Cosmology in Terms of Wave Geometry (VII) Some Characteristics of the Universe.

By

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§ 1. **Introduction and summary.** Cosmology in terms of wave geometry is characterized by the equations:

$$\frac{\partial\psi}{\partial x^i}=(\Gamma_i+\Sigma_i)\psi \tag{1.1}$$

$$u^i=\psi^\dagger A\gamma^i\psi \tag{1.2}$$

$$u^i\nabla_i u^j=Qu^j\,. \tag{1.3}$$

(1.1) is the fundamental equation of wave geometry. (1.2) is the definition of u^i, where ψ is the solution of (1.1) and A an hermitian matrix which makes $A\gamma^i$ hermitian.(1) (1.3) is the condition that u^i may generate a geodesic. From these equations it results that the universe is either (i) of de Sitter type or (ii) of Einstein type. We have here, on plausible grounds,(2) adopted the de Sitter type as representing the universe.

u^i thus defined has been identified with the momentum-density vector of a constituent matter of the universe, and the phenomena of red-shifts of the spectral lines in nebulae, i. e. Hubble's velocity-distance relation, have been successfully explained.

The physical interpretations of u^i, however, present some difficulties, which are considered in detail in § 2. In § 3 we proceed to further properties of u^i and other physical quantities in our cosmology, defining a material energy tensor analogously to relativistic hydrodynamics.

§ 2. **The physical meanings of u^i and M.** In cosmology in terms of wave geometry, a nebula was considered as having two aspects, one being that of a particle to detect the structure of the universe, the other being that of probability-existence, all the nebulae being regarded as almost smeared out all over the universe.(3) Therefore it seems natural to study the physical properties of the universe in a manner analogous to the gas theory or fluid mechanics in the classical theory.

(1) T. Sibata: this Journ., **8** (1938), 172, (W. G. No. 26).
(2) K. Itimaru: this Journ., **8** (1938), 240, (W. G. No. 31).
(3) Y. Mimura and T. Iwatsuki: this Journ., **8** (1938), 194 (W. G. No. 28).

Thus we shall suppose hereafter that the universe consists of minute particles uniformly distributed as the molecules in gaseous materials. In this model of the universe, we take a volume element, its volume and number of particles being dV_0 and dN_0 respectively with respect to the proper coordinates; then the number of particles in unit volume σ_0 is given by

$$(2.1) \qquad \sigma_0=\frac{dN_0}{dV_0},$$

and if dV, dN, and σ are the volume, the number of particles, and the number of particles per unit volume, respectively, as regards the coordinates whose origin is moving relative to the volume element, we have

$$(2.2) \qquad \sigma=\frac{dN}{dV}.$$

Now, the number of particles in the volume element must not change with the coordinate-system taken, so $dN=dN_0$. But the magnitude of volume changes with the coordinate-system as follows:

$$(2.3) \qquad dV=\frac{dV_0}{\beta}, \qquad \text{where} \qquad \beta=\frac{dt}{ds}.$$

Therefore,

$$(2.4) \qquad \sigma=\frac{dN_0}{dV_0}\beta=\beta\sigma_0.$$

We call σ_0 and σ "*proper particle-density*" and "*coordinate particle-density*," respectively.

Next, if the mass of a minute particle in the proper coordinates and in the relative coordinates is expressed by m_0 and m respectively, then densities d_0 and d for respective coordinate-systems are given by

$$(2.5) \quad d_0=m_0\sigma_0, \qquad \text{and} \qquad (2.6) \quad d=m\sigma=\beta^2 d_0 \qquad (\because \ m=\beta m_0).$$

Bearing this in mind, we shall now go into the meanings of u^i.

As was done in Itimaru's paper,(1) if we put

$$(2.7) \qquad u^i=\rho\frac{dx^i}{ds}, \qquad (\rho \text{ is an invariant quantity})$$

we have

$$(2.8) \qquad \frac{u^a}{u^4}=\frac{dx^a}{dx^4}=v^a \qquad (a=1, 2, 3),$$

and if we identify $x^1=r$, $x^2=\theta$, $x^3=\varphi$, $x^4=t$, then v^1, v^2, and v^3 represent the r-, θ-, and φ-component of velocity of nebular matters, the value being given by

(1) K. Itimaru: this Journal, 8 (1938), 242 (W. G. No. 31).

$$v^r = \frac{-pe^{-kt}+qe^{kt}}{pe^{-kt}+qe^{kt}} kr(1-k^2r^2)$$

$$v^\theta = v^\varphi = 0\,.$$

With the help of these expressions, Hubble's velocity-distance relation was satisfactorily deduced. As to the physical meanings of u^i, however, we see that it cannot be identified with momentum-density vector. For, since u^4 is given by

$$u^4 = \rho \frac{dx^4}{ds} \qquad \text{(where } \rho \text{ is a scaler)}$$

by transformation from the proper coordinate-system to the moving coordinate-system, u^4 behaves like σ, as shown in (2.4), on the other hand by the same transformation d is transformed as (2.6). So that u^4 must be identified with *particle-density* σ, but not with *density* d *in the ordinary sense.* Thus we conclude that u^i cannot be taken as momentum-density vector as has been done in papers hitherto published.[1]

To remedy this unsatisfactory state of affairs, we proceed to the following considerations.

