $\mathring{E}_{(a}\mathring{E}_{\beta)}=0$ we have

$$\left.\begin{aligned} &(\text{i}) && \mathring{\gamma}_aX_\lambda+X_\lambda\mathring{\gamma}_a=0, \\ &(\text{ii}) && \bar{\mathring{\gamma}}_aU_\lambda+U_\lambda\bar{\mathring{\gamma}}_a=0, \\ &(\text{iii}) && \mathring{\gamma}_\lambda Y_\lambda+Y_\lambda\bar{\mathring{\gamma}}_a=0, \\ &(\text{iv}) && \bar{\mathring{\gamma}}_aZ_\lambda+Z_\lambda\mathring{\gamma}_a=0. \end{aligned}\right\} \tag{9.4}$$

From (i) and (ii) we have $X_\lambda=U_\lambda=0$; and since $\bar{\mathring{\gamma}}_a$ can always be written as follows:

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(1) M. H. A. Newman: loc. cit.

(2) $\bar{\mathring{\gamma}}_a$ does not mean conjugate imaginary of $\mathring{\gamma}_a$, but another matrix satisfying $\bar{\mathring{\gamma}}_{(a}\bar{\mathring{\gamma}}_{\beta)}=\delta_{a\beta}1$.

$$\bar{\mathring{\gamma}}_i = T^{-1} h_i^r \mathring{\gamma}_r T, \qquad (\sum_{r=1}^{4} h_i^r h_j^r = \delta_{ij}, \; i, j, r = 1, 2, \ldots, 4)$$
$$\bar{\mathring{\gamma}}_5 = T^{-1} \mathring{\gamma}_5 T,$$

from (iii),

$$\mathring{\gamma}_i Y_\lambda + Y_\lambda T^{-1} h_i^k \mathring{\gamma}_k T = 0, \quad \text{i.e.} \quad \mathring{\gamma}_i = -Y_\lambda T^{-1} h_i^k \mathring{\gamma}_k T Y_\lambda^{-1},$$

$$\mathring{\gamma}_5 Y_\lambda + Y_\lambda T^{-1} \mathring{\gamma}_5 T = 0, \quad \text{i.e.} \quad \mathring{\gamma}_5 = -Y_\lambda T^{-1} \mathring{\gamma}_5 T Y_\lambda^{-1}.$$

(det. $|\mathring{E}_\lambda| \neq 0$, $\therefore$ det. $|Y_\lambda|$, det. $|Z_\lambda| \neq 0$). Then, by Theorem 1, TY_λ^{-1} must have the form S_2; therefore the matrix $\| h_j^i \|$ must be improper; so that $\mathring{\gamma}_i$ and $\bar{\mathring{\gamma}}_i$ are of opposite orientations. Then it is possible to find a 4-4 matrix V such that

$$\left.\begin{aligned} &\text{(i)} && \bar{\mathring{\gamma}}_i = T^{-1} h_i^r \mathring{\gamma}_r T = -V^{-1} \mathring{\gamma}_i V, \\ &\text{(ii)} && \bar{\mathring{\gamma}}_5 = T^{-1} \mathring{\gamma}_5 T = -V^{-1} \mathring{\gamma}_5 V. \end{aligned}\right\} \tag{9.5}$$

For, $VT^{-1} h_i^r \mathring{\gamma}_r T V^{-1} = -\mathring{\gamma}_i$ can always be solved with respect to TV^{-1} for given h_j^i and $\mathring{\gamma}_i$. And we see that the solution TV^{-1} satisfies (ii) of (9.5), because det. $|h_j^i| = -1$. When TV^{-1} is found, V is determined, i.e. there exists V satisfying (9.5). Then (iii) and (iv) of (9.4) become:

$$\left.\begin{aligned} &\text{(i)} && \mathring{\gamma}_a Y_\lambda - Y_\lambda V^{-1} \mathring{\gamma}_a V = 0, && \text{i.e.} \quad \mathring{\gamma}_a Y_\lambda V^{-1} - Y_\lambda V^{-1} \mathring{\gamma}_a = 0, \\ &\text{(ii)} && -V^{-1} \mathring{\gamma}_a V Z_\lambda + Z_\lambda \mathring{\gamma}_a = 0, && \text{i.e.} \quad \mathring{\gamma}_a V Z_\lambda - V Z_\lambda \mathring{\gamma}_a = 0, \end{aligned}\right\} \tag{9.6}$$

$$\therefore \qquad Y_\lambda V^{-1} = a_\lambda I, \qquad V Z_\lambda = b_\lambda I.$$

Therefore

$$\mathring{E}_a = \begin{pmatrix} \mathring{\gamma}_a & \\ & -V^{-1} \mathring{\gamma}_a V \end{pmatrix}, \qquad \mathring{E}_\lambda = \begin{pmatrix} & a_\lambda V \\ b_\lambda V^{-1} & \end{pmatrix},$$

and from $\mathring{E}_\lambda \mathring{E}_\lambda = I$, $a_\lambda b_\lambda = 1$, and from $\mathring{E}_\lambda \mathring{E}_\mu + \mathring{E}_\mu \mathring{E}_\lambda = 0$ $(\lambda \neq \mu)$, $a_\lambda b_\mu + b_\mu a_\lambda = 0$, i.e. $b_\lambda = \dfrac{1}{a_\lambda}$ and $a_\lambda^2 + a_\mu^2 = 0$. Thus we have:

$$\mathring{E}_\lambda = \begin{pmatrix} \mathring{\gamma}_a & \\ & -V^{-1} \mathring{\gamma}_a V \end{pmatrix}, \quad \mathring{E}_6 = \begin{pmatrix} & aV \\ \frac{1}{a} V & \end{pmatrix}, \quad \mathring{E}_7 = \begin{pmatrix} & \pm iaV \\ \mp \frac{i}{a} V & \end{pmatrix}. \tag{9.7}$$

Specially, if we put $V = \mathring{\gamma}_4$ and $a = 1$, we have

$$\mathring{E}_a + \begin{pmatrix} \mathring{\gamma}_a & \\ & \mathring{\gamma}_a \end{pmatrix}, \quad \mathring{E}_4 = \begin{pmatrix} \mathring{\gamma}_4 & \\ & -\mathring{\gamma}_4 \end{pmatrix}, \quad \mathring{E}_5 = \begin{pmatrix} \mathring{\gamma}_5 & \\ & \mathring{\gamma}_5 \end{pmatrix}, \quad \mathring{E}_6 = \begin{pmatrix} & \mathring{\gamma}_4 \\ \mathring{\gamma}_4 & \end{pmatrix}, \quad \mathring{E}_7 = \begin{pmatrix} & -i\mathring{\gamma}_4 \\ i\mathring{\gamma}_4 & \end{pmatrix}. \tag{9.8}$$

If we take $\mathring{\gamma}_5$ such that $\mathring{\gamma}_5 = \mathring{\gamma}_1 \mathring{\gamma}_2 \mathring{\gamma}_3 \mathring{\gamma}_4$, we have $\mathring{E}_1 \mathring{E}_2 \ldots \mathring{E}_6 = i\mathring{E}_7$; and (9.7) and (9.8) are the required form of E_λ.

Next we shall consider the general case when $E_{(\alpha} E_{\beta)} = g_{\alpha\beta} I$ $(\alpha, \beta = 1, 2, \ldots, 6)$. This exists then h_β^α, such that

$$\sum_{\gamma=1}^{6} h_\alpha^\gamma h_\beta^\gamma = g_{\alpha\beta}\,. \tag{9.9}$$

Any matrices E_α satisfying $E_{(\alpha}E_{\beta)}=g_{\alpha\beta}I$ are given by $E_\alpha=h_\alpha^\beta \mathring{E}_\beta$, where $\mathring{E}_\alpha$ are suitable 8-8 matrices satisfying $\mathring{E}_{(\alpha}\mathring{E}_{\beta)}=\delta_{\alpha\beta}I$. Then, by (9.1), the general forms of E_α are given as follows:

$$E_\alpha=S^{-1}h_\alpha^\beta \mathring{E}_\beta S\,. \tag{9.10}$$

Likewise, if we consider E_i such that $E_{(i}E_{j)}=g_{ij}I$ $(i, j=1, 2, 3, 4)$ the general forms of E_i are given as follows:

$$E_i=S^{-1}h_i^j \mathring{E}_j S\,, \tag{9.11}$$

where $\sum_{r=1}^{4} h_i^r h_j^r=g_{ij}$ and $\mathring{E}_i$ are any 8-8 matrices satisfying $\mathring{E}_{(i}\mathring{E}_{j)}=\delta_{ij}I$.

