Wave Geometry No. 45

Mathematical Foundation of Wave Geometry. II. A Generalization of Clifford Number.

By

Kakutaro MORINAGA.

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Summary.

In this continuation of an earlier paper[1] we shall investigate some properties of the iteration system.

In § 6, it will be proved that if an iteration system $\mathfrak{B}$ is M-divisible, there exists the unit element of $\mathfrak{B}$, and conversely; and the condition of h's being the unit element of $\mathfrak{B}$ is $h(M)\cdot M=M$.

In § 7, we shall investigate the relations between the numbers q and p (the degree and the order) of linearly independent basic elements of $\mathfrak{B}$ with the reference fields of $\{\mathfrak{K}, M\}$ and $\mathfrak{K}$ respectively, and the closed dimension n of $\mathfrak{B}$.

In § 8, we shall obtain the matrix-representation of $\mathfrak{B}$, and prove that the iteration systems are essentially classified in the two types.

In § 9, we shall investigate the relations between any two sets of the central bases a_i and a_i' $(i=1,\ldots\ldots, n)$ in an iteration system.

§ 6. Unit element.

We put forward the following two definitions.

Definition: When we can always conclude that, for any element $\beta_0 \in \mathfrak{B}$, $\beta_0=0$ from $M\beta_0=0$, $\mathfrak{B}$ is called *M-divisible.*

Definition: h is called the *unit element* of $\mathfrak{B}$ when the element h of $\mathfrak{B}$ satisfies the following relations for any element β of $\mathfrak{B}$;

$$h\beta=\beta\,, \qquad \beta h=\beta\,.$$

Then we have the following theorems concerning the unit element of $\mathfrak{B}$.

Theorem 21. *When $\mathfrak{B}$ is M-divisible, the necessary and sufficient condition for h's being the unit element of $\mathfrak{B}$ is that*

$$Mh=M$$

and as the definition for the unit element one of the relations $h\beta=\beta$ and $\beta h=\beta$ may be omitted.

(1) K. Morinaga, this Journal, **10** (1940), 215 (W.G. No. 40.) (We refer to this as **I.**)

Proof. Let h be the unit element of $\mathfrak{B}$; then

$$Mh=M \qquad (\text{for } M\in\mathfrak{B});$$

hence (6.1) is a necessary condition.

And, conversely, if $Mh=M$, then we have $hM=M$, because $M\in\mathfrak{N}$ (the zentrum of $\mathfrak{B}$). Multiplying these equations by a_i from left and right-hand side respectively, we have (M being contained in $\mathfrak{N}$)

$$M(ha_i-a_i)=0 \quad \text{and} \quad M(a_ih-a_i)=0\,.$$

Therefore, from the assumption, we have

$$ha_i=a_i \quad \text{and} \quad a_ih=a_i\,.$$

But since any element β of $\mathfrak{B}$ can be expressed in the form (cf. Theorem 7)

$$\beta=f_0(M)+\sum_{p=1}^{n}f^{i_1,\dots,i_p}(M)a_i\dots\dots a_{i_p} \qquad (1\leqq i_1<\dots\dots<i_p\leqq n),$$

we can conclude, from axiom **II**(1) (6.1), and (6.2), that

$$\beta h=f_0(M)h+\sum_{p=1}^{n}\left(f_{(M)}^{i_1,\dots,i_p}a_{i_1}\dots\dots a_{i_p}h\right)=f_0(M)+\sum_{p=1}^{n}f_{(M)}^{i_1,\dots,i_p}a_{i_1}\dots\dots a_{i_p}=\beta\,;$$

and similarly,

$$h\beta=\beta\,.$$

So, h is the unit element of $\mathfrak{B}$. Therefore (6.1) is the necessary and sufficient condition for h to be the unit element of $\mathfrak{B}$ which is M-divisible.

Next, if h satisfies the relation $h\beta=\beta$ for any element $\mathfrak{B}$, it must be true that $hM=M$ (for $M\in\mathfrak{B}$), and this is the necessary and sufficient condition for $h(M)$ to be the unit element of $\mathfrak{B}$; so that we have $\beta h=\beta$ for any element β of $\mathfrak{B}$. And similarly, if h satisfies the relation $\beta h=\beta$ for any element β of $\mathfrak{B}$, then we have $h\beta=\beta$ for any element β of $\mathfrak{B}$. Therefore, as the definition for the unit element, one of the relations $h\beta=\beta$ and $\beta h=\beta$ may be omitted. Q. E. D.

Since the unit element of $\mathfrak{B}$ belongs to the zentrum of $\mathfrak{B}$, by Theorem 10 h may be written in the form

$$h=h_{\epsilon_2}+\tilde{h}_{\epsilon_1}\overset{I}{A} \qquad (\epsilon_1+n-n_0=\text{odd number}); \tag{6.3}$$

hence (6.1) becomes

$$M(h_{\epsilon_2}+\tilde{h}_{\epsilon_1}\overset{I}{A})=M\,.$$

Theorem 22. *When $\mathfrak{B}$ is M-divisible and $n_0=0$,*(2) *the necessary and sufficient condition for $\mathfrak{B}$ having the unit element is that there exists a*

(1) I. p. 219.

(2) The condition $n_0=0$ is equivalent to $\mathring{a}_{(i}\mathring{a}_{j)}=\delta_{ij}=M$ $(i,j=1,\dots\dots,n)$.

relation at least among the basic elements (containing M) of $\mathfrak{B}$.(1)

Proof. Equation (6.1) is a relation among the basic elements; and it is not an identity, because $h \not\equiv \textcircled{I}$. Hence, the condition that there exists a relation among the basic elements is necessary for $\mathfrak{B}$ having the unit element.

Conversely, when $\mathfrak{B}$ is M-divisible and $n_0=0$ (i. e. $\mathfrak{B} \equiv \overset{I}{\mathfrak{B}}$),(2) if there is a relation among the basic elements of $\mathfrak{B}$, from Theorem 18 we know that the relation can be written in the following form:

$$\left.\begin{aligned} &\tilde{h}_2(M)\overset{I}{A}=0 \qquad \text{or } h_0(M)=0 \qquad \text{for } n=\text{even}, \\ &h_1(M)+\tilde{h}_2(M)\overset{I}{A}=0 \qquad \text{for } n=\text{odd}. \end{aligned}\right\} \tag{6.3$'$}$$

When n even, multiplying $\tilde{h}_2(M)\overset{I}{A}=0$ by $\overset{I}{A}$ we have $\tilde{h}_2M^n=0$; and from the M-divisibility of $\mathfrak{B}$ we have $\tilde{h}_2M=0$; hence, when n is even, if there is a relation among the basic elements of $\mathfrak{B}$, there exists the relation of M only.

When n is odd, using the corollary of Theorem 20 we know, from (6.3), that $\mathfrak{B}$ includes a relation of M: namely, if $\mathfrak{B}$ includes a relation among the basic elements (containing M), there always exists a relation with respect to M, i. e. $f(M)=0$. But from the construction of $\mathfrak{B}$ we know that the relation $f(M)=0$ can be written in the form: $M^p\big(M-Ml_1(M)\big)=0$, where l_1 does not include $\textcircled{I}$; hence, from the M-divisibility of $\mathfrak{B}$, we have

$$Ml_1(M)=M.$$

Consequently, from Theorem 21, $\mathfrak{B}$ has the unit element $l_1(M)$. Q. E. D.

For the case $n_0 \neq 0$, using similar reasoning to that in the proof above, we have

Corollary. *If $\mathfrak{B}$ is M-divisible, and $\mathfrak{B}$ contains a relation not including elements of $\overset{N}{\mathfrak{B}}$,*(3) *then $\mathfrak{B}$ has the unit element.*

Next, with respect to the M-divisibility of $\mathfrak{B}$ and existence of the unit element, we have the following theorem.

Theorem 23. *When $\mathfrak{B}$ is finite order and $\mathfrak{B} \equiv \overset{I}{\mathfrak{B}}$ (i. e. $n_0=0$), the conditions for existence of the unit element and the M-divisibility are equivalent.*

Proof. Let $\mathfrak{B}$ be M-divisible. Since $\mathfrak{B}$ is finite order, there exists a polynomial relation of M, i. e. $f(M)=0$. But since $\mathfrak{B}$ is M-divisible, from

(1) The basis (containing M) of $\mathfrak{B}$ means all elements such that M, a_i, $a_{[i_1}a_{i_2]}$, ... $a_{[i_1}\dots a_{i_n]}$ $(1 \leqq i_1 < i_2 < \cdots < i_n \leqq n)$; and the relation among the besic elements means their linear relation with the coefficients, polynomials of M.

(2) I. p. 223.

(3) $\overset{N}{\mathfrak{B}}$ is the ring composed of all elements $\mathring{a}_\tau$ in $[\mathring{a}_i]$ which satisfy $(\mathring{a}_\tau)^2=0$,

the corollary of Theorem 22 we know that $\mathfrak{B}$ has the unit element of the form $h(M)$.

Conversely, if there exists the unit element h, then from $\overset{I}{\mathfrak{B}} \equiv \mathfrak{B}$ (i. e. $n_0=0$) and (6.3), h can be written in the form

$$\left.\begin{aligned} &h=h_2(M) \quad \text{i. e. } h=p_2M+M\bar{h}_2(M) \quad \text{for } n=\text{even}\,; \\ &h=h_2(M)+h_1\overset{I}{A} \quad \text{i. e. } h=p_2(M)+M(\bar{h}_2+\bar{h}_1\overset{I}{A})+p_1\overset{I}{A} \quad \text{for } n=\text{odd}, \end{aligned}\right\} \tag{6.4}$$

in which $\bar{h}_1$ may contain $\textcircled{I}$ but $\bar{h}_2$ does not, and p's $\in \mathfrak{K}$. Consider a relation: $M\beta_0=0$ ($\beta_0 \in \mathfrak{B}$). When n even, by multiplying both sides of $M\beta_0=0$ by p_2 und $\bar{h}_2(M)$, and then adding the resulting relations, we have, from (6.4), $h(M)\beta_0=0$; but since h is the unit element, the relation $h\beta_0=0$ is reduced to $\beta_0=0$. Next, when n odd, by multiplying the relation $M\beta_0=0$ by p_2 and $\bar{h}_2+\bar{h}_1\overset{I}{A}$, and adding the resulting relations, we have, from (6.4),

$$(h-p_1\overset{I}{A})\beta_0=0\,, \qquad \text{i. e. } h\beta_0=p_1\overset{I}{A}\beta_0\,; \tag{6.4$'$}$$

hence $\beta_0=p_1\overset{I}{A}\beta_0$ (since h is the unit element). Multiplying this by $\overset{I}{A}$, we have

$$\overset{I}{A}\beta_0=p_1M^n\beta_0\,;$$

so that, in consequence of $M\beta_0=0$, it must follow that i. e. $\beta_0=0$ (since $\beta_0=p_1\overset{I}{A}\beta_0$ (6.4)$'$).

Accordingly $\mathfrak{B}$ is M-divisible. So we have proved the theorem.

Theorem 24. *If $h\left(=h_2(M, a_a)+h_1(M, a_a)\overset{I}{A}\right)$ is the unit element of $\mathfrak{B}$ in which $n-n_0$ is even (this means that $\mathfrak{B}$ has an even non-nilpotent central basis), then $2h_2-h_2^2$ is the unit element with the relation $h_2h_1=h_1$.*

Proof. Since the unit element $h\left(=h_2(M, a_a)+h_1(M, a_a)\overset{I}{A}\right)$ of $\mathfrak{B}$ must be an element of the zentrum ($a=n-n_0+1, \ldots\ldots, n$), we can conclude from Theorem 10 that, for $n-n_0$ even, h_1 is odd degree with respect to $\mathring{a}_a$ and $\mathring{a}_bh_1=0$. Hence

$$h_1^2=0 \quad \text{and} \quad h^2=h_2^2+2h_2h_1\overset{I}{A}\,. \tag{6.5}$$

But since h is the unit element of $\mathfrak{B}$, $h^2=h$ and $hM=M$. Therefore, from (6.5), we have

$$(h_2^2+2h_2h_1\overset{I}{A})M=M \tag{6.5$'$}$$

and

$$(h_2+h_1\overset{I}{A})M=M\,. \tag{6.6}$$

Multiplying (6.6) by h_2 from the right-hand side, and subtracting the resulting equation from (6.5)$'$ we have

$$(h_2h_1\overset{I}{A}+h_2)M=M\,;$$

and, using (6.6),

$$h_2 h_1 \overset{I}{A} M = h_1 \overset{I}{A} M .$$

Multiplying this by $\overset{I}{A}$ from the right-hand side, and using the M-divisibility of $\mathfrak{B}$ we have $h_2 h_1 = h_1$. So, (6.5)′ becomes

$$(h_2^2 + 2h_1 \overset{I}{A}) M = M .$$

From this and (6.6) we have

$$h_1 \overset{I}{A} = h_2 - h_2^2 .$$

Substituting this relation into (6.6), we get

$$(2h_2 - h_2^2) M = M ;$$

so that, from Theorem 21, we see that $2h_2 - h_2^2$ is the unit element of $\mathfrak{B}$. Q. E. D.

Remark 1. The necessary and sufficient condition for the element h of $\mathfrak{B}$ to be the unit element of a submanifold $\{P_C, P_C \mathfrak{B}\}$ of $\mathfrak{B}$, P_C being an element of zentrum of $\mathfrak{B}$, is

$$P_C h = P_C \qquad (\mathfrak{B} \text{ may be a non-}M\text{-divisible system}). \tag{6.7}$$

Proof. $P_C h = P_C$ is clearly the necessary condition for h to be the unit element of $\{P_C, P_C \mathfrak{B}\}$.

Conversely, if $P_C h = P_C$ for an element P_C belonging to the zentrum of $\mathfrak{B}$, then, for any element β_1 of $\{P_C. P_C \mathfrak{B}\}$, which is expressible in the form:

$$\beta_1 = P_C (p \mathrm{I} + \beta) \equiv (p \mathrm{I} \beta) P_C ,$$

we have

$$\beta_1 h = (p \mathrm{I} + \beta) P_C h = (p \mathrm{I} + \beta) P_C = \beta_1 ,$$

and similarly $h \beta_1 = \beta_1$. This shows that the element h satisfying the relation (6.7) is the unit element of $\{P_C, P_C \mathfrak{B}\}$.

According to this Remark, the relation (6.1) for h expresses the necessary and sufficient condition for h's being the unit element of $\{M, M\mathfrak{B}\}$, $\mathfrak{B}$ being not necessarily M-divisible.

Remark 2. If $\mathfrak{B}$ has the unit element, we may take this as the actually existing unit element instead of the ideal element I in all the expressions treated above.

§ 7. Order and dimension of $\mathfrak{B}$.