Let us take a 4-vector

$$\text{(2.9)} \qquad \lambda^i \equiv (\sigma v^1,\ \sigma v^2,\ \sigma v^3,\ \sigma) \qquad \left(v^a = \frac{dx^a}{dt}\right),$$

then λ^i can be rewritten as follows:

$$\lambda^i \equiv \left(\sigma_0 \frac{dt}{ds}\frac{dx^1}{dt},\quad \sigma_0 \frac{dt}{ds}\frac{dx^2}{dt},\quad \sigma_0 \frac{dt}{ds}\frac{dx^3}{dt},\quad \sigma_0 \frac{dt}{ds}\right)$$

or

$$\text{(2.10)} \quad \lambda^i \equiv \left(\sigma_0 \frac{dx^1}{ds},\quad \sigma_0 \frac{dx^2}{ds},\quad \sigma_0 \frac{dx^3}{ds},\quad \sigma_0 \frac{dt}{ds}\right) \equiv \sigma_0 \frac{dx^i}{ds} \quad (i=1, 2, 3, 4)\,.$$

The form of λ^i in (2.10) is identical with that of u^i in (2.7), and the transformation obeyed by λ^i is the same as that of u^i, so that we have ample reason to identify u^i in (2.2) with λ^i in (2.9). Here the first three components of (2.9) are the number of particles per unit volume multiplied by velocity-components, and the fourth component is the coordinate particle-density as defined above. Therefore, if we call the first three components of λ^i "particle-momentum," λ^i may be named "particle momentum-density vector." Accordingly *we may call* u^i "*particle momentum-density vector.*"

Next, by comparing (2.7) with (2.10), ρ can be identified with σ_0. On the other hand, since

(1) T. Iwatsuki, Y. Mimura, and T. Sibata: this Journ., **8** (1938), 188, and subsequent papers entitled "Cosmology in Terms of Wave Geometry" by Messrs. Y. Mimura and T. Iwatsuki; T. Sibata; H. Takeno; and K. Itimaru.

$$u^i u^j g_{ij} = \rho^2 \frac{dx^i}{ds} \cdot \frac{dx^j}{ds} g_{ij} = \rho^2 \quad \text{and} \quad u^i u^j g_{ij} = M^2 + N^2,$$ [(1)]

we have

(2.11) $$\rho^2 = M^2 + N^2,$$

where M and N are given by

(2.12) $M = \psi^\dagger A \psi$ (2.13) $N = \psi^\dagger A \gamma_5 \psi$.

Moreover, N in (2.11) is a constant depending on the integration constants of the fundamental equation (1.1), so without grave specialization we may put $N=0$; then (2.11) becomes $\rho^2 = M^2$, and therefore we have

(2.14) $$\rho = |M|.$$

Thus we can identify σ_0 with $|M|$.

Summarizing the considerations in this section, we have the schema:

$u^i \leftrightarrow \lambda^i \equiv$ particle momentum-density vector
$\lvert M\rvert \leftrightarrow \sigma_0 \equiv$ proper particle-density

It should be added, however, that in spite of the alternation of physical meanings of u^i, nothing needs amending in the essential conclusions hitherto obtained,[(2)] especially in the interpretation of Hubble's velocity-distance relation.

And further it must be added that in some physical systems, where constituent materials are regarded not to be smeared out, the physical meaning of u^i may be taken as 'probability flow-density vector' as in current quantum mechanics instead of 'particle momentum-density vector.'

§ 3. **Equation of continuity and law of conservation.** In the expression of the 4-vector u^i

(3.1) $$u^i = \psi^\dagger A \gamma^i \psi,$$

the factor A is an hermitian matrix which makes $A\gamma^i$'s also hermitian, as stated in § 1. And this A can be taken as $\mathring{\gamma}_4$ except for the real factor.[(3)]

Now, we put $A = \alpha\mathring{\gamma}_4$ and define $\mathring{u}^i$, $\mathring{M}$, u^i, and M as follows:

(3.2) $\mathring{u}^i = \psi^\dagger \mathring{\gamma}_4 \gamma^i \psi$, (3.3) $\mathring{M} = \psi^\dagger \mathring{\gamma}_4 \psi$,

(3.4) $u^i = \psi^\dagger \alpha \mathring{\gamma}_4 \gamma^i \psi$, (3.5) $M = \psi^\dagger A \mathring{\gamma}_4 \psi$.

Then we have

(3.6) $u^i = \alpha \mathring{u}^i$ and (3.7) $M = \alpha \mathring{M}$.

(1) T. Sibata: this Journ., **8** (1938), 175, (W. G. No. 26).
(2) K. Itimaru: loc. cit.
(3) T. Sibata: this Journ., **8** (1938), 172, (W. G. No. 26).