In 4-4 matrix, $I, \gamma_5, \gamma_i, \gamma_i\gamma_5$ and $\gamma_{[i}\gamma_{j]}$ from a basis, provided that $\gamma_5=\pm\frac{1}{\sqrt{\det.|g_{ij}|}}\gamma_{[1}\gamma_2\gamma_3\gamma_{4]}$, where $\gamma_{(i}\gamma_{j)}=g_{ij}I$. Likewise we know that, if $g_{\alpha\beta}$ is given, the basis of 8-8 matrix is obtained as follows:

$$I,\ E_\alpha,\ E_7,\ E_\alpha E_7,\ E_{[\alpha}E_{\beta]},\ E_{[\alpha}E_{\beta]}E_7,\ E_{[\alpha}E_\beta E_{\gamma]}\,,$$

where E_α are determined by (9.10) and $E_\beta=\pm\frac{i}{\sqrt{\det.|g_{\alpha\beta}|}}E_{[1}E_2\cdots E_{6]}$. But if g_{ij} is given, E_i are given by (9.11), and there more E_5, E_6, E_7 are added in the forms $S^{-1}E_5S$, $S^{-1}E_6S$, $S^{-1}E_7S$ respectively. Thus two cases occur: (1) when the fundamental tensor of 6-dimensional Riemannian space is given, (2) when the fundamental tensor of 4-dimensional Riemmannian space is given. But case (1) looks, at present, physically meaningless, therefore we shall describe only the result.

In case (1) we consider the space Γ constituted by all E_α $(=h_\alpha^\beta \mathring{E}_\beta)$ $(\alpha, \beta=1, 2, \ldots, 6)$ where $\mathring{E}_\alpha$ are fixed and h_β^α may take all the values satisfying $\sum_{\gamma=1}^{6} h_\alpha^\gamma h_\beta^\gamma=g_{\alpha\beta}$. As in Γ_4, F_8 splits into certain sub-spaces $R_8, S_8, \ldots$ such that elements of the same sub-space are related to one another by real orthogonal transformations. Then the set $\bar{\mathfrak{S}}$ of operators S which leave any sub-space, say R_8 invariant, i.e. $S^{-1}R_8S=R_8$, consists of two parts $\mathfrak{S}_1$ and $\mathfrak{S}_2$.[1] $\mathfrak{S}_1$ consists of S's which have the form $\Lambda I+\Lambda^7\mathring{E}_7+\Lambda^{\lambda\mu}\mathring{E}_\lambda\mathring{E}_\mu+\Lambda^{\lambda\mu7}\mathring{E}_\lambda\mathring{E}_\mu\mathring{E}_7\equiv S_1$, and whose coefficients $\Lambda, \Lambda^7, \Lambda^{\lambda\mu}$, and $\Lambda^{\lambda\mu7}$ satisfying the following relations:

$$\Lambda\Lambda^{\lambda\mu7}+\frac{i}{4}\mathring{\epsilon}^{\lambda\mu}_{\cdot\cdot\rho\sigma\omega\nu}\Lambda^{\rho\sigma}\Lambda^{\omega\nu}=0\,,$$

$$(\Lambda)^2\Lambda^7-\frac{i}{6}\mathring{\epsilon}_{\lambda\mu\omega\nu\rho\sigma}\Lambda^{\lambda\mu}\Lambda^{\omega\nu}\Lambda^{\rho\sigma}=0\,,$$

(1) The trivial one $S=0$ is excluded.

where $\mathring{\varepsilon}_{\lambda\mu\omega\nu\rho\sigma}=0$ when any two of $(\lambda\mu\omega\nu\rho\sigma)$ are equal,
$=1$ when $(\lambda\mu\omega\nu\rho\sigma)$ is an even permutation of $(12....6)$,
$=-1$ when $(\lambda\mu\omega\nu\rho\sigma)$ is an odd permutation of $(12....6)$;

$\mathfrak{S}_2$ consists of S's which have the form $\Lambda^\lambda\mathring{E}_\lambda+\Lambda^{\lambda7}\mathring{E}_\lambda\mathring{E}_7+\Lambda^{\lambda\mu\nu}\mathring{E}_\lambda\mathring{E}_\mu\mathring{E}_\nu\equiv S_2$, and whose coefficients Λ^λ, $\Lambda^{\lambda7}$, and $\Lambda^{\lambda\mu\nu}$ satisfy the following relations:

$$\Lambda^\lambda\Lambda^{\mu\nu\omega}\mathring{\varepsilon}_{\lambda\mu\nu\omega\rho\sigma}=0\,,$$

$$\Lambda_\lambda\Lambda^{\lambda7}=0\,,$$

$$\Lambda^\lambda\Lambda^{\mu7}-\frac{3}{2}i\mathring{\varepsilon}^{\lambda\mu}_{\cdot\cdot\tau\omega\nu\rho\sigma}\Lambda^{\tau\omega\nu}\Lambda^{\lambda\rho\sigma}=0. \qquad (\lambda\neq\mu)$$

In the equation above $\Lambda, i\Lambda^7, \Lambda^{\lambda\mu}, i\Lambda^{\lambda\nu7}$; or $\Lambda^\lambda, i\Lambda^{\lambda7}, \Lambda^{\lambda\mu\nu}$, are real except for a common factor. The elements S of $\mathfrak{S}_1$ reserves the orientation of E_λ of R_8, and S of $\mathfrak{S}_2$ changes the orientation of E_λ of R_8.

As in 4-4 matrix, if we consider the space $\widetilde{R}_8$ constituted by $E_a\,(=\overset{*}{h}{}^\beta_a\widetilde{E}_\beta)$ where $\widetilde{E}_a$ are fixed and $h^\gamma_a h^\delta_\beta g_{\gamma\delta}=g_{a\beta}$, the same result is obtained, $\Lambda, \Lambda^\lambda,....$ and $\mathring{\varepsilon}_{\lambda\mu\omega\nu\rho\sigma}$ being replaced by $\widetilde{\Lambda}, \widetilde{\Lambda}^\lambda,...$: and $\widetilde{\varepsilon}_{\lambda\mu\omega\nu\rho\sigma}=\pm\sqrt{\det.|g_{a\beta}|}\cdot\widetilde{\varepsilon}_{\lambda\mu\omega\nu\rho\sigma}$ respectively, where $\widetilde{\Lambda}, \widetilde{\Lambda}^\lambda,....$ are the coefficients of expansion of S with respect to $\widetilde{E}_\lambda$. The only difference is that

$$h^a_\lambda h^\beta_\mu\left(\widetilde{\Lambda}^\lambda\widetilde{\Lambda}^{\mu\eta}-\frac{3}{2}i\widetilde{\varepsilon}^{\lambda\mu}_{\cdot\cdot\tau\omega\nu\rho\sigma}\widetilde{\Lambda}^{\tau\omega\nu}\widetilde{\Lambda}^{\lambda\rho\sigma}\right)=0 \qquad (a\neq\beta)$$

instead of $\Lambda^\lambda\Lambda^{\mu7}-\frac{3}{2}i\mathring{\varepsilon}^{\lambda\mu}_{\cdot\cdot\tau\omega\nu\rho\sigma}\Lambda^{\tau\omega\nu}\Lambda^{\lambda\rho\sigma}=0$ $(\lambda\neq\mu)$ where

$$\widetilde{E}_a=h^\beta_a\mathring{E}_\beta\,,\quad \sum_{\lambda=1}^{6}h^\lambda_a h^\lambda_\beta=g_{a\beta}\,,\quad \text{and}\quad \mathring{E}_{(a}\mathring{E}_{\beta)}=\delta_{a\beta}I\,.$$

These theorems can be extended to 2^n-2^n matrix or to corresponding operator (n: positive integer) when the metric $g_{\lambda\mu}$ of $2n$-dimensional Riemannian space is given.(1) If the metric $g_{\lambda\mu}$ of $(2n+1)$-dimensional Riemannian space is given, $\gamma_1, \gamma_2, ..., \gamma_{2n}$ and $i^n\gamma_1\gamma_2....\gamma_{2n}=\gamma_{2n+1}$ play the rôles of $\gamma_1, \gamma_2, \gamma_3$, and γ_4 in 4-4 matrix, and consequently the problem is reduced to the case when the metric of even-number-dimensional Riemannian space is given.(2)

Case (2) is treated in the next section.

§ 10. Classification of 8-8 matrix S by the orientations of 4-dimensional vector space.