Definitions: *The number of linearly independent elements of $\mathfrak{B}$ which respect to $\mathfrak{K}$ is called the order of $\mathfrak{B}$. And if there is a relation $f(M)=0$ in reduced form (cf. Theorem 19), the degree of $f(M)$ with respect to M is called the index of $\mathfrak{B}$ with respect to M.*

Our iteration system $\mathfrak{B}$ is a hypercomplex number-system in which $\mathfrak{K}$ is the reference field and the number of its basic elements is the order.

However that may be, we can make another expression for $\mathfrak{B}$, i. e. $\mathfrak{B}$ is a number system in which $M, a_{i_1}, a_{[i_1}a_{i_2]}, \ldots, a_{[i_1}\ldots a_{i_n]}$ $(1 \leqq i_1, \ldots, i_n \leqq n)$ are its basic elements and $\mathfrak{K}' \equiv \{\mathfrak{K}, \mathfrak{M}\}$ or $\mathfrak{K}' \equiv \{\mathfrak{K}①, \mathfrak{M}\}$ is its reference field, $\mathfrak{M}$ being the principal ring with the basic element M in the field $\mathfrak{K}$; when $\mathfrak{B}$ does not contain the unit element, $\mathfrak{K}'$ is only an ideal symbolical field having no actual meaning in itself, but being defined to have meaning when operated to any element of $\mathfrak{B}$, e. g. $\big(k_1① + f_1(M)\big)\beta \equiv k_1\beta + f_1(M)\beta$. But here we cannot say that the bases: $M, a_{i_1}, a_{[i_1}a_{i_2]}, \ldots, a_{[i_1}\ldots a_{i_n]}$ are necessarily linearly independent in the reference field $\mathfrak{K}'$, and M is contained in its reference field and basic elements. However, hereafter, unless anything to the contrary is stated, we use the phrases "linearly independent" and "basis" in the latter sense.

With respect to the dependency of the basic element, we have to remark some important points. It may occur that $\mathfrak{K}'$ contains a definite element k_1' such that $k_1'\beta = 0$ for all element β of $\mathfrak{B}$. If $\mathfrak{K}'$ contains such an element k_1' $(\neq 0)$ we must therefore exclude the element k_1' from the coefficients which define the linear dependence of basic elements; for otherwise it is nonsense to consider the dependence of basic elements, because any element whatever vanishes by multiplying k_1'. Concerning the condition of existence of such k_1', we have:

Theorem 25. *When $\mathfrak{B}$ is M-divisible, the necessary and sufficient condition for the existence of a definite element k_1', satisfying the relation $k_1'\beta = 0$ for any element β of $\mathfrak{B}$, is that there exists a minimal relation of M: $f(M) = 0$.*

Proof. Let us assume that $k_1'\beta = 0$ $(k_1' \neq 0)$ for any $\beta(\in \mathfrak{B})$; then, by taking $\beta = M$, we have $k_1'M = 0$, which is a relation of M for $k_1' \in \mathfrak{K}'$; so there must exist a relation of M.

Conversely, if there exists a relation of M: $f(M) = 0$. Now, taking the minimal relation of M i. e. $f(M) = 0$, we have $f(M)\beta = 0$, i. e. $M\bar{f}(M)\beta = 0$ ($\bar{f}(M)$ must contain ①, as $\mathfrak{B}$ is M-divisible) and by the M-divisibility of $\mathfrak{B}$ we know that

$$\bar{f}(M)\beta = 0\,, \tag{7.1}$$

and, clearly, $$\bar{f} \neq 0 \qquad (\in \mathfrak{K}')\,. \qquad \text{Q. E. D.}$$

From the theorem above we know that when $\mathfrak{B}$ is M-divisible and the relation exists (in reduced form), $\bar{f}$ determined by $\bar{f}(M) \equiv f(M)$ must be excluded from the coefficients of the equation by which the linear independence of basic elements is considered.

Definition: *The number of linearly independent basic elements with* *reference field $\mathfrak{K}''$ which is composed of every element of $\mathfrak{K}'$ except for k_1', satisfying $k_1'\beta=0$ for any element of $\mathfrak{B}$, is called the degree of $\mathfrak{B}$.*

As the total number of bases $M, \alpha_{i_1}, \ldots\ldots, \alpha_{[i_1}\ldots\ldots, \alpha_{i_n]}$ is 2^n, the degree q of $\mathfrak{B}$ must satisfy the following inequality:

$$q \leqq 2^n .$$

Theorem 26. *When $\mathfrak{B}$ is M-divisible, and $n_0=0$ (i. e. $\mathfrak{B}=\overset{I}{\mathfrak{B}}$) the degree q of $\mathfrak{B}$ is given by*

$$q=2^n \qquad \textit{for } n \textit{ even}$$
$$q=2^n \quad \text{or} \quad 2^{n-1} \qquad \textit{for } n \textit{ odd},$$

and $q=2^{n-1}$ occurs when, and only when, there is a relation among the elements of $\mathfrak{B}$ which is not expressible by M only.

Proof. From Theorem 18, we know that when n even, there is no relation among the elements of $\mathfrak{B}$ including the basic elements other than M. Hence the degree of $\mathfrak{B}$ here is given by

$$q=2^n \qquad \text{(the number of bases of } \{[\alpha_i]\}) .$$

When n is odd, if there is no relation among the bases which includes the bases other than M, necessarily

$$q=2^n ;$$

and if there is a relation which is not expressible by M only, from Theorem 20 and $n_0=0$, the relation (in reduced form) is

$$g'(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0 \qquad (\tilde{g} \text{ contains } \textcircled{\scriptsize I}) \qquad (7.2)$$

If $g'\tilde{g}=0$ above, we have $\tilde{g}(M)\overset{I}{A}=0$; multiplying this by $\overset{I}{A}$, and using the M-divisibility of $\mathfrak{B}$, we have $M\tilde{g}(M)=0$; hence the relation (7.2) is reduced to $f(M)=0$. This contradicts the assumption. So, it must be true that $g'\tilde{g} \not\equiv 0$. Therefore, from (7.2), multiplying this by bases, say $\alpha_{i_{p+1}}\ldots\ldots\alpha_{i_{n-1}}$, which do not contain α_n, we have

$$\left.\begin{array}{c} \tilde{g}(M)\alpha_{i_1}\ldots\ldots\alpha_{i_p}\alpha_n M^{n-p-1}=\pm\alpha_{i_{p+1}}\ldots\ldots\alpha_{i_{n-1}}g'(M)\tilde{g}(M) \\ (i_j \neq i_k \text{ for } j \neq k). \end{array}\right\} \qquad (7.2)'$$

This shows that if there is a relation which is not expressible by M alone, every basis containing α_n is expressible linearly by the basis not containing α_n,[(1)] and the relation always contains α_n. Hence the basic elements not containing α_n are linearly independent among themselves. Therefore, if there is a relation among the bases, which is not expressible by M only, the degree q of $\mathfrak{B}$ is given by $q=2^{n-1}$. Q. E. D.

(1) The coefficient $\tilde{g}M^{n-p-1} \not\equiv 0$ since $\tilde{g}M \neq 0$.

When $\mathfrak{B}$ is M-divisible (n_0 is not necessarily equal to 0), by Theorems 19 and 20, and using again the method by which we proved Theorem 26, we can conclude that the degree q of $\mathfrak{B}$ satisfies the relation

$$2^{n-n_0} \leqq q \leqq 2^n \qquad \text{for } n-n_0=\text{even,}^{(1)}$$
$$2^{n-n_0-1} \leqq q \leqq 2^n \qquad \text{for } n-n_0=\text{odd.}$$

Remark. When $\mathfrak{B}$ is M-divisible and its index is $r+1$ with respect to M, we know from Theorems 17–20 that the order p and the degree q of $\mathfrak{B}$ satisfy the relations

$$\left.\begin{array}{ll} r2^{n-n_0} \leqq p \leqq r2^n & \text{for } n-n_0=\text{even,} \\ r2^{n-n_0-1} \leqq p \leqq r2^n & \text{for } n-n_0=\text{odd,} \end{array}\right\} \tag{7.3}$$

and

$$rq \leqq p\,.$$

Proof. Let the relation for M that gives the index $\gamma+1$ be $f(M)=0$. Then

$$M^sM,\, M^s a_i,\, \ldots\ldots,\, M^s a_{[i_1}\ldots\ldots, a_{i_{n-n_0-1}]} \tag{7.3$'$}$$
$$(1 \leqq i_1 < \cdots\cdots < i_{n-n_0-1};\ s=0,\ldots\ldots,\gamma-1\,;\ M^0 \equiv \text{Ⓘ})$$

are linearly independent in the reference field $\mathfrak{K}'$ (cf. Theorem 18), so we have

$$r2^{n-n_0-1} \leqq p\,.$$

And when n even, all the basic elements in (7.3)$'$ and $M^s a_{n-n_0}$, $M^s a_i a_{n-n_0}$, $\ldots\ldots, M^s a_{[i_1}\ldots\ldots a_{i_{n-n_0}]}$ together are linearly independent in $\mathfrak{K}$ (cf. Theorem 18); so, when $n-n_0=$even, we have

$$r2^{n-n_0} \leqq p\,.$$

But, since $\bar{f}(M)\beta=0$ $\big(\beta \in \mathfrak{B}, \bar{f}(M)M \equiv f(M)\big)$, any element of $\mathfrak{B}$ can be expressed by a linear combination of

$$M^sM,\, M^s a_i,\, \ldots\ldots,\, M^s a_{[a_1}\ldots\ldots a_{i_n]}$$
$$(s=0,\ldots\ldots,\gamma-1\,;\ M^0 \equiv \text{Ⓘ},\ 1 \leqq i_1 < \cdots\cdots < i_n \leqq n)$$

in the field $\mathfrak{K}$ (cf. Theorem 7); so necessarily $p \leqq \gamma 2^n$.

And if there are linearly independent basic elements with respect to $\mathfrak{K}''$, all resulting elements of multiplying these bases by M^s $(s=0,\ldots\ldots,\gamma-1)$ are linearly independent with respect to $\mathfrak{K}$; so we have $\gamma q \leqq p$. Thus we have proved the Remark.

Definition. *The relation among the basic elements reduced by Theorems 19 and 20:* $g'(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0$ *is called the reduced relation of* $\mathfrak{B}$,

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(1) When $n-n_0=$ even, the relation among the bases must contain $\overset{I}{A}$ and $a_a(a=n-n_0+1,\ldots\ldots n)$ in $h\,\epsilon$, since $\epsilon_2+\epsilon_1+n-n_0=$odd; i.e., it contains all a_i and a_{a_0} (a_0 is a certain number), so that $2^{n-n_0} \leqq q$.

and in the reduced relation $g'(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0$ *the degree of* $\tilde{g}(M)$ *with respect to* M *is called the degree of the relation, and* $\tilde{g}$ *is called the minimal coefficient of* $\mathfrak{V}$.

Theorem 27. *When* $\mathfrak{V}$ *is M-divisible, and there is at least a relation among the basic elements, there exists the following relation among its order* p, *degree* s *of the reduced relation and index* $r+1$ *with respect to* M *of* $\mathfrak{V}$:

$$p \geqq (r+s)2^{n-n_0} \quad \text{and} \quad r>s, \tag{7.4}$$

and $r+s$ *is then exactly the order of the zeutrum of* $\mathfrak{V}$. *And the equality hold good when, and only when,* g' *in the reduced relation:* $g'(M)g+\tilde{g}\overset{I}{A}=0$ *has the form* $g'\equiv M^{n-1}g''(M)$, *where* $g''(M)$ *does not contatn* $\textcircled{I}$.[1]

Proof. From Theorem 9 we know that the zentrum elements of $\mathfrak{V}$ are given in the form $h_2(M)+h_1(M)\overset{I}{A}$, is which $h_2(M)$ may contain r linearly indepedent elements $M, M^2, \dots, M^r$ with respect to $\mathfrak{R}$, because the index of $\mathfrak{V}$ with respect to M is $r+1$. And in the sets $\{M\}\overset{I}{A}, M^u\overset{I}{A}$ $(u=s, s+1, \dots, r)$ is linearly expressed by the elements in $\{M\}$ $\big($i. e. $M^u\overset{I}{A}\in\{h_2(M)\}\big)$, in which s is the degree of $\tilde{g}$ in consequence of $g'\tilde{g}+\tilde{g}\overset{I}{A}=0$. So that $M^v, M^t\overset{I}{A}$ $(v=1, \dots, r; t=0, \dots, s-1)$ are only elements linearly independent in $\{M\}$ and $\{M\}\overset{I}{A}$. So that $\mathfrak{V}$ contains exactly $r+s$ zentrum elements linearly independent of one another.

Next, when $\mathfrak{V}$ has the relation of degree s, from Theorem 20 and the process of the proof above, we know that the following terms are linearly independent with respect to $\mathfrak{R}$:

$$M^uM, M^ua_{i_1}\dots a_{i_j}, \dots, M^va_{i_1}\dots \sigma_{i_j}a_{n-n_0}$$
$$\begin{pmatrix}1\leqq i_1<\dots<j_j\leqq n-n_0-1; \\ u=0, \dots, r-1; v=0, \dots, s-1\end{pmatrix}^{(2)}.$$

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(1) We shall see in Theorem 31 that this relation is always satisfied by g' in the reduced relation. So that (7.4) is rewritten $p=(r+s)2^{n-1}$, $r=s$ precisely.

(2) From Theorem 25 we know that the $\bar{f}(\neq 0)$ determined by $\bar{f}\cdot M\equiv f(M)$ satisfies $\bar{f}\beta=0$ for every element of $\mathfrak{V}$, hence $\bar{f}\cdot a_{i_1}, \dots a_{i_p}=0$, so that $a_{i_1}\dots a_{i_p}$, $Ma_{i_1}\dots a_{i_p}, \dots, M^ra_{i_1}\dots a_{i_p}$ are linearly dependent with respect to $\mathfrak{R}$. On the other hand, if $a_{i_1}\dots a_{i_p}$, $\dots$, $M^{r-1}a_{i_1}\dots a_{i_p}$ are linearly dependent with respect to $\mathfrak{R}$, then $\bar{\bar{f}}a_{i_1}\dots a_{i_p}=0$, where degree $\bar{\bar{f}}$ is smaller than r, so that, multiplying by $a_{i_p}\dots a_{i_1}$, we have $\bar{\bar{f}}M^p=0$. Therefore $\bar{\bar{f}}\cdot M=0$ ($\bar{\bar{f}}$ may contain $\textcircled{I}$). Hence the index of $\mathfrak{V}$ is smaller than $r+1$. But this contradicts the assumption. Therefore $a_{i_1}\dots a_{i_p}$, $Ma_{i_1}\dots a_{i_p}, \dots M^{r-1}a_{i_1}\dots a_{i_p}$ are linearly independent; and, using Theorem 18, we know that

$$M^{v_1}, M^{v_2}a_{i_1}, \dots; M^{v_{n-n_0}}a_{i_1}\dots a_{i_{n-n_0}} (v_i=0, \dots, r-1; 1\leqq i_1<i_{n-n_0}\leqq n-n_0)$$

are linearly independent.