To study further properties of these quantities, we shall now define the energy tensor after the manner of current relativistic mechanics.

In relativistic mechanics, the material energy tensor is defined by

$$T^{ij}=(\rho_0+4p)\frac{dx^i}{ds}\frac{dx^j}{ds}-pg^{ij}. \tag{3.8}$$

In this expression, ρ_0 is the summation of density referring to the proper coordinate-system, so ρ_0 can be identified with $d_0=m_0\sigma_0=m_0|M|$. Therefore, taking (2.7) and (2.14) into account, we define our *material* energy tensor by

$$M^{ij}=(m_0|M|+4p)\frac{u^i}{|M|}\frac{u^j}{|M|}-pg^{ij}. \tag{3.9}$$

Now, restricting ourselves to the case when arbitrary motions of individual particles referred to the centre of mass do not exist, we can put $p=0$,[1] and by using (3.4), (3.5), (3.6), and (3.7) we have

$$M^{ij}=\frac{m_0}{|M|}u^iu^j=\frac{m_0 a}{|\mathring{M}|}\mathring{u}^i\mathring{u}^j. \tag{3.10}$$

Next, taking the divergence of M^{ij}, we have

$$\nabla_i M^{ij}=m_0\left\{-\frac{1}{\mathring{M}^2}(\nabla_i|\mathring{M}|)a\mathring{u}^i\mathring{u}^j+\frac{1}{|\mathring{M}|}(\varDelta_i a)\mathring{u}^i\mathring{u}^j+\frac{a}{|\mathring{M}|}(\nabla_i\mathring{u}^i)\mathring{u}^j+\frac{a}{|\mathring{M}|}\mathring{u}^i(\nabla_i\mathring{u}^j)\right\}, \tag{3.11}$$

and by using the following relations[2]:

$$\nabla_i|\mathring{M}|=k\mathring{u}_i, \qquad \mathring{u}^i\nabla_i\mathring{u}^j=k\mathring{u}^j, \qquad \varDelta_i\mathring{u}^i=4k|\mathring{M}|,$$

(3.11) can be rewritten as follows:

$$\nabla_i M^{ij}=\frac{m_0}{|\mathring{M}|}(\nabla_i u^i)\mathring{u}^j. \tag{3.12}$$

Therefore, if we normalize the arbitrary constant a such that

$$\nabla_i u^i=0, \tag{3.13}$$

then $\nabla_i M^{ij}=0$, and conversely; also, if $\nabla_i u^i\neq 0$, then $\nabla_i M^{ij}\neq 0$, and vice versa. In fact, a which satisfies (3.13) is given by

(1) Since, in our cosmological theory, all the nebulae were smeared over and accordingly the internal construction of a nebula was put aside out of consideration, it may be natural to proceed with this assumption. cf. Eddington: The Mathematical Theory of Relativity p. 121.

(2) H. Takeno: this Journ., 8 (1938), 223, (W. G. No. 30).

$$\alpha = \mathring{M}^{-4}\Phi\left(\frac{1}{pe^{-kt}+qe^{kt}}\sqrt{\frac{1-kr}{1+kr}}kr\right),$$

where Φ is an arbitrary function of the argument.

Next, (3.13) can be rewritten as follows:

$$\frac{\partial}{\partial x^1}(\sqrt{-g}\,\sigma v^1)+\frac{\partial}{\partial x^2}(\sqrt{-g}\,\sigma v^2)+\frac{\partial}{\partial x^3}(\sqrt{-g}\,\sigma v^3)=-\frac{\partial}{\partial x^4}(\sqrt{-g}\,\sigma),$$

particularly for proper coordinates,

$$\frac{\partial}{\partial x^1}(\sigma_0 v^1)+\frac{\partial}{\partial x^2}(\sigma_0 v^2)+\frac{\partial}{\partial x^3}(\sigma_0 v^3)=-\frac{\partial \sigma_0}{\partial x^4}.$$

Performing integration over the surface S of which the volume is V,

$$\iint_S \sigma_0 v_n dS = -\frac{\partial N_0}{\partial t}, \tag{3.14}$$

where N_0 is the number of particles in V, and v_n is the component of the velocity of particle outward normal to S.

In (3.14), the left-hand side shows the number of particles which flow out through the surface, and the right-hand side shows the number of particles which diminish in unit time in V. Therefore (3.14) shows that there occurs no annihilation or creation of particles in V. Moreover, we have $\iint_S \sigma_0 v_n dS+\frac{\partial N_0}{\partial t} \gtrless 0$, according as $\nabla_i u^i \gtrless 0$. In the upper case, there occurs the creation of particles in V, and in the lower case, the annihilation of particles.

Thus we have following conclusions:

(I) *According to the value of* α *which makes* $\nabla_i u^i \gtreqless 0$, *there occurs the creation, conservation, or annihilation of material particles.*

(II) *In our model of the universe, the conservation of material energy is equivalent with that of the number of particles.*

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