In § 9 we have seen that any 8-8 matrices E_i satisfying $E_{(i}E_{j)}=g_{ij}I$ are given by

(1) The general question for 2^n-2^n matrix is treated directly in the last section.
(2) Brauer and Weyl, loc. cit.

$$E_c = S^{-1} h_i^j \mathring{E}_j S \qquad (i, j = 1, 2, \ldots, 4), \tag{10.1}$$

where $\sum_{r=1}^{4} h_i^r h_j^r = g_{ij}$ and $\mathring{E}_{(i}\mathring{E}_{j)} = \delta_{ij} I$. In the following section we use the letters of indices as follows:

$$i, j, \ldots, r, s, \ldots = 1, 2, 3, 4,$$
$$x, y, \ldots = 5, 6,$$
$$\alpha, \beta, \ldots, \lambda, \mu, \ldots = 1, 2, \ldots, 6.$$

As in the case of 4-4 matrices, we consider the space Γ_4' constituted by all $E_i\ (= h_i^r \mathring{E}_r)$ where $\mathring{E}_i$ are fixed and h_j^i may take all the values satisfying the relations $\sum_{r=1}^{4} h_i^r h_j^r = g_{ij}$, and we investigate, the properties of S such that $'E_i = S^{-1} E_i S$ for any given E_i and $'E_i$ where $E_i = h_i^j \mathring{E}_j$ and $'E_i = k_i^j{}' \mathring{E}_j$. If we put $H \equiv \| h_j^i \|$ and $K \equiv \| k_j^i \|$, then $H^* H = K^* K = G\ (= \| g_{ij} \|)$, and $HK^{-1} \equiv A\ (= \| a_{ij} \|)$ becomes an orthogonal matrix. Here we assume that A is real, i. e. as in the case of 4-4 matrices we consider any one sub-space, say R_4'. We say that E_i and $'E_i$ have the same, or opposite, orientations according as A is proper or improper. From $'E_i = k_i^j \mathring{E}_j = S^{-1}\, 'E_i S = S^{-1} h_i^j \mathring{E}_j S$, we have

$$\mathring{E}_i = S^{-1} a_i^j \mathring{E}_j S \tag{10.2}$$

If we take $\mathring{E}_\lambda$ $(\lambda = 1, 2, \ldots, 6)$ and $i\mathring{E}_7 = \mathring{E}_1 \mathring{E}_2 \ldots \mathring{E}_6$, and expand S as follows:

$$S = \Lambda I + \Lambda^\lambda \mathring{E}_\lambda + \Lambda^7 \mathring{E}_7 + \Lambda^{\mu\lambda} \mathring{E}_\mu \mathring{E}_\nu + \Lambda^{\lambda 7} \mathring{E}_\lambda \mathring{E}_7 + \Lambda^{\lambda\mu\nu} \mathring{E}_\lambda \mathring{E}_\mu \mathring{E}_\nu + \Lambda^{\lambda\mu 7} \mathring{E}_\lambda \mathring{E}_\mu \mathring{E}_7, \tag{10.3}$$

where $\Lambda^{\lambda\mu} = -\Lambda^{\mu\lambda}$, $\Lambda^{\lambda\mu 7} = -\Lambda^{\mu\lambda 7}$ and $\Lambda^{\lambda\mu\nu} = \Lambda^{[\lambda\mu\nu]}$; then, substituting (10.3) into (10.2), and making use of the identities:

$$\mathring{E}_\lambda \mathring{E}_\mu \mathring{E}_\nu \mathring{E}_\omega = -\frac{i}{2} \epsilon_{\lambda\mu\nu\omega}{}^{\cdot\cdot\cdot\cdot\rho\sigma} \mathring{E}_\rho \mathring{E}_\sigma \mathring{E}_7, \qquad (\lambda, \mu, \nu, \omega \neq)$$

$$\mathring{E}_\lambda \mathring{E}_\mu \mathring{E}_\nu \mathring{E}_7 = \frac{i}{6} \mathring{\epsilon}_{\lambda\mu\nu}{}^{\cdot\cdot\cdot\rho\sigma\tau} \mathring{E}_\rho \mathring{E}_\sigma \mathring{E}_\tau, \qquad (\lambda, \mu, \nu \neq)$$

we have:

$$\left.\begin{array}{ll}
\text{(i)} & (a_i^k - \delta_i^k)\Lambda_k = 0, \\
\text{(ii)} & (a_i^\lambda - \delta_i^\lambda)\Lambda + 2(a_i^k + \delta_i^k)\Lambda_k^{\cdot\lambda} = 0, \\
\text{(iii)} & (a_i^k + \delta_i^k)\Lambda_k^{\cdot 7} = 0, \\
\text{(iv)} & (a_i^{[\lambda} + \delta_i^{[\lambda})\Lambda^{\mu]} + 3(a_i^k - \delta_i^k)\Lambda_k^{\cdot\lambda\mu} = 0, \\
\text{(v)} & (a_i^\lambda + \delta_i^\lambda)\Lambda^7 + 2(a_i^k - \delta_i^k)\Lambda_k^{\cdot\lambda 7} = 0, \\
\text{(vi)} & (a_i^{[\lambda} - \delta_i^{[\lambda})\Lambda^{\mu\nu]} + \dfrac{i}{6}(a_i^k + d_i^k)\Lambda^{\xi\rho 7} \mathring{\epsilon}_{k\xi\rho}{}^{\cdot\cdot\cdot\lambda\mu\nu} = 0, \\
\text{(vii)} & (a_i^{[\lambda} - \delta_i^{[\lambda})\Lambda^{\mu]7} - \dfrac{i}{2}(a_i^k + \delta_i^k)\Lambda^{\xi\rho\sigma} \mathring{\epsilon}_{k\xi\rho\sigma}{}^{\cdot\cdot\cdot\cdot\lambda\mu} = 0,
\end{array}\right\} \tag{10.4}$$

where $a_i^x=0$.

When $A=\|a_j^i\|$ is proper, since A is a real matrix, by Lemma 2 there exists a real orthogonal matrix $T\equiv\|t_j^i\|$ such that

$$T^{-1}AT=\widetilde{A}\equiv\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\dot{+}\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}.$$

Put $\widetilde{A}=\|\widetilde{a}_j^i\|$ and $t_i^j\mathring{\hat{E}}_j={}'\mathring{\hat{E}}_{ij}$; then ${}'\mathring{\hat{E}}_{(i}\mathring{\hat{E}}_{j)}=\delta_{ij}I$. Substituting this into (10.2), we have

$$'\mathring{\hat{E}}_i=S^{-1}\widetilde{a}_i^j\,'\mathring{\hat{E}}_jS. \tag{10.5}$$

If we put $i'\mathring{\hat{E}}_7={}'\mathring{\hat{E}}_1{}'\mathring{\hat{E}}_2\ldots\ldots{}'\mathring{\hat{E}}_6$ then ${}'\mathring{\hat{E}}_7=\epsilon\mathring{\hat{E}}_7$ where $\epsilon=\text{det.}\,|T|$. Therefore ${}'\mathring{\hat{E}}_a$ and ${}'\mathring{\hat{E}}_7$ can be regarded as the basis, and S can be expanded in ${}'\mathring{\hat{E}}_i$'s. We shall use the letters of indices as follows:

$$\begin{aligned} &i, j, \ldots\ldots, r, s, \ldots\ldots &&=1, 2, 3, 4,\\ &a, b, &&=1, 2,\\ &p, q, &&=3, 4,\\ &x, y, &&=5, 6.\end{aligned}$$

In what follows for simplicity dropping ripples and dashes, we write a_j^i and $\Lambda, \Lambda^\lambda\ldots\ldots$ instead of $\widetilde{a}_j^i$ and ${}'\Lambda, {}'\Lambda^\lambda, \ldots\ldots$ etc. (${}'\Lambda, {}'\Lambda^\lambda, \ldots\ldots$ are the coefficients of expansion of S with respect to ${}'\mathring{\hat{E}}_i$'s). Then, as with 4-4 matrices, as the general solutions of (10.5) we have:

$$\left.\begin{aligned}
&\Lambda:\ \text{arbitrary},\\
&\Lambda^i=0;\ \Lambda^x, \Lambda^7:\ \text{arbitrary},\\
&\Lambda^{ix}=\Lambda^{i7}=0;\quad \Lambda^{ap}=0,\quad \Lambda^{12}=\frac{1}{2}\tan\frac{\theta}{2}\Lambda,\quad \Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda,\\
&\Lambda^{56}=-\frac{i}{2}\cot\frac{\theta}{2}\cot\frac{\varphi}{2}\Lambda^7,\ \Lambda^{57}=i\cot\frac{\theta}{2}\cot\frac{\varphi}{2}\Lambda^6,\ \Lambda^{67}=-i\cot\frac{\theta}{2}\cot\frac{\varphi}{2}\Lambda^5,\\
&\Lambda^{ijk}=\Lambda^{ixy}=\Lambda^{ix7}=0;\quad \Lambda^{apx}=\Lambda^{ap7}=0,\\
&\qquad\Lambda^{125}=-\frac{1}{6}\cot\frac{\theta}{2}\Lambda^5;\quad \Lambda^{345}=-\frac{1}{6}\cot\frac{\varphi}{2}\Lambda^5,\\
&\qquad\Lambda^{126}=-\frac{1}{6}\cot\frac{\theta}{2}\Lambda^6;\quad \Lambda^{346}=-\frac{1}{6}\cot\frac{\varphi}{2}\Lambda^6,\\
&\qquad\Lambda^{127}=-\frac{1}{2}\cot\frac{\theta}{2}\Lambda^7;\quad \Lambda^{347}=-\frac{1}{2}\cot\frac{\varphi}{2}\Lambda^7,\\
&\qquad\qquad\Lambda^{457}=-\frac{i}{2}\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\Lambda.
\end{aligned}\right\} \tag{10.6}$$

(Calculations are found in Note.)