So, the oder p of $\mathfrak{B}$ satisfies the relation

$$p \geqq r2^{n-n_0-1}+s2^{n-n_0-1} \quad \text{i. e.} \quad p \geqq (r+s)2^{n-n_0-1}.$$

And since, from $g'\tilde{g}+\tilde{g}\overset{I}{A}=0$, we know that

$$M^{\tau}\tilde{g}a_{i_1}\dots\dots a_{i_{n-\tau-1}}a_{n-n_0}=\pm a_{i_{n-\tau}}\dots\dots a_{i_{n-n_0-1}}g'\tilde{g} \quad (\tau=0, 1, \dots\dots, n-n_0-1),$$

the equality above holds good when, and only when $g'(M)$ contains M^{n-n_0-1} as a factor. And, from the reduction in Theorem 20, we know that $r \geqq s$. So we have proved the theorem.

Definition. *A set of a_i's $(i=1, \dots\dots, n)$ is called the central basis when* $\mathfrak{B}=\{[a_i]\}$ $(i=1, \dots\dots, n)$.

There will be many sets of central bases defining the same iteration system. Concerning the numbers of the linearly independent elements in different central bases $\{[a_i]\}$, $\{[a'_j]\}$ which define the same iteration system, we have following theorem.

Theorem 28. *If $n_0=0$, $\mathfrak{B}=\{[a_i]\}=\{[a'_j]\}$ $(i=1, \dots\dots, n; j=1, \dots\dots, n')$, then $n=n'$ or $n=2m$ and $n'=2m+1$ when $n \neq n'$ (say $n<n'$). In the latter case, there is a relation among basic element of $\{[a'_j]\}$ (not only M'), and if $r+1$, $r'+1$ are indices of $\mathfrak{B}$ with respect to $M(=a_ia_i)$ and $M'(=a'_ia'_i)$ respectively, then*

$$r=r'+s',$$

where s is the degree of the reduced relation of $\{M'_1a'_j, \dots\dots, a'_{[j_1}\dots\dots a_{j_{n'}]}\}$, and then the minimal coefficient g' has M'^{n-n_0-1} as a factor.

Proof. (i) When n and n' are both even. From Theorems 7 and 19, we see that the zentrum of $\mathfrak{B}$ is composed of $\{M\}$, and the number of independent (with $\mathfrak{K}$) elements of the zentrum of $\mathfrak{B}$ is the index of $\mathfrak{B}$ with respect to M. Since n and n' are both even, we know that $\{M\}=\{M'\}$ (the zentrum of $\mathfrak{B}$), and the indices $r+1$ and $r'+1$ with respect to M and M' respectively are the same, i. e. $r=r'$. But in this case, since the order of $\{[a_i]\}$ and $\{[a'_i]\}$ is given by $r2^n$ and $r'2^{n'}$ respectively, we have $r2^n=r'2^{n'}$; therefore, of necessity, $n=n'$.

(ii) When n and n' are both odd. Among the respective indices of $\mathfrak{B}$ with respect to M and M' and the order p of $\mathfrak{B}$ we have the following relations (cf. Theorems 26, 27)

$$\left.\begin{aligned} r2^{n-1} \leqq p \leqq r2^n \\ r'2^{n'-1} \leqq p \leqq r'2^{n'}. \end{aligned}\right\} \tag{7.5}$$

So, from (7.5), we have

$$r'2^{n'-1} \leqq p \leqq r2^n. \tag{7.6}$$

Now we assume $n<n'$. The relation $n<n'$ now becomes $n \leqq n'-2$, so (7.6) becomes

$$r'2^{n+1} \leqq p \leqq r2^n, \qquad (7.6)'$$

hence
$$2r' \leqq r.$$

And since, from Theorem 27, we know that

$$r \geqq s, \qquad r' \geqq s', \qquad r+s=r'+s',$$

where s and s' are respective degrees of the reduced relations, we have

$$r \leqq 2r' \quad \text{and} \quad r' \leqq 2r. \qquad (7.7)$$

By this relation and (7.6), we have

$$2r'=r. \qquad (7.7)'$$

So, from (7.6), $n \leqq n'-2$, and the equation above, we have

$$n'-2=n \quad \text{and} \quad r'2^{n'-1}=p=r2^n. \qquad (7.8)$$

Hence, applying the Remark on pape 144 to this relation, we know that r is the order of zentraum of $\mathfrak{B}$; hence, from Theorem 27,

$$r2^{n'-1} \leqq p,$$

and, from (7.7)',

$$2r'2^{n'-1} \leqq p.$$

This relation contradicts (7.8). So it must be true that $n \not< n'$. So, when n and n' are not both odd, of necessity $n=n'$.

(iii) When n=even and n'=odd. From the Remark on page 144, and Theorem 27, we have

$$p=r2^n, \qquad r2^{n'-1} \leqq p \leqq r2^{n'}, \qquad (7.9)$$

where r is the order of zentrum of $\mathfrak{B}$; so that

$$r2^{n'-1} \leqq r2^n \leqq r2^{n'}; \qquad (7.10)$$

accordingly $n'-1=n$.

Therefore, when $n_0=0$, $\mathfrak{B}=\{[a_i]\}=\{[a'_j]\}$ $(i=1,\ldots,n; j=1,\ldots,n')$ and $n \neq n'$, one of n and n' is $2m$, and the other is $2m+1$. Also, in this case, the order of $\mathfrak{B}=\{[a'_j]\}$ $(j=1,\ldots,n')$ is $r2^{n'-1}$ (cf. (7.9)), where r is the order of the zentrum; hence there is a relation of the form $g'\tilde{g}+\tilde{g}\overset{I}{A}=0$, having the minimal degree s'; and $r=r'+s'$, where r' is the index of $\mathfrak{B}$ with respect to M'. Moreover, by using Theorem 27, we know that g' has the factor $M'^{n'-1}$. Q. E. D.

As the converse of this theorem we have the next corollary.

Corollary. *When $n_0=0$, $\mathfrak{B}$ is M-divisible, $\mathfrak{B}=\{[a_i]\}=\{[a'_j]\}$ $(i=1,\ldots, n; j=1,\ldots,n')$, and there is no relation among the basic elements of $\{[a_i]\}$ and $\{[a'_j]\}$, it must follow that $n=n'$.*

When $n_0=0$ and $\mathfrak{B}$ is M-divisible, from Theorem 28 we know that

(1) We have this case when there is a relation among the basic element of $\mathfrak{B}$.

there are two possible cases: (i) The numbers of the elements in all the different sets of central bases of $\mathfrak{B}$ are equal to one another, (ii) Some of the numbers may be equal to $2m$, and the others to $2m+1$, where m is a definite number.[1] So we advance the definition that in case (i) the number of the elements in a set of central basis is called *the dimension of* $\mathfrak{B}$; and in case (ii), if the number of the elements in two groups of all the sets of central basis of $\mathfrak{B}$ are equal to $2m$ and $2m+1$, the former is called *the minimal dimension,* and the latter *the closed dimension, of* $\mathfrak{B}$.

Remark 1. Putting together the results above, from Theorems 26, 27, and 28 we have the following four possible cases for the iteration system:

(i) When the order p of $\mathfrak{B}$ is $\omega 2^n$ and the dimension is n.

(ii) " " " p " $\mathfrak{B}$ is $\omega 2^{2m}$ " " " is $2m+1$, and $\mathfrak{B}$ has the minimal dimension.

(iii) When the order p of $\mathfrak{B}$ is $\omega 2^{2m}$ and the dimension is $2m+1$, and $\mathfrak{B}$ has not the minimal dimension.

(iv) When the order p of $\mathfrak{B}$ is $\omega 2^{2m} < p < \omega 2^{2m+1}$ and the dimension is $2m+1$, and $\mathfrak{B}$ has not the minimal dimension;

where ω is the order of the zentrum, i. e. ω is the index of $\mathfrak{B}$ for (i), and ω is the sum of index r and the minimal degree s for (ii)-(iv).

Two cases, (iii) and (iv), are distinguished according as g' in the reduced relation has or has not M^{2m} as a factor (cf. the Theorem 27). But, in § 8, we shall show that actually the case n=even in (i) is included in (ii); i. e., if $\mathfrak{B}=\{[a_i]\}$ $(i=1,\dots,2m)$, there exists a set of central bases such that $\mathfrak{B}=\{[a'_j]\}$ and $[a_i]\subset[a'_j]$ $(j=1,\dots,2m+1)$, and case (iv) does not exist.

Remark 2. In cases (ii)-(iv), the extension of the central manifold, i. e. $[a_i]\subset[a'_j]$ $(i=1,\dots,n;\ j=1,\dots,n,n+1)$, does not exist even when $\{[a_i]\}\subset\{[a'_j]\}$.

Proof. We assume that $[a'_j]$ is an extension of the central manifold $\{[a_i]\}$ i. e. $[a_i]\subset[a'_j]$. Then there is an element a such that a is contained in $[a'_j]$ $(j=1,\dots,n')$, but is not contained in $[a_i]$ $(i=1,\dots,n)$. If we take a_{i+1}, which is of the form $a-\frac{1}{2}\sum_{i=1}^{n}(a+a_i a a_i)$, then $[a_k]$ is a central manifold and

$$[a_i]\subset[a_k]\subseteqq[a'_j] \qquad (i=1,\dots,n;\ k=1,\dots,n,n+1;\ j=1,\dots,n+1)\,.$$

In cases (ii)-(iv), there exists a relation of the form

$$h_{\epsilon_2}+h_{\epsilon_1}\overset{I}{A}=0 \qquad (\overset{I}{A}=a_{[1}\dots\dots a_{n]}) \tag{7.11}$$

(for, otherwise, we have $p=\omega 2^{2m+1}$). On the other hand, since relation (7.11) must be a relation among the basic elements of $\{[a_j]\}$ $(j=1,\dots,n+1)$, (7.11) must be written in the form

$$h'_{\epsilon_2}+h'_{\epsilon_1}\overset{I}{A'}=0 \qquad (\overset{I}{A'}=a_{[1}\dots\dots a_{n+1]})\,. \tag{7.12}$$

For (7.11) to be expressible in the form (7.12), it must be true that $h_{e_2}=0$, $h_{e_1}\overset{I}{A}=0$ i. e. $h_{e_2}=h_{e_1}=0$; hence (7.11) is not a relation among the basic elements of $\mathfrak{B}$, but identity. This contradicts the fact that $\mathfrak{B}=\{[a_i]\}$ contains a relation among the basic elements. Therefore $n \nless n'$, i. e., $n=n'$, so that there is no extension of the central manifold $[a_i]$; so we have proved the Remark.

Remark 3. In cases (i) and (ii), in which the central manifold is expressed by the basis of minimal dimensions, we can extend the central manifolds.(1)

Proof. If there is an iteration system $\mathfrak{B}$ in which the central manifold and the reduced relation of M are $[a_i]$ $(i=1, \ldots., n)$ and $f(M)=0$, there exists another iteration system in which $[a'_j]$ $(j=1, \ldots., n, \ldots., n')$ and $f(M')=0$ are the central manifold and minimal relation of $M'(\equiv a'_j a'_j)$,(2) and $\{[a_i]\}$ and $\{[a'_i]\}$ $(i=1, \ldots., n)$ are simply isomorphic to each other— this will be proved in § 8. Since these two iteration system, simply isomorphic to each other, are abstractly regarded as the same, in our case the extension of the central manifold of $[a_i]$ is always possible abstractly.

§ 8. Matrix-representation of $\mathfrak{B}$.

In this section we shall consider the matrix-representations of the iteration system $\mathfrak{B}$. If the matrix-representation of $\mathfrak{B}$ for which the order is infinite be possible, the order of the matrix must be infinite. So, as the first step, we shall now consider the case when the order of $\mathfrak{B}$ is finite, i. e. the dimension of $\mathfrak{B}$ is finite, say n, and its index is finite, say $r+1$. (cf. Theorem 27 and p. 144). In this section we assume that $\mathfrak{K}$ is the field of complex number.

Remark 1. As preliminary let us consider the following matrix-equation of m-th degree

$$X^m = A\,, \tag{8.1}$$

A being a given matrix and X being unknown. Let

$$f(x)\equiv \prod_{i=1}^{N}(x-a_i)^{m_i} \qquad (a_i \neq a_j \text{ for } i\neq j) \tag{8.2}$$

be the minimal form of A. Then the equation $\prod_{i=1}^{N}(x-a_i)^{m_i}=0$ must be satisfied by X^m, so the minimum form of X is a factor of $\prod_{i=1}^{N}(x^m-a_i)^{m_i}$.

(1) This holds good when iteration systems which are simply isomorphic to each other are regarded as the same; and for case (iii) to be included in case (ii), the zentrum of $\mathfrak{B}$ must be a principal ring.

(2) The existence and actual form of such an iteration system will be proved in § 8.