So that S is given as follows:

$$S=S_0\left(I+\tan\frac{\theta}{2}\mathring{E}_1\mathring{E}_2+\tan\frac{\varphi}{2}\mathring{E}_3\mathring{E}_4-i\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\mathring{E}_5\mathring{E}_6\mathring{E}_7\right) \quad (10.7)$$

where

$$S_0=\Lambda I+2\Lambda^{56}\mathring{E}_5\mathring{E}_6+\Lambda^{57}\mathring{E}_5\mathring{E}_7+\Lambda^{67}\mathring{E}_6\mathring{E}_7. \quad (10.8)$$

Here $S_0^{-1}\mathring{E}_iS_0=\mathring{E}_i$, provided that there exists S_0^{-1}, and conversely, by putting $\theta=\varphi=0$ in (10.6),(1) S_0 such as $S_0^{-1}\mathring{E}_iS_0=\mathring{E}_i$ is obtained in the form (10.8). But, to avoid the indefiniteness of

$$I+\tan\frac{\theta}{2}\mathring{E}_1\mathring{E}_2+\tan\frac{\varphi}{2}\mathring{E}_3\mathring{E}_4-i\tan\frac{\theta}{2}\tan\frac{\varphi}{2}\mathring{E}_5\mathring{E}_6\mathring{E}_7$$

when θ or $\varphi\equiv\pi$ (mod. 2π), we rewrite (10.7) as follows:

$$S=S_0\cdot\alpha\left(I+\tan\frac{\theta}{2}\mathring{E}_1\mathring{E}_2+\tan\frac{\varphi}{2}\mathring{E}_3\mathring{E}_4-i\tan\frac{\theta}{2}\tan\frac{\theta}{2}\mathring{E}_5\mathring{E}_6\mathring{E}_7\right)$$

$$=S_0(\lambda I+2\lambda^{12}\mathring{E}_1\mathring{E}_2+2\lambda^{34}\mathring{E}_3\mathring{E}_4+\lambda^{567}\mathring{E}_5\mathring{E}_6\mathring{E}_8)^{(2)} \quad (10.9)$$

where $\lambda=\alpha$, $\lambda^{12}=\frac{\alpha}{2}\tan\frac{\theta}{2}$, $\lambda^{34}=\frac{\alpha}{2}\tan\frac{\varphi}{2}$, $\lambda^{567}=-i\alpha\tan\frac{\theta}{2}\tan\frac{\varphi}{2}$. Thus for any given E_i and $'E_j$ of $'R_4$, S such as $'E_i=S^{-1}E_iS$ is determined uniquely except for an element S_0. But $\Lambda, \Lambda^{\lambda}, \ldots, \lambda, \lambda^{12}, \ldots$ etc. here evaluated are in the dash-system; therefore, returning to the undashed system, we have:

$$S=S_0(\lambda I+\lambda^{ij}\mathring{E}_i\mathring{E}_j+\lambda^{567}\mathring{E}_5\mathring{E}_6\mathring{E}_7) \quad (10.10)$$

where $S_0=\Lambda I+2\Lambda^{56}\mathring{E}_5\mathring{E}_6+\Lambda^{57}\mathring{E}_5\mathring{E}_7+\Lambda^{67}\mathring{E}_6\mathring{E}_7$, and λ, λ^{ij}, $i\lambda^{567}$ are real except for a common factor. In the dash-system, eliminating θ, φ, we have $'\lambda'\lambda^{567}+4i'\lambda^{12}{}'\lambda^{34}=0$; therefore in the undashed system we have

$$\lambda\lambda^{567}+\frac{i}{2}\mathring{e}_{ijkl}\lambda^{ij}\lambda^{kl}=0. \quad (10.11)$$

As with 4-4 matrices, any S which transforms E_i to E'_i ($\in R'_4$) of the opposite orientation is expressed as a product a special S which interchanges the orientations, and general S which preserves the orientations; thus, for example, S is expressed as $S=S'\mathring{E}_4$ where S' mediates E_i's of the same orientations. Thus any S which changes the orientations has the following form:

$$S=S_0(\Lambda^i\mathring{E}_i+\Lambda^{ijk}\mathring{E}_i\mathring{E}_j\mathring{E}_k). \quad (10.12)$$

and Λ^i, Λ^{ijk} are real except for a common factor. Now,

$$\Lambda^i\mathring{E}_i+\Lambda^{ijk}-\mathring{E}_i\mathring{E}_j\mathring{E}_k=(\Lambda^4+\Lambda^a\mathring{E}_a\mathring{E}_4+3\Lambda^{4ab}\mathring{E}_a\mathring{E}_b-6i\Lambda^{123}\mathring{E}_5\mathring{E}_6\mathring{E}_7)\mathring{E}_4.$$

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(1) Cf. Note (N. 4).

(2) The fact that (10.6) is a general solution is destroyed when S is factorized as (10.7); but if we factorize S as (10.9), the generality of solution (10.6) is not destroyed. Cf. § 3.

Therefore, the condition[1] for S of the form $S_2=\Lambda^i\mathring{E}_i+\Lambda^{ijk}\mathring{E}_i\mathring{E}_j\mathring{E}_k$ to mediates E_i's of R_4' is obtained by substituting the equation above into (10.11) as follows:

$$\mathring{e}_{ijkl}\Lambda^i\Lambda^{jkl}=0\,. \qquad (10.13)$$

Next, to find the condition for the existence of the inverse of S of the form (10.10) or (10.12), put

$$S_0\equiv \Lambda I+\Lambda^{xy}\mathring{E}_x\mathring{E}_y, \qquad (x, y=5, 6, 7)$$
$$S_1\equiv \lambda I+\lambda^{ij}\mathring{E}_i\mathring{E}_j+\lambda^{567}\mathring{E}_5\mathring{E}_6\mathring{E}_7\,,$$
$$S_2\equiv \Lambda^i\mathring{E}_i+\Lambda^{ijk}\mathring{E}_i\mathring{E}_j\mathring{E}_k\,;$$

then

$$S_0(\Lambda I-\Lambda^{xy}\mathring{E}_x\mathring{E}_y)=\{(\Lambda)^2+2\Lambda^{xy}\}I\,, \qquad (10.14)$$

$$S_1(\lambda I-\lambda^{ij}\mathring{E}_i\mathring{E}_j+\lambda^{567}\mathring{E}_5\mathring{E}_6\mathring{E}_7)=\{(\lambda)^2-(\lambda^{567})^2+2\lambda^{ij}\lambda_{ij}A_{xy}\}I + \{2\lambda\lambda^{567}+i\mathring{e}_{ijhl}\lambda^{ij}\lambda^{kl}\}\mathring{E}_5\mathring{E}_6\mathring{E}_7 \qquad (10.15)$$

$$S_2(\Lambda^i\mathring{E}_i-\Lambda^{ijk}\mathring{E}_i\mathring{E}_j\mathring{E}_k)=\{\Lambda^i\Lambda_i+6\Lambda^{ijk}\Lambda_{ijk}\}I+2i\mathring{e}_{ijkl}\Lambda^i\Lambda^{kl}\mathring{E}_5\mathring{E}_6\mathring{E}_7\,. \qquad (10.16)$$

From (10.14) for S_0 such that $(\Lambda)^2+2\Lambda^{xy}\Lambda_{xy}=0$, we have $S_0^{-1}=\frac{1}{(\Lambda)^2+2\Lambda^{xy}\Lambda_{xy}}\times(\Lambda I-\Lambda^{xy}\mathring{E}_x\mathring{E}_y)$. The inverse of S_1 which satisfies relation (10.11) and whose coefficients $\lambda, \lambda^{ij}, i\lambda^{567}$ are real except for a common factor, from (10.15) is obtained as $S_1^{-1}=\frac{1}{\lambda^2-(\lambda^{567})^2+2\lambda^{ij}\lambda_{ij}}[\lambda]+\lambda^{567}\mathring{E}_5\mathring{E}_6\mathring{E}_7-\lambda^{ij}\mathring{E}_i\mathring{E}_j]$, $S=0$ being excluded. Next, for S_2 which satisfies the relation (10.12), and whose coefficients Λ^i, Λ^{ijk} are real except for a common factor, from (10.15) we have $S_2^{-1}=\frac{1}{\Lambda^i\Lambda_i+6\Lambda^{ijk}\Lambda_{ijk}}(\Lambda^i\mathring{E}_i-\Lambda^{ijk}\mathring{E}_i\mathring{E}_j\mathring{E}_k)$, $S=0$ being excluded. And $(S_0S_1)^{-1}$ or $(S_0S_2)^{-1}$ is equal to $S_1^{-1}S_0^{-1}$ or $S_2^{-1}S_0^{-1}$ respectively, i.e. the condition that there exists the inverse of S_0S_1 or S_0S_2 is that there exist the inverses of both S_0 and S_1 or both S_0 and S_2.

Thus we have the following theorem.

Theorem 3. *The set $\mathfrak{S}$ of operators S which leave the space R_4' invariant consists of two parts $\mathfrak{S}_1$ and $\mathfrak{S}_2$. $\mathfrak{S}_1$ consists of operators S which have the form S_0S_1 and consists of operators S which have the form S_0S_1, where*

$$S_0=\Lambda I+\Lambda^{xy}\mathring{E}_x\mathring{E}_y \quad (x, y=5, 6, 7)\,, \quad (\Lambda)^2+2\Lambda^{xy}\Lambda_{xy}\neq 0\,;$$

$S_1=\lambda I+\lambda^{ij}\mathring{E}_i\mathring{E}_j+\lambda^{567}\mathring{E}_5\mathring{E}_6\mathring{E}_7$, $\quad \lambda, \lambda^{ij}, \lambda^{567}$ *are real except for a common factor;*

$S_2=\Lambda^i\mathring{E}_i+\Lambda^{ijk}\mathring{E}_i\mathring{E}_i\mathring{E}_k$, $\quad \Lambda^i, \Lambda^{ijk}$ *are real except for a common factor;*

and their coefficients satisfy the following relations:

(1) Not the sufficient condition, but the necessary condition. Provided that the inverse of S exists, it is the necessary and sufficient condition.

$$\lambda\lambda^{567}+\frac{i}{2}\mathring{\epsilon}_{ijkl}\lambda^{ij}\lambda^{kl}=0\,, \qquad (S_1=0 \textit{ is excluded})$$

$$\mathring{\epsilon}_{ijkl}\Lambda^{i}\Lambda^{jkl}=0\,. \qquad (S_2=0 \textit{ is excluded})$$

Here $\hat{E}_{(i}\hat{E}_{x)}=0$, $\hat{E}_{(x}\hat{E}_{y)}I$, and $i\hat{E}_7=\hat{E}_1\hat{E}_2\ldots\hat{E}_6$. S_0 is an identical transformation in Γ_4'. The operator S of $\mathfrak{S}_1$ reserves the orientation of an element of R_4', and S of $\mathfrak{S}_2$ changes the orientation of an element of R_4'. For any two given elements E_i and $'E_i$ of R_4', S such that $S^{-1}E_iS='E_i$ is determined uniquely except for S_0.

With respect to $\hat{E}_\lambda$ given by (9.8) in which, as $\mathring{\gamma}_\lambda$ $(\lambda=1,2,\ldots,5)$, we take Dirac's matrices, the forms of S_0, S_1, S_2 are given as follows:

$$\left.\begin{array}{l} S_0=\left(\begin{array}{c|c} aI & \begin{array}{c|c} 0 & \times \\ \hline \times & 0 \end{array} \\ \hline \begin{array}{c|c} 0 & \times \\ \hline \times & 0 \end{array} & bI \end{array}\right), \text{ where } a \text{ and } b \text{ are arbitrary numbers,} \\ S_1=\left(\begin{array}{c|c} \begin{array}{c|c} \times & 0 \\ \hline 0 & \times \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \times & 0 \\ \hline 0 & \times \end{array} \end{array}\right), \qquad S_2=\left(\begin{array}{c|c} \begin{array}{c|c} 0 & \times \\ \hline \times & 0 \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} 0 & \times \\ \hline \times & 0 \end{array} \end{array}\right). \end{array}\right\} \quad (10.17)$$

Here $E_i=h_i^k\hat{E}_k=\begin{pmatrix}\gamma_i & \\ & \bar{\gamma}_i\end{pmatrix}$, where $\gamma_i=h_i^k\mathring{\gamma}_k$, $\bar{\gamma}_i=k_i^k\mathring{\gamma}_k$ and

$$k_i^a=h_i^a\,,\quad k_i^4=-h_i^4\quad a=1,2,3\quad\text{and}\quad i=1,2,3,4\,.$$

Provided that the factor S_0 is excluded, the relation between Cayley's parametrization of an orthogonal matrix and the spin-operator S,[(1)] and the meaning of condition (10.11) are quite the same as for 4-4 matrices. The set $\bar{\bar{\mathfrak{S}}}$ of S's obviously forms a group, and $\bar{\bar{\mathfrak{S}}}$ is homomorphic to the real orthogonal group $\mathfrak{G}$ as seen from (10.2). Then the quotient group $\mathfrak{S}'=\mathfrak{S}/\mathfrak{S}_0$ is isomorphic to $\mathfrak{G}$, where $\mathfrak{S}_0$ is the set of S_0's corresponding to the unit element of $\mathfrak{G}$. Now $\mathfrak{G}$ decomposes as $\mathfrak{G}=\mathfrak{H}+\tau\mathfrak{H}$, where $\mathfrak{H}$ is a proper orthogonal group; consequently its faithful representation $\mathfrak{S}'$ also decomposes as $\mathfrak{S}'=\mathfrak{T}+U\mathfrak{T}$. Comparing this with the result of Theorem 2, we see that $\mathfrak{T}$ consists of S of the form S_1, and $U\mathfrak{T}$ consists of S of the form S_2.[(2)]

(1) This is easily seen from (ii) and (vi) of (10.4).

(2) In the case of 4-4 matrices, $\mathfrak{S}_0$ consists of unit matrix with numerical multiple, and the representation $\mathfrak{S}'$ is irreducible. (Brauer and Weyl, loc. cit.) Then as Weyl shows (Weyl, The Classical Groups, p. 161), $\mathfrak{T}$ the representation of $\mathfrak{H}$, is irreducible, or breaks up into two irreducible parts of equal degree. When $\mathfrak{T}$ breaks up into two parts, by taking a suitable co-ordinate system $\mathfrak{T}$ can be written as $\mathfrak{T}=\left(\begin{array}{c|c}\times & 0\\ \hline 0 & \times\end{array}\right)$, and then U has the form $U=\left(\begin{array}{c|c}0 & \times\\ \hline \times & 0\end{array}\right)$. Then, by Theorem 1, we see that $\mathfrak{T}$ is reducible, and the form of $\mathfrak{T}$ and U is obtained by taking Dirac's matrices as $\mathring{\gamma}_i$.

If we consider the space $\tilde{R}'_4$ constituted by E_i $(=\tilde{h}_i^j\tilde{E}_j)$, where E_i are fixed and $\tilde{h}_i^r\tilde{h}_j^s g_{rs}=g_{ij}$, we see that exactly the same theorem as in R'_4 holds good, $\mathring{E}_\lambda$ and $\mathring{\epsilon}_{ijkl}$ being replaced by $\tilde{E}_\lambda$ and $\tilde{\epsilon}_{ijkl}=\det.|\mathring{H}|\cdot\epsilon_{ijkl}$ respectively.(1)

§ 11. Physical Meaning of (10.17).

In this section we shall consider the physical meaning of (10.17). Let us consider a physical system P composed of two particles A and B, each of which induces its field, and as a whole the system P also induces a field in the form of one particle. Now, according to the idea of wave geometry, if we regard the field induced by A, B, and P as being represented by matrix fields, i. e. γ_i, $\bar{\gamma}_i$, and E_i; then from the fundamental principle of wave geometry, these fields γ_i, $\bar{\gamma}_i$ and E_i determine the metrics of 4-dimensional space-time g_{ij}, $\bar{g}_{ij}$, and g'_{ij}, such that $\gamma_{(i}\gamma_{j)}=g_{ij}I$, $\bar{\gamma}_{(i}\bar{\gamma}_{j)}=\bar{g}_{ij}I$, and $E_{(i}E_{j)}=g'_{ij}I$. From the construction of the system P, it is natural to consider that, in the form of g_{ij}'s, the fields induced by A and B are the same and also coincide with the field g_{ij} induced by P, while they differ in the form of matrix.

Mathematically we might set down the statements above as follows:

$$\gamma_{(i}\gamma_{j)}=\bar{\gamma}_{(i}\bar{\gamma}_{j)}=g_{ij}I \qquad \text{for each particle, } A \text{ and } B$$

$$E_i=\begin{pmatrix}\gamma_i & \\ & \bar{\gamma}_i\end{pmatrix} \qquad \text{for the system } P.$$

Then it follows that $E_{(i}E_{j)}=g_{ij}I$. Here if we express E_i in the form $E_i=h_i^r\mathring{E}_r$ where $\mathring{E}_{(i}\mathring{E}_{j)}=\delta_{ij}I$, it must follow that $\mathring{E}_i=\begin{pmatrix}\mathring{\gamma}_i & \\ & '\mathring{\gamma}_i\end{pmatrix}$ and $\gamma_i=h_i^r\mathring{\gamma}_r$, $\bar{\gamma}_i=h_i^r{}'\mathring{\gamma}_r$, where $\mathring{\gamma}_i, '\mathring{\gamma}_i$ satisfy the relations $\mathring{\gamma}_{(i}\mathring{\gamma}_{j)}={}'\mathring{\gamma}_{(i}{}'\mathring{\gamma}_{i)}=\delta_{ij}I$ and have opposite orientations.(2) So that γ_1 and $\bar{\gamma}_i$ have opposite orientations. Then A and B can be regarded as particles that induce fields of opposite orientations; in other words A and B are in the states of opposite signs such that the field induced by particles with positive or negative charge.

Now, the general solution $\mathring{E}_\lambda$ satisfying $\mathring{E}_{(\lambda}\mathring{E}_{\mu)}=\delta_{\lambda\mu}I$ is given by (9.7), but, for simplicity, we take the form (9.8), i. e.

$$\mathring{E}_a=\begin{pmatrix}\mathring{\gamma}_a & \\ & \mathring{\gamma}_a\end{pmatrix},\quad \mathring{E}_4=\begin{pmatrix}\mathring{\gamma}_4 & \\ & -\mathring{\gamma}_4\end{pmatrix},\quad \mathring{E}_5=\begin{pmatrix}\mathring{\gamma}_5 & \\ & \mathring{\gamma}_5\end{pmatrix},\quad \mathring{E}_6=\begin{pmatrix} & \mathring{\gamma}_4\\ \mathring{\gamma}_4 & \end{pmatrix},\quad \mathring{E}_7=\begin{pmatrix} & -i\mathring{\gamma}_4\\ i\mathring{\gamma}_4 & \end{pmatrix}. \tag{11.1}$$

Next we shall consider the problem: What changes of inner construction or inner states of P can be allowed, provided that the field by P is unchanged?