And the matrix z whose minimum form is a factor of $\prod_{i=1}^{N}(x^m - a_i)^{m_i}$ and includes the following form as a factor,

$$\prod_{i=1}^{N}(x - a_{i,\mu_i})^{m_i}, \tag{8.3}$$

where $$(a_{i,\mu_i})^m = a_i \qquad (1 \leqq \mu_i \leqq m),$$

satisfies equation (8.1); but the matrix whose minimum form is a factor of $\prod_{i=1}^{N}(x^m - a_i)^{m_i}$ and does not contain the following form as a factor,

$$\prod_{i=1}^{N}(x - a_{i,\nu_i})^{m_i},$$

does not satisfy (8.1). So the general independent solution of (9.1) is given by the matrix, except for the equivalent matrices whose minimum form is given by

$$\left.\begin{array}{c} \prod_{i=1}^{N}(x - a_{i,\mu_i})^{m_i} \prod_{i=1}(x - a_{i,\nu_i})^{l_1} \ldots\ldots (x - a_{i,\nu_{m-1}})^{l_{m-1}} \\ \left(\begin{array}{l} \nu_i \neq \nu_j \text{ for } i \neq j;\ \nu_i \neq \mu_i; \\ 1 \leqq \mu_i,\ \nu_i < m;\ l\text{'s} = 0, \ldots\ldots, m_i \end{array}\right) \end{array}\right\} \tag{8.4}$$

where N is the number of different roots of $f(x) = 0$; hence there are at least m^{N-1} inequivalent solutions of (8.1).[1]

Accordingly, the normal forms $\mathring{A}$ and $\mathring{X}$ of A and X (i.e. $S\mathring{X}S^{-1} = X$ and $U\mathring{A}U^{-1} = A$) are given by

$$\mathring{A} = \begin{bmatrix} \overset{1}{J}_1 & & 0 \\ & \ddots & \\ 0 & & \overset{1}{J}_N \end{bmatrix} \quad \textit{and} \quad \mathring{X} = \begin{bmatrix} \overset{2}{J}_1 & & 0 \\ & \ddots & \\ 0 & & \overset{2}{J}_N \end{bmatrix}$$

where

$$\overset{1}{J}_i = \left[\begin{array}{c|c|c} \overbrace{\begin{matrix} a_i & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 0 \\ & & & & 1 \\ & & & & a_i \end{matrix}}^{m_i} & 0 & 0 \\ \hline 0 & \overbrace{\begin{matrix} \epsilon a_i & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & 0 \\ & & & & 1 \\ & & & & \epsilon a_i \end{matrix}}^{\bar{m}_i} & 0 \\ \hline & 0 & \end{array}\right] \quad \textit{and} \quad \overset{2}{J}_i = \left[\begin{array}{c|c} \begin{matrix} a_{i,\nu_i} & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & 0 \\ & & & & 1 \\ & & & & a_{i,\nu_i} \end{matrix} & 0 \quad 0 \\ \hline 0 & \begin{matrix} \bar{\epsilon} a_{i,\mu_i} & 1 & \\ & \ddots & \\ & & \bar{\epsilon} a_{i,\mu} \end{matrix} \\ \hline & 0 \end{array}\right] \quad \left(\begin{array}{l} m_i \geqq \bar{m}_i,\ \bar{\bar{m}}_i; \\ \epsilon,\ \bar{\epsilon} = 0, 1 \end{array}\right)$$

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(1) We say that matrices transformable to one another are equivalent. And the order of the matrix is no less than $\sum_{i=1}^{N} m_i$; so there exists a matrix whose minimum form is given by $\prod_{i=1}^{N}(x - a_{i,\mu_i})^{m_i}$ $(1 \leqq \mu_j \leqq m)$, and there are m^{N-1} different forms of $\prod^{N}(x - a_{i,\mu_i})^{m_i}$ at least because a_1 may be 0.

Moreover, the order of A must be equal to $\sum m_i$ (the degree of the minimum form of A), or larger.

Remark 2. For the purpose of obtaining the matrix-representation of the iteration system $\mathfrak{B}$ which does not include relations among the basic elements, we shall consider the matrix-equation

$$X^2 = A .$$

If the minimum form of A is given by

$$f(x) \prod_{i=1}^{N} (x-a_i)^{m_i} ,$$

where $a_i \neq a_j$ for $i \neq j$, then, from Remark 1, we know that there exist at least 2^{N-1} inequivalent matrices z which satisfy $f(x^2)=0$ and whose normal form is written as follows:

$$\mathring{Z} = \begin{bmatrix} \mathring{J}_1 & & O \\ & \ddots & \\ O & & \mathring{J}_N \end{bmatrix}, \quad \textit{where} \quad \mathring{J}_i = \begin{bmatrix} \begin{matrix} \pm\sqrt{a_i} & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ O & & & \ddots & 1 \\ & & & & \pm\sqrt{a_i} \end{matrix} & O \\ O & \overbrace{\begin{matrix} \epsilon\sqrt{a_i} & 1 & 0 & \cdots & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ & & & & \epsilon\sqrt{a_i} \end{matrix}}^{\bar{m}_i} \\ & & \ddots \end{bmatrix} \quad \begin{pmatrix} \bar{m}_i \leqq m_i ; \\ \epsilon = 0, \pm 1 \end{pmatrix}$$

Remark 3. For the purpose of obtaining the matrix-representation of the iteration system $\mathfrak{B}$ which includes the reduced relation $g(M)\tilde{g}(M) + \tilde{g}(M)\overset{I}{A} = 0$ among the basic elements of $\mathfrak{B}$, we shall consider the following simultaneous matrix-equations:

$$\left. \begin{aligned} & f(z^2) = 0 \\ & g(z^2)\tilde{g}(z^2) + \tilde{g}(z^2) z^n = 0 , \end{aligned} \right\} \tag{8.5}$$

where n odd and f, g satisfy the relations

$$\left. \begin{aligned} & f(x^2) \equiv \bar{\gamma}(x^2)\tilde{g}(x^2) \\ & \left(g^2(x^2) - (-1)^{n-1} x^{2n}\right)\tilde{g}(x^2) \equiv \gamma(x^2) f(x^2) . \end{aligned} \right\} \tag{8.5$'$}$$

From assumption (8.5)$'$, for f and g we have

$$\left. \begin{aligned} \left(g^2(x^2) - (-1)^{n-1} x^{2n}\right)\tilde{g}(x^2) &= \left(g(x^2) + (-1)^{\frac{n-1}{2}} x^n\right)\left(g(x^2) - (-1)^{\frac{n-1}{2}} x^n\right)\tilde{g}(x^2) \\ &\equiv \gamma(x^2) f(x^2 \equiv \gamma(x^2) \prod_{i=1}^{N} (x^2 - a_i)^{m_i} \\ \text{and} \qquad \tilde{g}(x^2) = \prod_{\tau=1}^{N} (x^2 - a_\tau)^{m'_\tau} & \qquad (m'_\tau \leqq m_\tau , \quad \text{and} \quad \textstyle\sum m'_\tau < \sum m_\tau) . \end{aligned} \right\} \tag{8.6}$$

provided that $f(x)=\prod_{\tau=1}^{N}(x-a_\tau)^{m_\tau}$.[1]

But, since n odd, from the equation above we have[2]

$$\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)=\prod_{\nu=1}^{N}(x\pm\sqrt{a_\nu})^{m_\nu-m'_\nu}\prod(x-b_j)$$

where $b_j\neq\pm\sqrt{a_\nu}$; the double signs being taken as corresponding to those of $(x\pm\sqrt{a_\nu})^{m_\nu-m'_\nu}$.

Accordingly

$$\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x^2)\equiv\prod_{\tau=1}^{N}(x^2-a_\tau)^{m'_\tau}\prod_{\nu=1}^{N}(x\pm\sqrt{a_\nu})^{m_\nu-m'_\nu}\prod(x-b_j)\,. \tag{8.6$'$}$$

Now let us consider a matrix z whose minimum form is given by the second product $\left(\text{a factor of } f(x^2)\right)$ of (8.6)$'$ multiplied by x^ϵ in which $\epsilon=0$ for $m'_1=0$ and $\epsilon=-1$ for $m'_1\neq 0$, i. e.

$$x^\epsilon\prod_{\tau=1}^{N}(x^2-a_\tau)^{m'_\tau}\prod_{\nu=1}^{N}(x\pm\sqrt{a_\nu})^{m_\nu-m'_\nu}\,. \tag{8.7}$$

Actually, such a matrix z is given by (in the normal form)

$$\begin{pmatrix}\mathring{J}_1 & & 0\\ & \ddots & \\ 0 & & \mathring{J}_N\end{pmatrix} \quad \textit{where} \quad J_l=\left(\begin{array}{c|c|c} \begin{matrix}\mp\sqrt{a_l} & 1 & \cdots & 0\\ & \ddots & \ddots & \vdots\\ & & \ddots & 1\\ 0 & & & \mp\sqrt{a_l}\end{matrix} & 0 & 0\\ \hline 0 & \begin{matrix}\pm\sqrt{a_l} & 1 & \cdots & 0\\ & \ddots & \ddots & \vdots\\ & & \ddots & 1\\ 0 & & & \pm\sqrt{a_l}\end{matrix} & 0\\ \hline & 0 & \end{array}\right)$$

From Remark 1, we know that the matrix z above-obtained is a solution of (8.5), and $f(x)$ is the minimum form of z^2. Moreover, such z has the following property: When $\tilde{g}(x^2)=\mathring{g}(x^2)\tilde{g}'(x^2)$ and $\mathring{g}(x^2)$ is not constant, there exists the inequality:

$$\left(g'(z^2)+(-1)^{\frac{n-1}{2}}z^n\right)\tilde{g}'(z^2)\neq 0 \tag{8.8}$$

for any polynomial $g'(x)$ of x. For if the following relation is satisfied by z:

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(1) The relation $\sum m'_\tau<\sum m_\tau$ follows from the meaning of minimum coefficient of $\mathfrak{B}$.

(2) It must be true that $\left(g(x^2)+x^n\right)\left(g(x^2)-x^n\right)$ contains $\prod_{\nu=1}^{N}(x^2-a_\nu)^{m'_\nu-m_\nu}$ as a factor; so if $(x+\sqrt{a_i})$ $(a_i\neq 0)$ is not contained in $\left(g(x^2)+x^n\right)$, then $(x-\sqrt{a_i})$ is not contained in $\left(g(x^2)+x^n\right)$ because of n's being odd.

$$\left(g'(z^2)+(-1)^{\frac{n-1}{2}}z^n\right)\tilde{g}'(z^2)=0\,,$$

we know that by using the same reasoning as in obtaining (8.6) from (8.5) (because $f(x)$ is the minimum form of z^2), the following relation will be found to exist:

$$\left(g'(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}'(x^2)=\prod_{\tau=1}^{N}(x^2-a)^{m''_\tau}(x\pm\sqrt{a_\mu})^{m_\mu-m''_\mu}\prod(x-b'_j) \qquad (8.9)$$

where $\tilde{g}'(x^2)=\prod_{\tau=1}^{N}(x^2-a_\tau)^{m''_\tau}$, $m''_\tau\leqq m'_\tau$, $\sum m''_\tau<m'_\tau$ and $b'_j\neq\pm\sqrt{a_\mu}$;

the double signs responding to those of $(x\pm\sqrt{a_\mu})^{m_\mu-m''_\mu}$. Accordingly, the minimum form of z must be a factor of $\prod_{\tau=1}^{N}(x^2-a_\tau)^{m''_\tau}\prod_{\nu=1}^{N}(x\pm\sqrt{a_\nu})^{m_\nu-m''_\nu}$ (the greatest common factor of (8.7) and (8.9)), and it is an essential factor of (8.7) because of $m''_\tau\leqq m'_\tau$ and $\sum m''_\tau<m'_\tau$. This contradicts the minimum form of z as given by (8.7).

Therefore, $$\left(g'(z^2)+(-1)^{\frac{n-1}{2}}z^n\right)\tilde{g}'(z^2)\neq 0\,. \qquad \text{Q. E. D.}$$

Using these Remarks 2, 3, we shall find the matrix-representation of $\mathfrak{B}$ when $\mathfrak{B}$ is M-divisible and $n_0=0$ (i. e. $\mathfrak{B}$ does not contain null central base).

As regards the E-number systems having $2p$ and $2p+1$ (minimum) dimension respectively,(1) we are well aquainted with the fact that there exist matrix-representations of 2^p- and 2^{p+1}-th order respectively,(2) and when the dimension of $\mathfrak{B}$ is $2p$, there is a simple isomorphism between $\mathfrak{B}$ and the matrix-manifold of $2p$-th order; but when the dimension (minimal) of $\mathfrak{B}$ is $2p+1$, there exists a simple isomorphism between $\mathfrak{B}$ and a part of the matrix-manifold of 2^{p+1}-th order.

Let γ_i $(i=1,\dots,n)$ be the central bases of a matrix-representation of E-number system whose dimension is n. Using this γ_i, we shall consider a representation of central bases in the iteration system $\mathfrak{B}$ and shall obtain a matrix-representation of $\mathfrak{B}$.

Here, we shall consider the problem in two cases separately: I. There is no relation, II. There is a relation, among the basic elements of $\mathfrak{B}$.

(1) E-number system is the ring $\{[\gamma_i]\}$ with complex number as reference field characterized by $\gamma_{(i}\gamma_{j)}=\delta_{ij}I$ (I is the unit element) $(i=1,\dots,n)$. So it is a special case of the iteration system such that $M=I$ and $n_0=0$ and $\mathfrak{K}$ is a complex number system (i. e. $f(M)\equiv M^2-M$ is the minimal relation of that iteration system).

(2) When $p=1$, in fact $\gamma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\gamma_2=\begin{pmatrix}0&i\\-i&0\end{pmatrix}$ is the central basis of a two-dimensional E-number system (open). And if $\gamma_1,\dots,\gamma_{2p}$ is the matrix presentation of central bases of a $2p$-dimensional E-number system, then $\begin{pmatrix}I&0\\0&-I\end{pmatrix}$, $\begin{pmatrix}0&\gamma_i\\\gamma_i&0\end{pmatrix}$, $\begin{pmatrix}0&iI\\-iI&0\end{pmatrix}$ gives a matrix representation of central bases of a $2(p+1)$-dimensional E-number system. So by induction, we have a matrix-representation of n-dimensional E-number system.

Case I. When there is no relation among the basic elements of $\mathfrak{B}=\{[a_i]\}$ $(i=1,\ldots\ldots,n)$.

Let $r+1$ be the index of $\mathfrak{B}$ with respect of M, and let

$$f(M)\equiv\prod_{i=1}^{N}(M-a_i)^{m_i} \tag{8.10}$$

be its minimal relation of $\mathfrak{B}$ with respect to M.(1)

In the first instance, choose ν and q such that $r+1\leqq 2^{\nu}$ and $q-n=2^{\nu}$; and let $\gamma_1,\ldots\ldots,\gamma_n,\ldots\ldots,\gamma_{q+1}$ be a matrix-representation of the central bases in an E-number system whose minimal dimension is $q+1$. Then the set of $\gamma_{q+1}\gamma_{n+1},\ldots\ldots,\gamma_{q+1}\gamma_q$ becomes a central basis of the E-number system whose dimension is $q-n$. Accordingly, from (8.10) and the theory of matrix-representation of E-number systems, we know that

$$\mathfrak{B}_1=\{[\gamma_{q+1}\gamma_{n+1},\ldots\ldots,\gamma_{q+1}\gamma_q]\}$$

is simply isomorphic with the manifold of all the matrices of 2^{ν}-th order. But from (8.10) and Remarks 1, 2, we know that there exists a matrix $\bar{z}$ of $2^{\nu}(\geqq r+1)$-th order such that its minimum form is

$$\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i} \tag{8.11}$$

and $f(x)$ $\left(\equiv\prod_{i=1}^{N}(x-a_i)^{m_i}\right)$ is the minimum form of $\bar{z}^2$. And, in our case, since $f(x^2)$ has the form

$$\bar{f}(x^2)\,x^2\,, \tag{8.11$'$}$$

and $\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i}$ is a factor of $\bar{f}(x^2)x$, $\bar{z}$ satisfies (by (8.11))

$$\bar{f}(x^2)x=0\,. \tag{8.11$''$}$$

Let z be the element in $\mathfrak{B}_1$ which corresponds to $\bar{z}$ in the above-mentioned simple isomorphism between $\mathfrak{B}_1$ and the manifold of all the matrices of 2^{ν}-th order. Then we know that $\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i}$ is the minimum form of z, and z satisfies

$$\bar{f}(x^2)\,x=0\,. \tag{8.12}$$

Moreover, $f(x)$ is the minimum form of z^2.