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(1) $\tilde{E}_i=\mathring{h}_i^j\mathring{E}_j$, $\mathring{E}_{(i}\mathring{E}_{j)}=\delta_{ij}I$ and $\mathring{H}=\|h_j^i\|$.

(2) Here we assume that $E_5=\begin{pmatrix}\gamma_5 & \\ & \gamma_5\end{pmatrix}$. Then from the discussion in §9, $\mathring{\gamma}_i$ and $'\mathring{\gamma}_i$ must be of opposite orientations.

Now, there are two kinds of transformations of E_i, one being the coordinate transformation, the other the spin transformation. But so far as we are concerned with the orthogonal coordinate transformations with respect to g_{ij}, the coordinate transformations are reducible to spin transformations.[1] Therefore the action by which the inner construction of P is changed must be regarded as representable by a spin transformation of E_i. Thus we may set down the condition answering our problem in the following equation:

$$S^{-1}E_iS=E_i\,.$$

By (10.17), S satisfying (11.2) can be obtained as follows:

$$S=S_0=\left(\begin{array}{c|c} aI & \begin{smallmatrix} & \times \\ \times & \end{smallmatrix} \\ \hline \begin{smallmatrix} & \times \\ \times & \end{smallmatrix} & bI \end{array}\right)=U+V\,,$$

where $U=\begin{pmatrix} aI & \\ & bI \end{pmatrix}$ and $V=\left(\begin{array}{c|c} & \begin{smallmatrix} & \times \\ \times & \end{smallmatrix} \\ \hline \begin{smallmatrix} & \times \\ \times & \end{smallmatrix} & \end{array}\right)$. By U, the positions as well as the orientations of A and B remain unchangeable, but by V the positions and the orientations of A and B interchanged.

Next, by means of the spin transformation of E_i, we can also consider the transformation of orientation of P as a whole. That is to say, if we solve the equation $S^{-1}E_iS=E'_i$, we have, excluding S_0 which satisfies $S_0^{-1}E_iS_0=E_i$,

$$S=\left(\begin{array}{c|c} \begin{smallmatrix} \times & \\ & \times \end{smallmatrix} & \\ \hline & \begin{smallmatrix} \times & \\ & \times \end{smallmatrix} \end{array}\right)\equiv S_1 \quad \text{or} \quad S=\left(\begin{array}{c|c} \begin{smallmatrix} & \times \\ \times & \end{smallmatrix} & \\ \hline & \begin{smallmatrix} & \times \\ \times & \end{smallmatrix} \end{array}\right)\equiv S_2\,.$$

The transformation S_1 leaves the orientation of P, the position and orientations of A and B, unchanged; and S_2 changes the orientations of P, while it leaves the positions of A and B but changes their orientations.

As the actual model represented by such P as stated above we might mention a deuteron consisting of one proton and one neutron.

§ 12. Remarks.

The discussion above concerning the transformation: ${}'\gamma_i=S^{-1}\gamma_iS$ can be extended to the case of matrix of degree 2^n by the analogous method.

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(1) Cf. § 9.

But in the calculation of S for given γ_i and $'\gamma_i$, i.e. for given orthogonal matrix $\|a^i_j\|$, the properties of spinors have not been used explicity. If we avail ourselves of the properties of spinors, the calculation for S becomes very simple; moreover, the general case of matrix of degree 2^n can be easily treated. We explain them in the following paragraphs.

We consider the operators $\mathring{\gamma}_i$ $(i=1, 2, \ldots, n)$ satisfying

$$\mathring{\gamma}_{(i}\mathring{\gamma}_{j)}=\delta_{ij}I \tag{12.1}$$

where I denotes the unit operator. The problem, in general form, becomes to determine S such that

$$a^q_p\mathring{\gamma}_q=S\mathring{\gamma}_pS^{-1} \qquad (p, q=1, 2, \ldots r, r: \text{ even}) \tag{12.2}$$

for any real proper orthogonal matrices transformation (a^q_p). We consider such a set of S as an algebla Π consisting of all linear combinations of the 2^n units

$$l_{a_1a_2\ldots a_n}=\mathring{\gamma}_1^{a_1}\mathring{\gamma}_2^{a_2}\ldots\mathring{\gamma}_n^{a_n} \qquad (a_1, a_2, \ldots, a_n \text{ integers mod. 2}).$$

If we take any orthogonal matrix $T=\|t^i_j\|$, then

$$'\mathring{\gamma}_i=t^j_i\mathring{\gamma}_j \qquad (i, j=1, 2, \ldots, n)$$

also satisfy the same relation (12.1), and belong to the algebra Π; also, the units of Π can be constructed from $'\mathring{\gamma}_i$. Therefore we can make the reduction as in § 10.

First we shall find an element S of Π such that

$$\mathring{\gamma}_x=S\mathring{\gamma}_xS^{-1} \qquad (x=s+1, s+2, \ldots, n); \tag{12.3}$$

s is arbitrary integer such that $0\leqq s<n$. Then $\mathring{\gamma}_xS\mathring{\gamma}_x=S$. Expanding S in the bases: $S=\lambda I+{}^{i_1}\mathring{\gamma}_{i_1}+\lambda^{i_1i_2}\mathring{\gamma}_{i_1}\mathring{\gamma}_{i_2}+\cdots+\lambda^{i_1i_2\ldots i_n}\mathring{\gamma}_{i_1}\mathring{\gamma}_{i_2}\ldots\mathring{\gamma}_{i_n}$ where $\lambda^{i_1i_2}=\lambda^{[i_1i_2]}, \ldots, \lambda^{i_1\ldots i_n}=\lambda^{[i_1i_2\ldots i_n]}$, and making use of the relations: if p is even and $i_1, i_2, \ldots, i_p \neq$,

$$\mathring{\gamma}_x\mathring{\gamma}_{i_1}\mathring{\gamma}_{i_2}\ldots\mathring{\gamma}_{i_p}\mathring{\gamma}_x=\mathring{\gamma}_{i_1}\mathring{\gamma}_{i_2}\ldots\mathring{\gamma}_{i_p} \quad \text{when } \mathring{\gamma}_{i_1}\ldots\mathring{\gamma}_{i_p} \text{ does not contain } \mathring{\gamma}_x \text{ as factor,}$$

$$=-\mathring{\gamma}_{i_1}\mathring{\gamma}_{i_2}\ldots\mathring{\gamma}_{i_p} \quad \text{when } \mathring{\gamma}_{i_1}\ldots\mathring{\gamma}_{i_p} \text{ contains } \mathring{\gamma}_x \text{ as factor;}$$

if p is odd, the result is the reverse, we see that S contains only the terms of the product of even number of $\mathring{\gamma}_i$ not containing $\mathring{\gamma}_x$, and the terms of the product of odd number of $\mathring{\gamma}_i$ containing all $\mathring{\gamma}_x$. And conversely, if S is so, if satisfies the relations (12.3) provided that the inverse of S exists.

Next, to solve equation (12.2), we make the reduction as in § 10, by which $A=\|a^q_p\|$ can be set in the form;

$$A=\begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix}\dot{+}\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\dot{+}\cdots\cdot. \tag{12.4}$$

Here if we put

$$A_1\equiv\begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix}\dot{+}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\dot{+}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\dot{+}\cdots\cdot\equiv\|\overset{1}{a}{}^p_q\|$$

$$A_2\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\dot{+}\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\dot{+}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\dot{+}\cdots\cdot\equiv\|\overset{2}{a}{}^p_q\|,$$

....

then $A=A_1A_2\ldots.$ Now a special solution of (12.2) for $\|a^p_q\|$ given by (12.4) is given by $S=S_1S_2\ldots.$, where S_λ are the solutions of

$$\overset{\lambda}{a}{}^q_p\mathring{\gamma}_q=S_\lambda\mathring{\gamma}_pS_\lambda^{-1}\qquad\left((\lambda=1,2,\ldots\ldots,\frac{r}{2}\right). \tag{12.5}$$

In actual the equation (12.5) becomes as follows:

(i) $\overset{\lambda}{a}{}^q_p\mathring{\gamma}_q=S_\lambda\mathring{\gamma}_pS_\lambda^{-1}$, $\quad(p, q=2\lambda-1, 2\lambda)$

(ii) $\mathring{\gamma}_x=S_\lambda\mathring{\gamma}_xS_\lambda^{-1}$. $\quad(x=1, 2,\ldots.2\lambda-2, 2\lambda+1,\ldots\ldots, r)$. (12.6)

But, since S_λ satisfying (ii) of (12.6) can be put in the form $S=aI+b\mathring{\gamma}_{2\gamma-1}\mathring{\gamma}_{2\lambda}$, (not the necessary form), by substituting this into (i) of (12.6) we have

$$S_\lambda=a_\lambda\left(I+\tan\frac{\theta_\lambda}{2}\mathring{\gamma}_{2\lambda-1}\mathring{\gamma}_{2\lambda}\right); \tag{12.7}$$

so that (12.7) is a special solution of (12.5). Therefore, as a special solution of (12.2), we have

$$U=a\left(I+\tan\frac{\theta_1}{2}\mathring{\gamma}_1\mathring{\gamma}_2\right)\left(I+\tan\frac{\theta_2}{2}\mathring{\gamma}_3\mathring{\gamma}_4\right)(\ldots.)\ldots.(\ldots.). \tag{12.8}$$

Next, to find the general solution S of (12.2), from $a^q_p\mathring{\gamma}_q=S\mathring{\gamma}_pS^{-1}=U\mathring{\gamma}_pU^{-1}$, if we put $S_0\equiv U^{-1}S$, we have $S_0\mathring{\gamma}_pS_0^{-1}=\mathring{\gamma}_p$. Therefore the general form of S_0 is obtained as the sum of the terms of the product of even number of $\mathring{\gamma}_i$ not containing $\mathring{\gamma}_p$, and the terms of the product of odd number of $\mathring{\gamma}_i$ containing all $\mathring{\gamma}_p$. Thus the general solution S of (12.2) is given by

$$S=S_0\left(I+\tan\frac{\theta_1}{1}\mathring{\gamma}_1\mathring{\gamma}_2\right)\left(I+\tan\frac{\theta_2}{2}\mathring{\gamma}_3\mathring{\gamma}_4\right)\ldots., \tag{12.9}$$

using the commutativity of S_0 and U.

If $r=n$, $S_0=aI$. Therefore, when $n=r=4$, we obtain the result of § 3, and when $n=r=6$, we obtain the result of § 9. When $n\doteq6$, and $r=4$.

$$S_0=\alpha I+\beta\mathring{\gamma}_5\mathring{\gamma}_6+\gamma\mathring{\gamma}_1\ldots\ldots\mathring{\gamma}_4\mathring{\gamma}_5+\delta\mathring{\gamma}_1\ldots\ldots\mathring{\gamma}_4\mathring{\gamma}_6$$

which can be written in the form:

$$S_0=\alpha I+\beta^{xy}\mathring{\gamma}_x\mathring{\gamma}_y \qquad (x, y=5, 6, 7),$$

where $\mathring{\gamma}_7=\pm i\mathring{\gamma}_1\mathring{\gamma}_2\ldots\ldots\mathring{\gamma}_6$. That is to say, we obtain the result of § 10.

This method of finding S for given $A=\|a^p_q\|$ is a finite formulation of the infinitesimal method used by Cartan, etc.[1] In other word, if we take $\theta_1, \theta_2, \ldots\ldots$ as infinitesimal, we have $A=I+\tau$, where $\tau=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\theta_1\dot{+}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\theta_2\dot{+}\cdots\cdots$; and corresponding to this, from (12.9) we have: $S=S_0(I+\lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j)$, where $\sigma\equiv\|\lambda^{ij}\|=\begin{pmatrix}0&\frac{1}{4}\\-\frac{1}{4}&0\end{pmatrix}\theta_1\dot{+}\begin{pmatrix}0&\frac{1}{4}\\-\frac{1}{4}&0\end{pmatrix}\theta_2\dot{+}\cdots\cdots=\frac{1}{4}\tau$, which is the form of infinitesimal spin-transformation corresponding to an infinitesimal rotation obtained by Cartan and others. If we take into account that the finite formulation of $A=I+\tau$ may be regarded as Cayley's parametrization of the form $A=(I-T)(I+T)^{-1}$, the relation between S expressed by (12.9) and Cayley's parametrization T corresponding to the relation $\sigma=\frac{1}{4}\tau$ in the infinitesimal case is obtainable. In actuality since

$$A=\begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix}\dot{+}\begin{pmatrix}\cos\theta_2 & \sin\theta_2\\ -\sin\theta_2 & \cos\theta_2\end{pmatrix}\dot{+}\cdots\cdots,$$

$T=\begin{pmatrix}0 & -\tan\frac{\theta_1}{2}\\ \tan\frac{\theta_1}{2} & 0\end{pmatrix}\dot{+}\begin{pmatrix}0 & -\tan\frac{\theta_2}{2}\\ \tan\frac{\theta_2}{2} & 0\end{pmatrix}\dot{+}\cdots\cdots$ is obtained from $A=(I-T)(I+T)^{-1}$; and if we compare this T with $S=S_0(I+\lambda^{ij}\mathring{\gamma}_i\mathring{\gamma}_j+\cdots\cdots)$ expressed by (12.9), we have the relation: $R=\|\lambda^{ij}\|=-\frac{1}{2}T$. This is no other than the finite formulation of the relation $\sigma=\frac{1}{4}\tau$.[2] Furthermore, we can see that the relation $R=-\frac{1}{2}T$ holds good in general coordinates, though it has been obtained in a special coordinate.[3]

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(1) Cartan, Bull. Soc. Math. d. France, **41** (1913); Pauli, ibid.; Brauer & Weyl, loc. cit.

(2) If we choose Cayley's parametrization such that it coincides with τ when A is infinitesimal rotation, A is written as $A=\left(1+\frac{T}{2}\right)\left(1-\frac{T}{2}\right)^{-1}$, so that the relation $R=-\frac{1}{2}T$ is replaced by $R=\|\lambda^{ij}\|=\frac{1}{4}T$.

(3) Reversing the reduction of A into the form expressed by (12.4), we can prove that the relation $R=-\frac{1}{2}T$ holds good in special coordinates.

Note.

We consider the problem in the following cases:

Case I. $|A+I| \neq 0$, and $|A-I| \neq 0$,

Case II. $|A+I| = 0$, and $|A-I| \neq 0$,

Case III. $|A+I| \neq 0$, and $|A-I| = 0$,

Case IV. $|A+I| = 0$, and $|A-I| = 0$,

Case I. $|A+I| \neq 0$, $|A-I| \neq 0$, i.e. $\theta, \varphi \not\equiv 0, \pi \pmod{2\pi}$.

From (i) $\Lambda_k = 0$; from (iii) $\Lambda_k^{\cdot 7} = 0$.

In (ii), by putting $\lambda = x$, $\Lambda_k^{\cdot x} = 0$, and by putting $i = a$, $\lambda = p$, $\Lambda_a^{\cdot p} = 0$, and by putting $i = a$, $\lambda = b$, $\Lambda^{12} = \frac{1}{2} \tan \frac{\theta}{2} \Lambda$. Likewise, $\Lambda^{34} = \frac{1}{2} \tan \frac{\varphi}{2} \Lambda$.

In (iv) by putting $\lambda = x$, $\mu = y$, $\Lambda_k^{\cdot xy} = 0$, and by putting $\lambda = j$, $\mu = k$, $\Lambda_i^{\cdot jk} = 0$, (iv) can be written as follows:

$$(a_i^j + \delta_i^j) V^x + 6(a_i^k - \delta_i^k) \Lambda_k^{\cdot jx} = 0\,;$$

By putting $i = a$, $j = p$, $\Lambda_a^{\cdot px} = 0$, and by putting $i = a$, $j = b$, $\Lambda^{12x} = -\frac{1}{6} \cot \frac{\theta}{2} \Lambda^x$. In like manner, $\Lambda^{34x} = -\frac{1}{6} \cot \frac{\varphi}{2} \Lambda^x$.

In (v), by putting $\lambda = x$, $\Lambda^{ix7} = 0$, and by putting $i = a$, $\lambda = p$, $\Lambda^{ap7} = 0$, and by putting $i = a$, $\lambda = b$, $\Lambda^{127} = -\frac{1}{2} \cot \frac{\theta}{2} \Lambda^7$. Similarly $\Lambda^{347} = -\frac{1}{2} \cot \frac{\varphi}{2} \Lambda^7$.

In (vi), by putting $i = a$, $\lambda = b$, $\mu = x$, $\nu = y$, we have $\Lambda^{56} = i \cot \frac{\theta}{2} \Lambda^{347} = -\frac{i}{2} \cot \frac{\theta}{2} \cot \frac{\varphi}{2} \Lambda^7$; and by putting $i = a$, $\lambda = b$, $\mu = p$, $\nu = q$, $\Lambda^{567} = -\frac{i}{2} \tan \frac{\theta}{2} \tan \frac{\varphi}{2} \Lambda$.

In (vii), by putting $i = a$, $\lambda = b$, $\mu = x$, we have $\Lambda^{57} = i \cot \frac{\theta}{2} \cot \frac{\varphi}{2} \Lambda^6$, and $\Lambda^{67} = -i \cot \frac{\theta}{2} \cot \frac{\varphi}{2} \Lambda^5$. In the other cases we have identities. Thus we have established (10.6).

Case II. $|A+I| = 0$, $|A-I| \neq 0$.