But, since all the elements of $\mathfrak{B}_1(\equiv\{[\gamma_{q+1}\gamma_{n+1},\ldots\ldots,\gamma_{q+1}q_q]\})$ are the matrices of $2^{\frac{q+1+\epsilon}{2}}\left(\epsilon=\begin{cases}1 \text{ for } q+1=\text{odd}\\ 0 \text{ for } q+1=\text{even}\end{cases}\right)$th order, z is the matrix of $2^{\frac{q+1+\epsilon}{2}}$th order.

Next, we consider the matrices

$$\gamma_i z=\mu_i \qquad (i=1,\ldots\ldots,n)\,.$$

(1) We assume that in $f(x)\equiv\prod_{i=1}(M-a_i)^{m_i}$, $a_i\neq a_j$ for $i\neq j$ and $a_1=0$, $m_1=1$.

We can prove that *the set of matrices* μ_i $(i=1,\ldots\ldots,n)$ *forms a central basis of the iteration system* $\mathfrak{B}$, *the minimum form of* $\mu_i\mu_i=z^2=\mu$ *is* $f(x)\left(\equiv\prod_{i=1}^{N}(x-a_i)^{m_i}\right)$, *and moreover* $\{[\mu_i]\}$ *is* μ- *divisible.*

Proof. Elements of $\mathfrak{B}_1(\equiv\{[\gamma_{q+1}\gamma_{n+1},\ldots\ldots\gamma_{q+1}\gamma_q]\}$ are all even degree with respect to the representants of $\gamma_\tau(\tau=n+1,\ldots\ldots,q+1)$ of the central basis in an E-number system, and do not include the other basic elements γ_i $(i=1,\ldots\ldots,n)$ in the same central basis of the E-number system. Hence γ_i $(i=1,\ldots\ldots,n)$ is commutative with all elements of $\mathfrak{B}_1$. And since z belongs to $\mathfrak{B}_1$, z is commutative with all γ_i. So we have

$$\mu_{(i}\mu_{j)}=\gamma_{(i}z\gamma_{j)}=\gamma_{(i}\gamma_{j)}z^2=\delta_{ij}z^2\,.$$

That is, μ_i's are anticommutative to each other,(1) and $\mu_i\mu_i=\mu$ $(i=1,\ldots\ldots,n)$. Therefore, the set of μ_i forms a central basis of n-dimension, and $\{[\mu_i]\}$ is an n-dimensional iteration system. And from Remarks 1 and 2, and the method of obtaining z, we know that $f(x)\left(\equiv\prod_{i=1}^{N}(x-a_i)^{m_i}\right)$ is the minimum form of $\mu\equiv z^2$.

Next, we shall prove that $\{[\mu_i]\}$ is μ-divisible. Since any element b_0 in this iteration system can be written in the form

$$b_0=f_0(\mu)+\sum_{i_1\ldots\ldots i_p}^{n}f_0^{i_1\ldots\ldots i_p}\mu_{i_1}\ldots\ldots\mu_{i_p}\qquad(1\leqq i_1<\cdots\cdots<i_p\leqq n)\,,$$

we have, using $\mu_i=\gamma_i z$,

$$\left.\begin{array}{c}\mu b_0=\mu f_0+\sum\mu f_0^{i_1\ldots\ldots i_p}\mu_{i_1}\ldots\ldots\mu_{i_p}=z^2f_0+\sum z^{p+2}f_0^{i_1\ldots\ldots i_p}\gamma_{i_1}\ldots\ldots\gamma_{i_p}\\(1\leqq i_1<\cdots\cdots<i_p\leqq n)\,.\end{array}\right\}\quad(8.13)$$

But, since z has the expansion of the following form:

$$z=\sum_{n<j}^{q+1}g^{j_1\ldots\ldots j_r}\gamma_{j_1}\ldots\ldots\gamma_{i_r}\qquad(n+1\leqq j_1<\cdots\cdots<j_2\leqq q+1)\qquad(8.13)'$$

and there is no relation among the basic elements γ_i $(i=1,\ldots\ldots,q+1)$, because of γ_i's being minimal basis, therefore the elements $z^2f_0^{i_1\ldots\ldots i_p}z^p\gamma_{i_1}\ldots\ldots\gamma_{i_p}$ (not summed by i's) and $z^2f_0^{i'_1\ldots\ldots i'_{p'}}\gamma_{i'_1}\ldots\ldots\gamma_{i'_{p'}}$ (not summed by i''s) $(1\leqq i_1<i_2<\cdots\cdots<i_p\leqq n;\,1\leqq i'_1<\cdots\cdots<i'_{p'}\leqq n)$ have common terms in the expansion with respect to $\gamma_{\omega_1},\gamma_{\omega_1}\gamma_{\omega_2},\ldots\ldots,\gamma_{\omega_1}\ldots\ldots\gamma_{\omega_{q+1}}$ (substituting the expansion (8.13)′ for z) $(1\leqq\omega\text{'s}\leqq q+1)$ when, and only when, $p=p'$, $i_1=i'_1$,

(1) μ_i's $(i=1,\ldots\ldots,n)$ are linearly independent of one another, because of being anticommutative to one another (non nilpotent). For, if there exist a linear relation $a\mu_j=\sum_\tau a^\tau\mu_\tau$ $(\tau\neq j;\ a\neq 0)$, then the left-hand side of this relation is commutative with μ_j, but the right-hand side is anticommutative with μ_j. So $a\mu_j=0$, i.e. $a=0$. Therefore there is no linear relation.

$\ldots, i_p = i'_p$. So, if $\mu b_0 = 0$, applying to (8.13) the linear independence of the basic elements, we have

$$\left.\begin{array}{l} z^2 f_0^{i_1 \ldots i_p}(z^2) z^p \gamma_{i_1} \ldots \gamma_{i_p} = 0 \qquad \text{(not sum by } i\text{'s)} \\ z^2 f_0(z^2) = 0 \,; \end{array}\right\} \tag{8.14}$$

and, multiplying the first equation above by $\gamma_{i_p} \ldots \gamma_{i_1}$, we get

$$z^2 f_0^{i_1 \ldots i_p}(z^2) z^p = 0 \,, \tag{8.15}$$

where $f_0^{i_1 \ldots i_p}$ may contain the term Ⓘ. Since $f_0(z^2)$ belongs to $\{[\mu_i]\}$, it can be written in the form

$$f_0(z^2) \equiv f'_0(z^2) z^2 \,.$$

So, from (8.14) and (8.15),[1] we have

$$\left.\begin{array}{l} f_0(x^2) \equiv r_0(x^2) f(x^2) \\ x^2 f_0^{i_1 \ldots i_p}(x^2) \equiv r^{i_1 \ldots i_p}(x^2) f(x^2) \,, \end{array}\right\} \tag{8.16}$$

where Ⓘ may be included in r's. Therefore, from (8.11)$''$, it follows that

$$x f_0^{i_1 \ldots i_p}(x^2) \equiv r^{i_1 \ldots i_p}(x^2) \bar{f}(x^2) x \,. \tag{8.17}$$

But, since z satisfies $\bar{f}(x^2)x = 0$ and $f(x^2) = 0$ (cf. (8.11)$''$), we have, from (8.16) and (8.17),

$$f_0(z^2) = 0 \quad \text{and} \quad z f_0^{i_1 \ldots i_p}(z^2) = 0 \,. \tag{8.18}$$

Accordingly $f_0^{i_1 \ldots i_p}(z^2) \mu_{i_1} \ldots \mu_{i_p} = 0$ (because $\mu_i = z\gamma_i$), so that

$$b_0 = f_0(z^2) + \sum f_0^{i_1 \ldots i_p}(z^2) \mu_{i_1} \ldots \mu_{i_p} = 0 \,.$$

Therefore $\{[\mu_i]\}$ is μ-divisible. Q. E. D.

Since $\{[\mu_i]\}$ $(\equiv \{[\gamma_i z]\})$ is a subset of $\{[\mu_i]\}$ $(\tau = 1, \ldots, q+1)$, and, as we have mentioned, γ_τ's are matrices of $2^{\frac{q+1+\epsilon}{2}}$th order $\left(\epsilon = \begin{cases} 0 \text{ for } q+1 = \text{even} \\ 1 \text{ for } q+1 = \text{odd} \end{cases}\right)$, $\{[\mu_i]\}$ consists of the matrices of $2^{\frac{q+1+\epsilon}{2}}$th order. So we have

Theorem 29. *For the M-divisible iteration system $\mathfrak{B}$ which has no relation among the basic elements, there exists always a representation by matrices of the $2^{\frac{q+1+\epsilon}{2}}$th order where q is determined by the relations $r+1 \leqq 2^\nu$ and $2\nu = q - n$ when $r+1$ and n are the index and dimension of $\mathfrak{B}$, respectively, and ν is a certain integer satisfying the relation $r+1 \leqq 2^\nu$. (The method of securing the actual form of the representation is given in the deduction of this theorem).*

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(1) For, in $f(x) \equiv \prod_{i=1}^{N} (x - a_i)^{m_i}$, $a_i \neq a_j$ for $i \neq j$ and $a_1 = 0$, $m_1 = 1$.

Case II. When there exists a relation among the basic elements of $\mathfrak{B}$. Since there exists a relation among the basic elements, n must be odd. Now we shall find a matrix-representation of $\mathfrak{B}$ whose index is $r+1$, and whose reduced relation and minimal relation are given by

$$\left.\begin{array}{l} g(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0 \\ f(M)=0 \end{array}\right\} \qquad (8.19)$$

and

respectively.

In this case, from the corollary of Theorem 20, necessarily

$$f(x)\equiv g_0(x)\tilde{g}(x)\,.$$

Multiplying (8.19) by $\overset{I}{A}$, we have

$$\overset{I}{A}g\tilde{g}+(-1)^{\frac{n+1}{2}}\tilde{g}M^n=0\,;$$

eliminating $\overset{I}{A}$ from (8.19) and the relation above, we have

$$\left(g^2(M)-(-1)^{\frac{n-1}{2}}M^n\right)\tilde{g}(M)=0\,.$$

But, since $f(x)=0$ is the minimal form of M, it must be true that

$$\left(g^2(x)-(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x)\equiv\gamma(x)f(x)\,.$$

So, by the same reasoning as in Remark 3 (p. 29), we know that

$$\left(g(x^2)+(-1)^{\frac{n-1}{4}}x^n\right)\equiv\prod_{\tau=1}^{N}(x\pm\sqrt{a_\tau})^{m_\tau-m'_\tau}\prod(x-b_\omega)\,, \qquad (8.20)$$

where

$$f(x)=\prod_{i=1}^{N}(x-a_i)^{m_i}\,,\qquad \tilde{g}(x)=\prod_{i=1}^{N}(x-a_i)^{m'_i}\,,$$

$$m'_\tau\leqq m_\tau \quad\text{and}\quad \sum m'_\tau<\sum m_\tau\,.$$

Now let us consider the matrix $\bar{z}$ whose minimal form is

$$x^{-\eta}\prod_{\nu=1}^{N}(x\pm\sqrt{a_\nu})^{m_\nu-m'_\nu}\prod_{i=1}^{N}(x^2-a_i)^{m'_i}\,, \qquad (8.21)$$

where $\eta=0$ for $m'_1=0$, $\eta=1$ for $m'_1\neq 0$, and the minimal order of such a matrix $\bar{z}$ is $r+1+\sum_{i=1}^{N}m'_i-\eta$ (because the form (8.21) is $r+1+\sum_{i=1}^{N}m'_i-\eta$th degree[1]). As in Case I, we take the matrix-representation of the central basis of E-number system whose minimal dimension is q, i. e., $\gamma_1,\ldots,\gamma_{n-1}$, $\gamma_{n+1},\ldots,\gamma_{q+1}$. If we put

$$\gamma_n\equiv(-1)^{\frac{n-1}{2}}\gamma_1\ldots\gamma_{n-1}\,, \qquad (8.22)$$

(1) The degree of $\prod_{\nu=1}^{N}(x\pm\sqrt{a_\nu})^{m_\nu}$ is $(r+11)$.

since n is odd, γ_i's $(i=1,\dots,n-1)$ are anticommutative with γ_n, and

$$\gamma_n^2=\gamma_1^2\dots\gamma_{n-1}^2=I.$$

So we see that $[\gamma_i]$ $(i=1,\dots,n)$ is a closed central basis of $(n-1)$-dimensional E-number system.

By the same method as in Case I, we obtain the matrix z, which corresponds to $\bar{z}$ above, $\left(z\subset\{[\gamma_i]\}\ (i=1,\dots,q+1)\right)$, using the γ's above and (8.21) in place of γ's and (8.11) in Case I. (cf. p. 151).

Taking the following matrices

$$\gamma_i z=\mu_i \qquad (i=1,\dots,n)$$

and making use of Remark 3, by the similar reasoning to that in the proof of Case I, we know that $\{[\mu_i]\}$ forms an n-dimensional (closed) μ-divisible iteration system[1] and $f(x)\left(\equiv\prod_{i=1}^{N}(x-a_i)^{m_i}\right)$ is the minimal form of $\mathfrak{B}$ with respect to $\mu_i\mu_i=\mu$. Moreover, from Remark 3 we know that $g(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0$ is the reduced relation of $\{[\mu_i]\}$, and the order of matrices in $\{[\mu_i]\}$ is $2^{\frac{q+\epsilon}{2}}\left(\epsilon=\begin{cases}0 \text{ for } q=\text{even}\\ 1 \text{ for } q=\text{odd}\end{cases}\right)$, where q is determined by $\gamma+1+\sum_{i=1}^{N}m_i-\eta\leqq 2^\nu$ and $2\nu=q-n$ $\left(\eta=\begin{cases}0 \text{ for } m_1'=0\\ 1 \text{ for } m_1'\neq 0\end{cases}\right)$.[2] Therefore, combining the result above and Theorem 29, we have

Theorem 30. *There always exists a matrix-representation of the iteration system* $\mathfrak{B}$ *which is M-divisible and* $n_0=0$ *(i. e. not having nilpotent central basis), and its order of matrix is* $2^{\frac{q+\epsilon}{2}}$ $\left(\epsilon=\begin{cases}0 \text{ for } q=even\\ 1 \text{ for } q=odd\end{cases}\right)$ *where* q *is determined by the relations* $r+1+\sum_{i=1}^{N}m_i'-\eta\leqq 2^\nu$ *and* $2\nu\leqq q-n\leqq 2\nu+1$ *when* $r+1$ *is its index with respect to* M, *and* m_i' *is the multiplicity of different roots of minimal coefficient:* $\tilde{g}\equiv\prod_{i=1}^{N}(x^2-a_i)^{m_i}$ *and* $\eta=\begin{cases}0 \text{ for } m_1'=0\\ 1 \text{ for } m_1'\neq 0,\end{cases}$ *and* ν *is a certain number satisfying the first relation* $(r+1+\sum_{i=1}^{N}m_1'-$

(1) From Remark 3 we know that the matrix $\bar{z}$ satisfies $g(x^2)\tilde{g}(x^2)+g(x^2)x^n=0$, and does not satisfy the equation of lower degree than this (by 8.8). So

$$g(z^2)\tilde{g}(z^2)+\tilde{g}(z^2)z^n=0 \tag{A}$$

is the minimal relation of the matrix z whose representation is $\bar{z}$ in the isomorphism. Accordingly, using the relation $\gamma_1\dots\gamma_n=I$, we have

$$g(z^2)\tilde{g}(z^2)+\tilde{g}(z^2)z^n\gamma_1\dots\gamma_n=0,$$

and this becomes, from $z\gamma_i=\gamma_i z=\mu_i$,

$$g(z^2)\tilde{g}(z^2)+\tilde{g}\mu_1\dots\mu_n=0 \quad \text{i. e.} \quad g(\mu)\tilde{g}(\mu)+\tilde{g}(\mu)\mu_1\dots\mu_n=0.$$

Moreover, from (A) we know that this is the minimal relation of $\{[\mu_i]\}$.