Here $\theta \equiv \pi$ or $\varphi \equiv \pi \pmod{2\pi}$. First we assume $\theta \equiv \pi \pmod{2\pi}$ and $\varphi \not\equiv \pi \pmod{2\pi}$. Then, from $|A-I| \neq 0$, $\varphi \not\equiv 0 \pmod{2\pi}$. Therefore $a_a^b = -\delta_a^b$, and $|a_q^p \pm \delta_q^p| \neq 0$. From (i) $\Lambda_k = 0$; therefore from (iv) $\Lambda^{ixy} = 0$, $\Lambda^{ijk} = 0$, $\Lambda_a^{\cdot px} = 0$, $\Lambda^{12x} = 0$, and $\Lambda^{34x} = -\frac{1}{6} \cot \frac{\varphi}{2} \Lambda^x$. In (ii), by putting $i = a$, $\Lambda = 0$, and therefore $\Lambda_p^{\cdot \lambda} = 0$, i.e. $\Lambda^{ap} = \Lambda^{pq} = \Lambda^{px} = 0$. From (iii). $\Lambda_p^{\cdot 7} = 0$.

In (v), by putting $i=a$, $\Lambda_a^{\cdot\lambda 7}=0$, i.e. $\Lambda^{ab7}=\Lambda^{ap7}=\Lambda^{ax7}=0$, and by putting $\lambda=x$, $\Lambda^{ix7}=0$, and by putting $i=p$, $\Lambda^{347}=-\frac{1}{2}\cot\frac{\varphi}{2}\Lambda^7$. In (vi), by putting $i=a$, $\delta_a^{[\lambda}\Lambda^{\mu\nu]}=0$, i.e. $\Lambda^{x\lambda}=0$, i.e. $\Lambda^{ix}=\Lambda^{xy}=0$, and by putting $i=p$, $\Lambda^{567}=-i\tan\frac{\varphi}{2}\Lambda^{12}$. In (vii), by putting $i=a$, $\Lambda^{\lambda 7}=0$, and therefore $\Lambda^{\lambda\mu\nu}$ except $\Lambda^{pq\lambda}$ are zero. In the other cases we have identities. Thus we have:

$$\left.\begin{array}{l} \Lambda=0 \\ \Lambda^i=0\,;\quad \Lambda^x, \Lambda^7 \text{ are arbitrary,} \\ \Lambda^{x\lambda}=\Lambda^{p\lambda}=0\,;\quad \Lambda^{\lambda 7}=0\,;\quad \Lambda^{12} \text{ is arbitrary,} \\ \Lambda^{ijk}=\Lambda^{ixy}=\Lambda^{ix7}=0\,;\quad \Lambda^{apx}=\Lambda^{a\lambda 7}=0\,;\quad \Lambda^{12x}=0\,, \\ \qquad \Lambda^{34x}=-\frac{1}{6}\cot\frac{\varphi}{2}\Lambda^x\,,\qquad \Lambda^{347}=-\frac{1}{2}\cot\frac{\varphi}{2}\Lambda^7\,, \\ \qquad \Lambda^{567}=-i\tan\frac{\varphi}{2}\Lambda^{12}\,. \end{array}\right\} \qquad (\text{N. }1)$$

This is the same as (10.6) in which $\theta=\pi$ (mod. 2π) in the same sense as in § 3.

If φ is also congruent π mod. $\pi 2$, i.e. $a_i^\lambda=-\delta_i^\lambda$, then we have:

$$\left.\begin{array}{l} \Lambda=0 \\ \Lambda^i=0\,;\quad \Lambda^x, \Lambda^7 \text{ are arbitrary,} \\ \Lambda^{\lambda\mu}=0\,;\quad \Lambda^{\lambda 7}=0\,, \\ \Lambda^{\lambda\mu\nu}=\Lambda^{i\lambda 7}=0\,,\qquad \Lambda^{567} \text{ is arbitrary.} \end{array}\right\} \qquad (\text{N. }2)$$

This is the same as (N. 1) in which $\varphi\equiv\pi$ (mod. 2π), i.e. the same as (10.6) in which $\theta, \varphi\equiv\pi$ (mod. 2π).

Case III. $|A+I|\neq 0$, $|A-I|=0$.

Here $\theta\equiv 0$ or $\varphi\equiv 0$ (mod. 2π). First we assume that $\theta\equiv 0$ (mod. 2π) and $\varphi\not\equiv 0$ (mod. 2π). Therefore $a_a^b=\delta_a^b$, $|a_p^q\pm\delta_p^q|\neq 0$. From (i) $\Lambda_p=0$, and from (ii) $\Lambda^{a\lambda}=0$, $\Lambda^{px}=0$, and $\Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda$. From (iii) $\Lambda^{i7}=0$. In (iv), by putting $i=a$, $\Lambda^\lambda=0$, and by putting $i=p$, $\Lambda^{p\lambda\mu}=0$. In (v), by putting $i=a$, $\Lambda^7=0$; therefore $\Lambda^{p\lambda 7}=0$. In (vi), by putting $i=a$, $\Lambda^{\xi\rho 7}\mathfrak{e}_{a\xi\rho}^{\cdot\cdot\cdot\lambda\mu\nu}=0$, so that $\Lambda^{\xi\rho 7}$ except Λ^{127} are zero; and by putting $i=p$, $\lambda=x$, $\mu=y$, $\nu=q$, we have $\Lambda^{127}=-i\tan\frac{\varphi}{2}\Lambda^{56}$. In (vii), by putting $i=a$, $\Lambda^{\xi\rho\sigma}\mathfrak{e}_{a\xi\rho\sigma}^{\cdot\cdot\cdot\cdot\lambda\mu}=0$, therefore $\Lambda^{\xi\rho\sigma}$ except Λ^{abx} are zero, because $\Lambda^{p\lambda\mu}=0$. Putting $i=p$, $\lambda=q$, $\mu=x$, we have $\Lambda^{126}=\frac{i}{6}\tan\frac{\varphi}{2}\Lambda^{57}$, $\Lambda^{125}=-\frac{i}{6}\tan\frac{\varphi}{2}\Lambda^{67}$. In the other cases we have identities. Thus we have:

$$
\left.\begin{array}{l}
\Lambda \text{ is arbitrary,}\\
\Lambda^{\lambda}=0\,;\quad \Lambda^{7}=0\,,\\
\Lambda^{a\lambda}=\Lambda^{px}=0\,;\quad \Lambda^{i7}=0\,;\quad \Lambda^{34}=\frac{1}{2}\tan\frac{\varphi}{2}\Lambda,\quad \Lambda^{56},\Lambda^{57},\Lambda^{67}\text{ arbitrary,}\\
\Lambda^{p\lambda\mu}=\Lambda^{axy}=0\,;\quad \Lambda^{p\lambda7}=\Lambda^{xy7}=\Lambda^{ax7}=0\,;\quad \Lambda^{125}=-\frac{i}{6}\tan\frac{\varphi}{2}\Lambda^{67},\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Lambda^{126}=\frac{i}{6}\tan\frac{\varphi}{2}\Lambda^{57},\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Lambda^{127}=-i\tan\frac{\varphi}{2}\Lambda^{56},
\end{array}\right\}\quad (\text{N. }3)
$$

This result is the same as (10.6) in which $\theta\equiv 0$ (mod. 2π). If φ is also congruent 0 mod. 2π, i. e. $a^{i}_{j}=\delta^{i}_{j}$, we have:

$$
\left.\begin{array}{l}
\Lambda:\ \text{arbitrary,}\\
\Lambda^{\lambda}=\Lambda^{7}=0\,,\\
\Lambda^{i\lambda}=\Lambda^{i7}=0\,;\quad \Lambda^{56},\Lambda^{57},\Lambda^{67}:\ \text{arbitrary,}\\
\Lambda^{\xi\rho\sigma}=\Lambda^{\xi\rho7}=0\,.
\end{array}\right\}\quad (\text{N. }4)
$$

This result is the same as (N. 3) in which $\varphi\equiv 0$ (mod. 2π), i. e. (10.6) in which $\theta, \varphi\equiv 0$ (mod. 2π).

Case IV. $|A+I|=0,\ |A-I|=0.$

He we can assume, without loss of generality, that $\theta\equiv 0$ (mod. 2π) and $\varphi\equiv\pi$ (mod. 2π), i. e. $a^{\lambda}_{a}=\delta^{\lambda}_{a}$ and $a^{\lambda}_{p}=-\delta^{\lambda}_{p}$. Then we have:

$$
\left.\begin{array}{l}
\Lambda=0\\
\Lambda^{\lambda}=0\,;\quad \Lambda^{7}=0\,,\\
\Lambda^{a\lambda}=\Lambda^{px}=\Lambda^{xy}=0\,;\quad \Lambda^{\lambda7}=0\,,\quad \Lambda^{34}:\ \text{arbitrary,}\\
\Lambda^{p\lambda\mu}=\Lambda^{axy}=0\,;\quad \Lambda^{p\lambda7}=\Lambda^{ax7}=\Lambda^{xy7}=0\,;\quad \Lambda^{125},\Lambda^{126},\Lambda^{127}:\ \text{arbitrary.}
\end{array}\right\}\quad (\text{N. }5)
$$

This result is the same as (N. 3) in which $\varphi\equiv\pi$, i. e. the same as (10.6) in which $\theta\equiv 0$, $\varphi\equiv\pi$ (mod. 2π).

This problem was discussed at a special Seminar of Geometry and Theoretical Physics in the Hirosima University, and research into it has been carried on under the Scientific-Research Expenditure of the Department of Education.

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