(2) Cf. footnote (1) on p. 156.

$\eta \leqq 2^{\nu}$). (*The methods of obtaining the representation are also given in the deduction of this theorem*).

Next, using this theorem, we shall prove that when any function $f(x)$ and its factor $\tilde{g}(x)$ and arbitrary odd number n are given, there exists one, and only one, M-divisible iteration system except for isomorphic iteration systems, the reduced relation being given by the form $g\tilde{g}+\tilde{g}(M)\overset{I}{A}=0$ (g being a certain function of M) and the minimal relation being $f(M)=0$. In this case the reduced relation can be reduced to the linear form, i. e.,

$$M^{n-1}\tilde{g}(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0\,.$$

To show this we shall first prove

Lemma 1. *If a function $f(x)$, its factor $\tilde{g}$, and an odd number n are arbitrarily given, by choosing a suitable function $g(x)$ there exists a matrix having $f(x)$ and $\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x^2)$ as minimum forms of z^2 and z.*

Proof. From Remarks 1 and 3, we know that the necessary and sufficient condition for $f(x)$ and $\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)g(x^2)$ to be the minimum forms of z^2 and z is that there exist constants k' and b_j' and a function $\gamma(x)$ satisfying the identity

$$\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x^2)\equiv k'\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i}\prod(x-b_j')+\gamma(x)f(x^2)\,,$$

where $$f(x)\equiv k_1\prod_{i=1}^{N}(x-a_i)^{m_i}\,.$$

But, since $\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i}$ (either sign of $\pm\sqrt{a_i}$ is to be taken) is a factor of $f(x^2)$ $\left(\equiv\prod_{i=1}^{N}(x^2-a_i)^{m_i}\right)$, the identity is reduced to

$$\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x^2)\equiv k'\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i}\prod(x-b_\lambda'')\,, \tag{8.23}$$

and using the relation:

$$\tilde{g}(x^2)\equiv k_1\prod_{i=1}^{N}(x^2-a_i)^{m_i'}\,, \tag{8.23$'$}$$

we have, from (8.23), for suitable k, b_j,

$$\left(g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\equiv k\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i-m_i'}\prod(x-b_j)\,. \tag{8.24}$$

We can prove that there exist b_j and $g(x^2)$ which satisfy (8.24).

Let $2r+2$ and $2r'$ be the degree of $f(x^2)$ and $\tilde{g}(x^2)$; then then the degree of $\prod_{i=1}^{N}(x-a_i)^{m_i-m_i'}$ is $r-r'+1$, and when the degree of $\prod(x-b_j)$ is r'', the degree of the right-hand side of (8.24) is $r-r'+1+r''$. On the other hand, since the function $g(x^2)$ in the left-hand side of (8.24) is a certain polynomial in x^2, and the coefficients of odd degree in $g(x^2)+(-1)^{\frac{n-1}{2}}x^n$ with respect to x are 0 or 1 when n is odd, from which we know that there must exist relations among the constants k and b_j for the sake of the identity (8.24), and that the number of these relations is given by

$$\frac{1}{2}(r-r'+1+r''+\epsilon)$$

$$(\epsilon=0 \text{ for } r-r'+1+r''>n \quad \text{and} \quad \epsilon=1 \text{ for } r-r'+1+r''\leqq n).$$[1]

So, in general, when the number of the constants b_j is greater than $\frac{1}{2}(r-r'+1+r''+\epsilon)$ i.e. $r-r'+1+\epsilon\leqq r''$, we can determine such b_j and accordingly $g(x^2)$ through the identity (8.24). But since there exist unbounded numbers of such polynomial $\Pi(x-b_j)$ whose degree r'' satisfies $r-r'+1+\epsilon\leqq r''$, there always exist b_j and $g(x^2)$ satisfying (8.24). And for such $g(x^2)$, $f(x^2)$ is a factor of $\left(g^2(x^2)-(-1)^{n-1}x^{2n}\right)\tilde{g}(x^2)$.[2] So, from Remarks 1 and 3, there exists a matrix $\bar{z}$ such that the minimal forms of $\bar{z}$ and $\bar{z}^2$ are $x^{-\eta}\left(g(x^2)+(-1)^{n-1}x^n\right)\tilde{g}(x^2)$ and $f(x)$, respectively, where $\eta=0$ for $m_1'=0$ and $\eta=1$ for $m_1'\neq 0$. So we have proved Lemma 1.

Hence, if we obtain a function $g(x^2)$ by the above-mentioned method for a given $f(x)$, its factor $\tilde{g}(x)$, and an odd number n; then, by Theorem 30, we have an iteration system in matric form whose minimal form and minimal coefficient are $f(x)$ and $\tilde{g}(x)$ respectively. So we have

Theorem 31. *When $f(x)$, its factor $\tilde{g}(x)$, and an odd number n are given, there always exists an iteration system in matric form whose minimum form and minimal coefficient are given by $f(x)$ and $\tilde{g}(x)$; and there are at least 2^{N-1} different iteration systems, which are not transformable but isomorphic to each other, where N is the number of different roots of $f(x)=0$.*

Lemma 2. *When $f(x)$, its factor $\tilde{g}(x)$, numbers n (odd), and $\bar{\bar{n}}(<n)$ are given, for a function $\bar{g}(x)$ suitably chosen, there exists a matrix $\bar{z}$ such that the minimum forms of $\bar{z}^2$ and $\bar{z}$ are given by $f(x)$ and $\left(x^{2\bar{\bar{n}}-2}g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x^2)$ respectively.*

Proof. We shall use the same notation as in Lemma 1. In this case, the identity corresponding to (8.24) becomes

(1) When $r-r'+1+r''>n$, the degree of $g(z^2)+(-1)^{\frac{n-1}{2}}x^n$ is the same as that of $g(x^2)$; and when $r-r'+1+r''\leqq n$, the degree of $g(x^2)+(-1)^{\frac{n-1}{2}}x^n$ is the same as that of x^n.

(2) Substituting $-x$ for x in (8.24), we have $\left(g(x^2)-(-1)^{\frac{n-1}{2}}x^n\right)=\pm k\prod_{i=1}^{N}(x\mp\sqrt{a_i})\,\Pi(x+b_j)$ as n is odd. So the product of $g(x^2)-(-1)^{\frac{n-1}{2}}x^n$ (8.24) and (8.23)' contains $f(x^2)$ as factor.

$$\prod_{i=1}^{N}(x\pm\sqrt{a_i})^{m_i-m_i'}\prod(x-b_j)\equiv x^{2\bar{\bar{n}}-2}\bar{g}(x^2)+x^n. \tag{8.25}$$

Since the degree of the left-hand side above is $r-r'+1+r''$, and the polynomial $\bar{g}(x^2)$ in the right-hand side is of even degree, the coefficients of the terms of lower degree than $2\bar{\bar{n}}-2$ and of the terms of odd degree in the right-hand side must be 0 or 1. This condition becomes, using identity (8.25),

$$\frac{1}{2}(r-r'+r''+1+\epsilon)+\bar{\bar{n}}$$

$$(\epsilon=0 \text{ for } r-r'+1+r''>n \quad \text{and} \quad \epsilon=1 \text{ for } r-r'+1+r''<n)$$

relations among the constants b_j. Since we can always make the number r'' of constants b_j satisfy the inequality $r-r'+2\bar{\bar{n}}+1+\epsilon\leqq r''$, we can choose the constants b_j satisfying the relations among the constants b_j. So, there exist b_j and $\bar{g}(x^2)$ satisfying (8.25).

Therefore, taking such a $\bar{g}(x^2)$ from (8.25), for the matrix $\bar{z}$ whose minimum form is

$$x^{-\eta}\prod_{j=1}^{N}(x\pm\sqrt{a_j})^{m_j-m_j'}\prod_{i=1}^{N}(x^2-a_i)^{m_i'} \tag{8.26}$$

where $\eta=0$ for $m_1'=0$ and $\eta=1$ for $m_1'\neq 0$, we know from Remark 3 that $f(x)$ and $\left(x^{2\bar{\bar{n}}-2}g(x^2)+(-1)^{\frac{n-1}{2}}x^n\right)\tilde{g}(x^2)$ are the minimum form of $\bar{z}^2$ and $\bar{z}$ respectively. So, we have proved the lemma. Therefore, by the same reasoning as that employed to obtain Theorem 31 from Lemma 1, we have the following theorem for Lemma 2.

Theorem 32. *When $f(x)$, its factor $\tilde{g}(x)$, n (odd) and $\bar{\bar{n}}(<n)$ are arbitrarily given, there exists an iteration system* $\mathfrak{B}$ *in matric form whose minimum relation and reduced relation are $f(M)=0$ and $M^{\bar{\bar{n}}-1}\bar{g}(M)\tilde{g}(M)+\tilde{g}(M)\overset{I}{A}=0$ respectively.*

Forms (8.24) (in Theorem 31) and (8.26) (in Theorem 32), by which z is determined, depend only on $f(x)$ and its factor $\tilde{g}(x)$.[2] So, when two isomorphic iteration systems are regarded as the same from Theorems 31 and 32 we can say that the iteration system of dimension n is uniquely determined by the minimal form $f(x)$ and the minimal coefficient $\tilde{g}$ which is a factor of $f(x)$ and its dimension n.

So that, from Theorems 31 and 32, we have

Theorem 33. *If* $\mathfrak{B}$ *contains a relation among the basic elements, then the reduced relation is given by the form $M^{n-1}\bar{g}(M)\tilde{g}+\tilde{g}(M)\overset{I}{A}=0$.*

(1) $\frac{1}{2}(r-r'+1+r''+\epsilon)$ and $\bar{\bar{n}}$ are numbers of conditions among coefficients of odd and even degree respectively.

(2) The iteration systems $\{[\mu_i]\}$ obtained from matrices $\bar{z}$ determined from two choices of sign in $\pm\sqrt{a_i}$ are clearly isomorphic.

And, from Theorem 28, we have

Corollary. *When $r+1$ and $s(\neq 0)$ are index and degree of relation of an M-divisible and non-nilpotent iteration system $\mathfrak{B}$, the order of $\mathfrak{B}$ is given by*

$$(r+s)2^{n-1}.$$

When $\mathfrak{B}$ is M-divisible and $n_0=0$, from the corollary above we know that case (iv) in Remark I (p. 148) does not occur. So we have

Theorem 34. *Iteration systems are essentially classified into two types:*

I. *The order p of $\mathfrak{B}$ is $\omega 2^n$, and its dimension is n.*

II. *The order p of $\mathfrak{B}$ is $\omega 2^{2m}$, and its dimension is $2m+1$, but it is impossible for $\mathfrak{B}$ to be expressed by $2m$-dimensional basic elements, where ω is the order of the zentrum of $\mathfrak{B}$.*

§ 9. On the transformations of the central basis.

In this section we shall consider the relations between the sets of basis α_i and α'_j when $\mathfrak{B}=\{[\alpha_i]\}=\{[\alpha'_j]\}$ $(i=1,\ldots,n; j=1,\ldots,n')$. For this purpose we shall prove some lemmas.

Lemma 1. If $\mathfrak{B}(=\{[\alpha_i]\})$ is M-divisible, and there exists $\bar{\alpha}_i$ in $\mathfrak{B}$ such that $\alpha_i S=S\bar{\alpha}_i$ and $T\alpha_i=\bar{\alpha}_i T$(1) for certain S and T in $\mathfrak{B}$ $(i=1,\ldots,n)$, then $ST=P_C$, where P_C is a zentrum-element. Moreover, if such P_C has no common factor with the reduced relation $f(M)=0$, then $\{[\bar{\alpha}_i]\}=\mathfrak{B}$, and TS is a zentrum element of $\mathfrak{B}$.

Proof. From $\alpha_i S=S\bar{\alpha}_i$ and $T\alpha_i=\bar{\alpha}_i T$ we have

$$\alpha_i ST=S\bar{\alpha}_i T=ST\alpha_i \quad \text{and} \quad T\alpha_i S=TS\bar{\alpha}_i=\bar{\alpha}_i TS \quad (i=1,\ldots,n) \tag{9.1}$$

So, ST is commutative with all the elements α'_i's; therefore ST belongs to the zentrum of $\mathfrak{B}$,(2), i.e. $ST=P_C$. Further, from $\alpha_i S=S\bar{\alpha}_i$, we have $\alpha_j\alpha_i S=\alpha_j S\bar{\alpha}_i=S\bar{\alpha}_j\bar{\alpha}_i$. So we have, in general, $\beta S=S\bar{\beta}$, where $\beta\equiv F(\alpha_i)$ and $\bar{\beta}\equiv F(\bar{\alpha}_i)$. Now, if $\beta_1 S=S\bar{\beta}_1$ and $\beta_2 S=S\bar{\beta}_1$ then $(\beta_1-\beta_2)S=0$; hence $(\beta_1-\beta_2)ST=0$, i.e. $(\beta_1-\beta_2)P_C=0$. But, since $\mathfrak{B}$ is M-divisible, by applying the elimination-method (cf. Theorem 19) to

$$(\beta_1-\beta_2)P_C=0\,, \quad (\beta_1-\beta_2)f(M)=0\,, \quad \text{and} \quad (\beta_1-\beta_2)f(M)\overset{I}{A}=0\,,$$

we have the reduced form $(\beta_1-\beta_2)P_C=0$, where $P_C(M)$ is a common factor of P_C and $f(M)$. Hence, if P_C and $f(M)$ have no common factor, then $\beta_1=\beta_2$. And, since β may be any element of $\mathfrak{B}$ and $\bar{\beta}\in\mathfrak{B}$ and the order of $\mathfrak{B}$ is finite, $\bar{\beta}$ can cover all the elements of $\mathfrak{B}$. Accordingly, the correspondence of β and $\bar{\beta}$ defines an automorphism in $\mathfrak{B}$. So we nave $\bar{\alpha}_{(i}\bar{\alpha}_{j)}=0$ $(i\neq j)$, because $\alpha_{(i}\alpha_{j)}=0$ $(i\neq j)$ and $\alpha'^2_i=M'$. Therafore $[\bar{\alpha}_i]$ is a central

(1) We do not assume that $[\bar{\alpha}_i]$ is a central manifold of $\mathfrak{B}$.

(2) Cf. this Journal, **10** (1940), p. 226.

manifold of $\mathfrak{B}$, i. e. $\{[\bar{a}_i]\}=\mathfrak{B}$. And from (9.1) we see that TS is also commutative with all elements of $\mathfrak{B}$ i. e. $TS=P_C$. Q. E. D.

Definition: *When $\mathfrak{B}$ is M-divisible and $STM=M$, T is called the inverse of S.*

Remark: We know from § 6 that such ST is the unit element of $\mathfrak{B}$, and accordingly $TS^{(1)}$ is also that unit element of $\mathfrak{B}$. So, by the definition above, S is the inverse of T.

Lemma 2. *If $a_1S=S\bar{a}_i$, $a_iS'=S'\bar{a}_i$, and there exists the inverse T of S, then $S'=P_CS$.*

Proof. Multiplying $a_iS=S\bar{a}_i$ by T, the inverse of S, from both sides, we have

$$Ta_i=\bar{a}_iT\,; \tag{9.2}$$

multiplying $a_iS'=S'\bar{a}_i$ by T from the left-hand side, we have $Ta_iS'=TS'\bar{a}_i$, so, by the use of (9.2) this becomes

$$\bar{a}_iTS'=TS'\bar{a}_i\,. \tag*{(9.2)$'$}$$

Since, in this case, ST does not contain a factor of the minimal function $f(M)$ of M (because TS is the unit element), from (9.2) and Lemma 1 we have $\{[\bar{a}_i]\}=\{[\bar{a}_i]\}=\mathfrak{B}$. From this result and (9.2); we know that TS' belongs to the zentrum of $\mathfrak{B}$ i. e., $TS'=P_C$. Therefore, multiplying this equation by S from the left-hand side, we have

$$S'=P_CS\,. \qquad \text{Q. E. D.}$$

Lemma 3. *If $f(M)=0$ and $f'(M')=0$ are the minimum relations for M and M' respectively and moreover $M=k(M')$, and $M'=k'(M)$, then the following relations hold good:*

$$k(a_i')=a_j\,,\qquad k'(a_{i'})=a_j'\,,$$

for the non-null roots a_i and $a_j'(a_i, a_j'\in\mathfrak{R})$ of $f(x)=0$ and $f'(x)=0$ $(x\in\mathfrak{R})$ respectively, i. e.,

$$f(x)\equiv\prod_{i=1}^{N}(x-a_i)^{m_i}\,,\qquad f'(x)\equiv\prod_{j=1}^{N'}(x-a_j')^{m_j'}\,.$$

Proof. From the assumption, we have $f\big(k(M')\big)=0$ and $f'\big(k'(M)\big)=0$, i. e.

$$\left.\begin{aligned} f\big(k(x)\big)&\equiv\tau(x)f'(x)\\ f'\big(k'(x)\big)&\equiv\tau'(x)f(x)\end{aligned}\right\} \tag{9.3}$$

---

(1) We assume that $STM=M$. If we write $T\beta S=\beta'$, then, multiplying T from the right-hand side, we have $T\beta=\beta'T$ (for ST is the unit element). So from Lemma 1 we know that the correspondence between β and β' defines an automorphism with in $\mathfrak{B}$ in which the unit element corresponds to itself clearly. So, if we substitute ST for β in $T\beta S=\beta'$, it follows that $T(ST)S=ST$, i. e. $TS=ST$ (for ST is the unit element). Therefore TS is the unit element of $\mathfrak{B}$ and S is the inverse of T.

On ther other hand, we have

$$M=k\big(k'(M)\big)\,,\quad \text{i. e.}\quad x\equiv k\big(k'(x)\big)+l(x)f(x)\,. \tag{9.3)'$$

From this identity, we know that $k'(x)$ and $f(x)$ have no common factor except x.(1)

Hence $k(a_i')\neq 0$, and the conditions of (9.3) can be written by $k(a_i')=a_j$, $k'(a_{j'})=a_j'$, and $a_j, a_j', \neq 0$ $(j, i'=1,\ldots\ldots, N;\ i, j'=1,\ldots\ldots, N')$, where $f(x)\equiv \prod_{i=1}^{N}(x-a_i)^{m_i}$ and $f'(x)\equiv\prod_{j=1}^{N}(x-a_j')^{m_j'}$ $(a_i, a_j'\in\mathfrak{K})$. Q. E. D.

When the base M satisfies the minimum relation $f(M)=0$, and a_1 is a root of $f(x)=0$ $(a_1\neq 0)$ the relation $Mf_1(M)=a_1f_1(M)$ obtains, provided that $f(x)\equiv(x-a_1)f_1(x)$. So if we put $\overline{M}_1=\dfrac{M}{a_1}$ and $f_1(a_1M_1)=g_1(\overline{M})$, then $\overline{M}_1g_1(\overline{M})=g_1(\overline{M}_1)\,\big(g_1(\overline{M})\not\equiv 0\big)$.

If we take such $g(\overline{M})$'s for different roots of $f(x)$, we can show that they are essentially different; for, if $f_1(M)$ $\big($or $g_1(\overline{M}_1)\big)$ and $f_2(M)$ $\big($or $g_2(\overline{M}_2)\big)$ for a_1, a_2 $(a_1\neq a_2)$ are essentially equivalent, then it follows that $b_1f_1+b_2f_2=0$, where b's are constants; but this contradicts the assumption that $f(M)=0$ is the minimal relation of M. Accordingly, there are different $g(\overline{M})$'s as many as the number of different non-null roots of $f(x)=0$. Conversely, in an iteration system $\mathfrak{B}$, if there exists $g(\overline{M})$ such that $\overline{M}g(\overline{M})=g(\overline{M})$— the relation among the bases of $\mathfrak{B}$—, from Theorem 19 and the corollary of Theorem 20 there exists the minimal relation, say $f(M)=0$. So we have

Lemma 4. *In an iteration system $\mathfrak{B}$, the necessary and sufficient condition for the existence of a $g(\overline{M})$ and $\overline{M}$ satisfying the relation $\overline{M}g(\overline{M})=g(\overline{M})$, $\big(g(\overline{M})\not\equiv 0\big)$, there exists the relation $f(M)=0$ of base M.*

Now, when $\mathfrak{B}$ is M-divisible and $n_0=0$, using the Lemmas above let us consider the problem of the isomorphic correspondence of the two sets of bases, α_i and α_i', of $\mathfrak{B}(\mathfrak{B}=\{[\alpha_i]\}=\{[\alpha_i']\})$:

$$\alpha_iS=S\alpha_i'\quad(\text{or }\beta S=S\beta')\,. \tag{9.6}$$

For this purpose, we shall treat the problem in two separate cases: I. $M=M'$, and II. $\{M\}=\{M'\}$.

Case I. $M=M'$.

If we put $\bar\alpha_i=\dfrac{\alpha_i}{\sqrt{a_l}}$ and $\alpha_i'=\dfrac{\alpha_i'}{\sqrt{a_l}}$ (l is any difinite number), then (9.6) becomes

$$\bar\alpha_iS=S\bar\alpha_i'\,. \tag{9.7}$$

Multiplying this by $g_l(\overline{M}_l)\bar\alpha_i$ from the left-hand side, we have

(1) From the construction of $\mathfrak{B}$ and $k(M')$, it follows that $k(M')=a_1M'+a_2M'^2+\cdots\cdots$; hence, $k(k'(M))=a_1k'(M)+a_2k'^2(M)+\cdots\cdots$. So if $f(x)$ and $k'(x)$ have a common factor $x-a$ other than x $(a\neq 0)$, substituting a for x into (9.3)' we have $a=0$; which contradicts $a\neq 0$. Accordingly $k'(x)$ and $f(x)$ have no commen factor except x.

$$\overline{M}_l\overline{S}_l=\overline{a}_i\overline{S}_l\overline{a}'_i\,,\quad \text{i. e.}\quad \overline{S}_l=\overline{a}_i\overline{S}_l\overline{a}'_i \tag{9.8}$$

where $\overline{S}_l=g_l(\overline{M}_l)S$; hence $\overline{S}_l=\overline{a}_i\overline{S}_l\overline{a}'_i$. (Hereafter we shall write $g(\overline{M})$ for $g_l(\overline{M})$ when there is no confusion.) So $\overline{S}=\overline{a}_i\overline{S}\overline{a}'_i$. Now we assume that S includes the term of the 0-th degree, say $s_0(M)$, with respect to a_i; i. e., $\overline{S}$ includes $\overline{s}_0(M)$ of the form $s_0(M)g(\overline{M})$. Then the right-hand side of (9.8) includes the terms $\overline{s}_0(M)\overline{a}_{i_1}\overline{a}'_{i_1}$; so the term $\overline{s}_0(M)\overline{a}_{i_1}\overline{a}'_{i_1}$ must be included in the left-hand side $\overline{S}$ $(i_1=1,\dots,n)$. Hence, if we put $i=i_2$ in (9.8), the right-hand side includes the term $\overline{s}_0(M)\overline{a}_{i_2}\overline{a}_{i_1}\overline{a}'_{i_1}\overline{a}'_{i_2}$ $(i_1<i_2)$; so the term $\overline{s}_0(M)\overline{a}_{i_2}\overline{a}_{i_1}\overline{a}'_{i_1}\overline{a}'_{i_2}$ must be included in the left-hand side of (9.8), i. e., $\overline{S}$. By continuing this reasoning, we see that $\overline{S}$ includes the term

$$\overline{s}_0(M)\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p}\qquad (1\leqq i_1<i_2<\dots<i_p\leqq n)\,.$$

So $\overline{S}$ must include all their sum

$$\overline{S}_0=\overline{s}_0(M)\sum_{i_1\dots i_p}^{n}\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p}\qquad (0\leqq i_1<i_2<\dots<i_p\leqq n)\,.$$

On the other hand, we can prove that *the* $\overline{S}_0$ *satisfies the relation:*

$$\overline{S}_0=\overline{a}_i\overline{S}_0\overline{a}'_i\,.$$

Proof: First we shall consider the reduction of a term $\overline{s}_0\overline{a}_i(\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p})\overline{a}'_i$ which appears in $\overline{a}_i\overline{S}_0\overline{a}'_i$, separating it into two cases.

When i of $\overline{s}_0\overline{a}_i(\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p})\overline{a}'_i$ is any one of $i_1,\dots,i_p$, say i_ϵ, we have

$$\overline{s}_0\overline{a}_i(\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p})=\overline{s}_0\overline{M}\overline{M}\overline{a}_{i_p}\dots\overline{a}_{i_{\epsilon+1}}\overline{a}_{i_{\epsilon-1}}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_{\epsilon-1}}\overline{a}'_{i_{\epsilon+1}}\dots\overline{a}'_{i_p}.$$

But from Lemma 4,

$$\overline{s}_0\overline{M}\overline{M}=s_0g(\overline{M})\overline{M}\overline{M}=\overline{s}_0g(\overline{M})=\overline{s}_0\,.$$

So the term $s_0(M)\overline{a}_i(\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p})\overline{a}'_i$ in $\overline{a}_iS_0\overline{a}'_i$ becomes an other term in $\overline{S}_0$ such that

$$\overline{s}_0(\overline{M})\overline{a}_{i_p}\dots\overline{a}_{i_{\epsilon+1}}\overline{a}_{i_{\epsilon-1}}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_{\epsilon-1}}\overline{a}'_{i_{\epsilon+1}}\dots\overline{a}'_{i_n}\,.$$

When i is not included in $i_1,\dots,i_p$ as i_ϵ, $\overline{s}(\overline{M})\overline{a}_i\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p}\overline{a}'_i$ becomes another terms of $\overline{S}_0$ containing $\overline{a}_i\overline{a}'_i$ as factor, i. e.,

$$\overline{s}_0(\overline{M})\overline{a}_{i_p}\dots\overline{a}_i\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_i\dots\overline{a}'_{i_p}\,.$$

Accordingly, $\overline{s}_0\overline{a}_i(\sum\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{i_p})\overline{a}'_i=\overline{s}_0\sum\overline{a}_{i_p}\dots\overline{a}_{i_1}\overline{a}'_{i_1}\dots\overline{a}'_{j_p}$. Therefore, we have

$$\overline{S}_0=\overline{a}_i\overline{S}_0\overline{a}'_i\,.$$

Multiplying this relation by $\overline{a}_i$ from the left-hand side, and taking into account that $\overline{M}\overline{S}_0=\overline{S}_0$, we have

$$\bar{a}_i\bar{S}_0=\bar{S}_0\bar{a}'_i \quad \text{i. e.} \quad a_i\bar{S}_0=\bar{S}_0a'_i\,.$$

So we have proved the proposition.

Similarly, for an element $\bar{T}_0$ in $\mathfrak{B}$ of the form

$$\bar{T}_0=\bar{t}_0(\bar{M})\sum_{i_1,\dots,i_p}^{n}\bar{a}'_{i_p}\dots\bar{a}'_{i_1}\bar{a}_{i_1}\dots\bar{a}_{i_p} \qquad (0\leqq i_1<\dots<i_p\leqq n)\,,$$

where $\bar{t}_0=t_0g(\bar{M})$ $(t_0\in\mathfrak{K})$, we can prove that $\bar{T}_0$ satisfies the relation:

$$\bar{T}_0a_i=a'_i\bar{T}_0\,.$$

So we have

Theorem 35. *When there exists a relation $f(M)=0$ and $\mathfrak{B}=\{[a_i]\}=\{[a'_i]\}$, any elements $\bar{S}$ and $\bar{T}$ given by*

$$\bar{S}=\sum_\tau p_\tau q_\tau(\bar{M}_\tau)\sum_{i_1,\dots,i_p}^{n}\bar{a}_{i_p}\dots\bar{a}_{i_1}\bar{a}'_{i_1}\dots\bar{a}'_{i_p}$$

and $$\bar{T}=\sum_\tau q_\tau g_\tau(\bar{M}_\tau)\sum_{i_1,\dots,i_p}^{n}\bar{a}'_{i_p}\dots\bar{a}'_{i_1}\bar{a}_{i_1}\dots\bar{a}_{i_p} \qquad (0\leqq i_1<\dots<i_p\leqq n)\,,$$

define the following isomorphism:

$$a_i\bar{S}=\bar{S}a'_i \quad \text{and} \quad \bar{T}a_i=a'_i\bar{T} \tag{9.9}$$

where $g_\tau(\bar{M}_\tau)$ are defined by $f(M)\equiv(M-a_\tau)g_\tau(\bar{M}_\tau)$ and $\bar{a}_i=\dfrac{a_i}{\sqrt{a_\tau}}$ and p_τ, q_τ are any number of $\mathfrak{K}$. And, from (9.9) and Lemma 1, we have

Corollary. *$\bar{S}\bar{T}$ and $\bar{T}\bar{S}$ are elements of the zentrum in $\mathfrak{B}$.*

Now we shall consider the problem: What is the necessary and sufficient condition for a basis a_i of $\mathfrak{B}$, a'_i given by $Ta_iS=a'_i$, to be also a basis of $\mathfrak{B}$ i. e. $\{[a'_i]\}=\{[a_i]\}$? To answer this we shall prove the next lemma.

Lemma 5. *An element $A(\in\mathfrak{B})$ satisfying the relation $l_1(M)Aa_i=a_il_2(M)A$ $(i=1,\dots,n)$ is zero, if there is no relation $l_1=l_2$ or $l_1=-l_2$ and $l_1(x)+l_2(x)$ and $l_1(x)-l_2(x)$ are both prime to the minimum function $f(x)$ of $\mathfrak{B}$. And when $l_1(x)+l_2(x)$ is prime to $f(x)$ and $l_1(x)=l_2(x)$,*

$A=a_0(M)+a_2(M)\overset{I}{A}$ *when the dimension of $\mathfrak{B}$ is odd,*

$A=a_0(M)$ *when the dimension of $\mathfrak{B}$ is even;*

when $l_1(x)-l_2(x)$ prime to $f(x)$ and $l_1(x)=-l_2(x)$,

$A=0$ *when the dimension of $\mathfrak{B}$ is odd*

$A=a_2\overset{I}{A}$ *when the dimension of $\mathfrak{B}$ is even.*

Proof. Multiplying the relation

$$l_1(M)Aa_i=a_il_2(M)A \tag{9.13}$$

by a_i from the left-hand side, we have

$$l_1a_iAa_i=Ml_2A\,. \tag{9.13$'$}$$

We separate the terms of A into two groups, A_{i+} and A_{i-}, such that one group A_{i+} consists of all the terms of odd degree (with respect to α_i) including α_i as a factor, and of all the terms of even degree not including α_i as a factor, and the other group A_{i-} consists of all the remaining terms of A. Then (9.13)$'$ is rewritten by this A_{i+} and A_{i-} as follows:

$$l_1(A_{i+}-A_{i-})=l_2(A_{i+}+A_{i-})\,.$$

Multiplying (9.13) by α_i from the right-hand side, we have

$$l_1(A_{i+}+A_{i-})=l_2(A_{i+}-A_{i-})\,.$$

So, from these two relations, we have

$$\left.\begin{matrix}(l_1-l_2)A_{i+}=0\\ (l_1+l_2)A_{i-}=0\end{matrix}\right\}\ (i=1,\ldots\ldots,n)\,. \tag{9.14}$$

When $l_1(x)+l_2(x)$ and $l_1(x)-l_2(x)$ are both prime to the reduced function $f(x)$ of $\mathfrak{B}$, from (9.14) we can conclude that $l_1\neq\pm l_2$; accordingly $A_{i+1}=A_{i-1}=0$ i. e. $A=0$; and if $l_1=l_2$, then $A_{i-}=0$ $(i=1,\ldots\ldots,n)$, so that

$$\left.\begin{matrix}A=a_0(M)+a_2(M)\overset{I}{A} & \text{when the dimension of } \mathfrak{B} \text{ is odd}\\ A=a_0(M) & \text{when the dimension of } \mathfrak{B} \text{ is even,}\end{matrix}\right\} \tag{9.15}$$

if $l_1=-l_2$, then $A_{i+1}=0$ $(i=1,\ldots\ldots,n)$; so

$$\left.\begin{matrix}A=0 & \text{for dimension } n \text{ odd}\\ A=a_2(M)\overset{I}{A} & \text{for} \quad ,, \qquad ,, \text{ even.}\end{matrix}\right\} \tag{9.15$'$}$$

Conversely, for A given by (9.15) and (9.15)$'$ we can easily see that it satisfies the relation (9.13). So we have proved the lemma.

Theorem 36. *When $\mathfrak{B}$ has the unit element and $T\alpha_iS=\alpha_i'$ $(i=1,\ldots\ldots,n)$ $(T,S\in\mathfrak{B})$, the necessary and sufficient condition for $\{[\alpha_i]\}=\{[\alpha_i']\}$ is that TS has the form $TS=h_2(M)+h_1(M)\overset{I}{A}$ and*

$$\{h_2^2(M)+(-1)^{\frac{n}{2}}h_1^2(M)M\}=\{M\} \quad \textit{when the dimension } n \textit{ is even,}$$

$$\{(TS)^2M,\ (TS)^n\overset{I}{A}\}=\{M,\ \overset{I}{A}\} \quad \textit{when the dimension } n \textit{ is odd.}$$

Proof. Let us assume that

$$T\alpha_iS=\alpha_i' \quad \text{and} \quad \{[\alpha_i]\}=\{[\alpha_i']\}\,; \tag{9.16}$$

then α_i' has the inverse.(1) Accordingly, S and T are non-singular, i. e.(2)

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(1) The unit element h of $\mathfrak{B}$ has the form $h_2(M')+h_1(M')\overset{I}{A}{}'$. So we can express h as follows: $h\equiv\alpha_i'\alpha_i'h_2'+\alpha_i'h_1\alpha_1'\ldots\ldots\alpha_{i-1}'\alpha_{i+1}'\ldots\ldots\alpha_n'$; hence $\alpha_i'(\alpha_i'h_2'+h_1\alpha_1'\ldots\ldots\alpha_{i-1}'\alpha_{i+1}'\ldots\ldots\alpha_n')=h$. Therefore α_i' has the inverse $(\alpha_i'h_2'+h_1'\alpha_i'\ldots\ldots\alpha_{i-1}\alpha_{i+1}\ldots\ldots\alpha_n')$.

(2) If $S(\beta_1-\beta_2)=0$, then from (9.16) $\alpha_i'(\beta_1-\beta_2)=0$. So, multiplying this relation by the inverse w_i' of α_i' from the left-hand side, we have $\beta_1-\beta_2=0$. Therefore $S\mathfrak{B}=\mathfrak{B}$ (because the order of $\mathfrak{B}$ is finite). Since $\mathfrak{B}$ includes a unit element h, so there exists S^{-1} such that $SS^{-1}=h(S^{-1}\in\mathfrak{B})$.

So if we write $U=TS$ and $S^{-1}a_iS\equiv\bar{a}_i$, then the relation $Ta_iS=a'_i$ becomes $US^{-1}a_iS=a'_i$, i. e. $U\bar{a}_i=a'_i$, and $\bar{a}_i\bar{a}_i=M$ and U has its inverse. Therefore we have

$$U\bar{a}_iU\bar{a}_i=a'_ia'_i=M'=P_C \tag{9.16$'$}$$

and

$$U\bar{a}_iU\bar{a}_j+U\bar{a}_jU\bar{a}_i=a'_ia'_j+a'_ja'_i=0\,,$$

i. e.

$$\bar{a}_iU\bar{a}_j+\bar{a}_jU\bar{a}_i=0 \qquad (i\neq j)\,.$$

Multiplying this by $\bar{a}_i$ and $\bar{a}_j$ from the left- and right-hand sides respectively, we have

$$M^2U=\bar{a}_j(\bar{a}_iU\bar{a}_i)\bar{a}_j\,.$$

Then, by applying the notation used in § 5,(1) this relation can be written in

$$M^2U=M^2(U^{(i)(j)}_{1\ 1}+U^{(i)(j)}_{2\ 2}-U^{(i)(j)}_{1\ 2}-U^{(i)(j)}_{2\ 1})\,. \tag{9.16$''$}$$

On the other hand, U is written in

$$U\equiv U^{(i)(j)}_{1\ 1}+U^{(i)(j)}_{2\ 2}+U^{(i)(j)}_{1\ 2}+U^{(i)(j)}_{2\ 1}\,.$$

So, from (9.16)$''$ and the identity above, we have

$$U^{(i)(j)}_{1\ 2}+U^{(i)(j)}_{2\ 1}=0 \qquad U^{(i)(j)}_{1\ 1}+U^{(i)(j)}_{2\ 2}=0 \qquad (i\neq j,\ i,j=1,\ldots\ldots,n)\,.$$

So the terms of odd degree in U are the sum of two parts, the one composed of terms (in $U^{(i)(j)}_{1\ 1}$) which contain $\bar{a}_i\bar{a}_j$ as a factor, and the other composed of terms (in $U^{(i)(j)}_{2\ 2}$) which contain neither $\bar{a}_i$ nor $\bar{a}_j$. By giving j the values from 1 to n for a fixed i, we know that the term of odd degree in U contains $\bar{a}_1\ldots\ldots\bar{a}_n$ as a factor.(2) And similarly, the terms of even degree in U are composed of two parts, one not containing $\bar{a}_j$ ($j=1, 2,\ldots\ldots,n$) (in $M^{(i)(j)}_{1\ 1}$), the other containg $\bar{a}_1\bar{a}_2\ldots\ldots\bar{a}_n$ as a factor (in $M^{(i)(j)}_{2\ 2}$). So we have

$$U=h'_2(M)+h'_1(M)\bar{a}_1\ldots\ldots\bar{a}_n=h_2(M)+h_1(M)\overset{I}{\bar{A}} \quad (\overset{I}{\bar{A}}\equiv\bar{a}\ldots\ldots\bar{a}_n). \tag{9.17}$$

Further, substituting U above into (9.16), we have

$$\left(h_2(M)+h_1(M)\overset{I}{\bar{A}}\right)\left(h_2+(-1)^{n-1}h_1\overset{I}{\bar{A}}\right)M=P_C=M'\,: \tag{9.18}$$

by actual calculation, we have

$$\text{when}\quad n=\text{even}\quad \left(h_2^2-(-1)^{\frac{n}{2}}h_1^2M^n\right)M=M'$$

$$\text{when}\quad n=\text{odd}\quad \left(h_2^2+(-1)^{\frac{n-1}{2}}h_1^2M^n+2h_1h_2\overset{I}{\bar{A}}\right)M=M'\,.$$

Of course, the zentrums of $\{[a_i]\}$ and $\{[a'_i]\}$ must be the same. So we have

(1) Cf. this Journal, **10** (1940), 236.

(2) The term of odd degree which does not contain a_j ($j=1,\ldots\ldots,n$) at all as a factor must vanish.

$$\left\{\left(h_2^2-(-1)^{\frac{n}{2}}h_1^2M^n\right)M\right\}=\{M\} \quad \text{when } n \text{ is even.} \tag{9.19}$$

If $n=\text{odd}$, $M'=U^2M$ and $\overset{I}{A}'=U^n\overset{I}{A}$ $(\overset{I}{\bar{A}}\equiv\overset{I}{A})^{(1)}$; U and $\overset{I}{A}$ belong to the zentrum of $\mathfrak{B}$; so, from Theorem 10, it follows that

$$\{U^2M_1U^n\overset{I}{A}\}=\mathfrak{N}(M,\overset{I}{A}). \tag{9.19)'$$

Conversely, if we take U in (9.17) which satisfies (9.19) when n even, and (9.19) when n odd, then a_i' given by $US^{-1}a_iS=a_i'$ for any element S of $\mathfrak{B}$, (i. e. $U\bar{a}_i\equiv a_i'$) satisfies the relations

$$a'_{(i}a'_{j)}=\delta_{ij}M'.$$

Accordingly a_i''s constitute a set of central bases of an iteration system $\mathfrak{B}'$ $(\equiv\{]a_i']\})$. And, by (9.19) or (9.19)', the zentrums of $\{[a_i]\}$ and $\{[a_i']\}$ are equal. So, from Remark 1 (on p. 144) and Theorem 27, $\{[a_i]\}$ and $\{[a_i']\}$ contain the same number of linearly independent basic elements with respect to $\mathfrak{K}$ and $\{[a_i']\}\subset\{[a_i[\}$. So we have proved the theorem.

Theorem 37. *When $\mathfrak{B}$ contains the unit element and $\{[US^{-1}a_iS]\}=\{[a_i]\}$ and the norm is defined in $\mathfrak{K}$, the necessary and sufficient condition for U satisfying the relation above to have an infinitesimal operator (i. e. $h(M)+\epsilon A$, $\epsilon\ll 1$) is that U is given in the following form:*

$$U=e^{l(M)\overset{I}{A}} \qquad \textit{when } n \textit{ is even}$$

and

$$U=h(M) \qquad \textit{when } n \textit{ is odd,}$$

where $h(M)$ is the unit element of $\mathfrak{B}$.

Proof. If we substitute $h(M)+\epsilon A$ $(\epsilon\ll 1)$ for U in (9.16)' where h is the unit element of $\mathfrak{B}$, we have

$$\left.\begin{aligned} &M(h+\epsilon A)a_i=P_Ca_i(h-\epsilon A)\\ &a_iAa_j+a_jAa_i=0\,. \end{aligned}\right\} \tag{9.20}$$

So, $Ma_i=P_Ca_i$. Multiplying this relation by a_i from the right-hand side, we have $M=P_C$.

Substituting this into (9.20), we have

$$Aa_i=-a_iA\,.$$

So, from Lemma 5, $A=0$ when n is odd, and $A=l(M)\overset{I}{A}$ when $n=\text{even}$. Therefore, if U is the result of repetions of an infinitesimal operator, it

(1) When n odd, since $a_1\ldots\ldots a_n$ is commutative with $a_i, a_1\ldots\ldots a_n$ is an element in the zentrum of $\mathfrak{B}$. Hence the relation $\bar{a}_1\ldots\ldots\bar{a}_n=S^{-1}a_1\ldots\ldots a_nS$ becomes $\bar{a}_1\ldots\ldots\bar{a}_n=S^{-1}Sa_1\ldots\ldots a_n$, so we have

$$\overset{I}{\bar{A}}\equiv\overset{I}{A}.$$

follows that $U=e^{l(M)\overset{I}{A}}$ (for h is the unit element) provided that e^0 is the unit element h(M).

Conversely, if n even and $U=e^{l(M)\overset{I}{A}}$, we have

$$U\bar{a}_i=e^{\frac{1}{2}l(M)\overset{I}{A}}\bar{a}_i e^{-\frac{1}{2}l(M)\overset{I}{A}}=U^{\frac{1}{2}}\bar{a}_i U^{-\frac{1}{2}}.$$

Accordingly, we know that

$$\{[a_i]\}=\{[U\bar{a}_i]\}.$$

So we have proved the theorem.

This problem was discussed at a special Seminar of Geometry and Theoretical Physics in the Hirosima University, and research into it has been carried on under the Scientific-Research Expenditure of the Department of Education.

Mathematics Institute, Hirosima University.