Relative Dimensionality in Operator Rings.

By

Fumitomo MAEDA.

(Received Jan. 15, 1941.)

In a Hilbert space, let **M** be a ring containing 1. We write $\mathfrak{M} \sim \mathfrak{N}$ $(\ldots . \mathbf{M})$ if a partially isometric operator $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$ and $\mathfrak{N}$ respectively. When **M** is a factor, F. J. Murray and J. v. Neumann have proved the following comparability theorem: "If $\mathfrak{M}, \mathfrak{N} \eta \mathbf{M}$, then either $\mathfrak{M} \sim \mathfrak{N}' \subset \mathfrak{N}$ or $\mathfrak{N} \sim \mathfrak{M}' \subset \mathfrak{M}$."(1)

In the present paper I shall investigate the case where **M** is not a factor, and obtain the same results (cf. Theorems I–IV below) as those in reducible continuous geometry.

From this fact we may conjecture that with respect to dimensionality there is a lattice theory which contains both the continuous geometry and the operator rings.

1. In a Hilbert space $\mathfrak{H}$, let **M** be a ring containing 1. Denote by **E** the set of all projections E belonging to **M**. When $EF=FE=E$, we write $E \leqq F$. Let $\mathfrak{M}, \mathfrak{N}$ be the ranges of E, F respectively, then $E \leqq F$ if and only if $\mathfrak{M} \subset \mathfrak{N}$. Hence **E** is a partially ordered system with the order $\leqq$. Since $E=P_{\mathfrak{M}} \in \mathbf{E}$ if and only if $U\mathfrak{M}=\mathfrak{M}$ for every unitary $U \in \mathbf{M}'$,(2) it is evident that E is a lattice, where the join $P_{\mathfrak{M}} \vee P_{\mathfrak{N}}$ is the projection whose range is $[\mathfrak{M}, \mathfrak{N}]$, and the meet $P_{\mathfrak{M}} \wedge P_{\mathfrak{N}}$ is the projection whose range is $\mathfrak{M} \cdot \mathfrak{N}$. $E \vee F=E+F$ if and only if $EF=0$ or $FE=0$, and in this case $E \perp F$.(3) $E \wedge F=EF$ if and only if, $EF=FE$. 0 and 1 are the zero and unit elements of **E**. If $E \leqq F$, then $F-E$ belongs to **E**. And

$$E \vee (F-E)=F, \qquad E \wedge (F-E)=0 .$$

Hence **E** is a complemented lattice. But **E** is not necessarily modular. For example, when **M** is the set of all bounded operators in $\mathfrak{H}$, then **E** is not modular.(4)

We write $\mathfrak{M} \sim \mathfrak{N}$ $(\ldots . \mathbf{M})$, and for $E=P_{\mathfrak{M}}$, $F=P_{\mathfrak{N}}$, $E \sim F$ $(\ldots . \mathbf{M})$, if a partially isometric $U \in \mathbf{M}$ exists, the initial and final sets of which are $\mathfrak{M}$

(1) Murray and v. Neumann [1], Lemma 6.2.3. The numbers in square brackets refer to the list given at the end of this paper.

(2) Murray and v. Neumann [1], 141.

(3) $E \perp F$ means that the ranges of E and F are orthogonal.

(4) Cf. G. Birkhoff and J. v. Neumann, *The Logic of Quantum Mechanics*, Annals of Math. **37** (1936), 832.

and $\mathfrak{N}$ respectively. We say that $\mathfrak{M}, \mathfrak{N}$ or E, F have the same relative dimension with respect to **M**. (If no misunderstanding is possible we will omit the (.... **M**).) The relation $\sim$ is reflexive, symmetric, and transitive.(1)

DEFINITION 1.1. The *center* of **E** is the set of projections $E \in \mathbf{E}$ which is carried into $[1, 0]$ under an isomorphism of **E** with a product.(2)

DEFINITION 1.2. A projection $E \in \mathbf{E}$ is *neutral* if and only if the triple $\{E, E_1, E_2\}$ generates a distributive sublattice of **E** for every $E_1, E_2 \in \mathbf{E}$.(3)

DEFINITION 1.3. Let $E_1, E_2 \in \mathbf{E}$. If $E \wedge (E_1 \vee E_2) = (E \wedge E_1) \vee (E \wedge E_2)$ for every $E \in \mathbf{E}$, then we write $(E_1, E_2)D$.

LEMMA 1.1. *When $E, F \in \mathbf{E}$ and $E \wedge F = 0$, there exists a complement E' of E such that $F \leqq E'$.*

PROOF. Let $F_1 = 1 - (E \vee F)$ and $E' = F \vee F_1 = F + F_1$. Then $E \vee E' = 1$. Let f be any element in the range of $E \wedge E'$. Then $f = E'f = (F + F_1)f$. Since $f = Ef$ and $E \perp F_1$, we have $F_1 f = 0$. Therefore $f = Ff$. On the other hand, $f = Ef$ and $E \wedge F = 0$. Hence $f = 0$. Consequently $E \wedge E' = 0$.

THEOREM I. *Let $\bar{\mathbf{E}}$ be the set of all projections in $\mathbf{M} \cdot \mathbf{M}'$. Then the following assertions are equivalent: (α) $E \in \bar{\mathbf{E}}$, (β) E is in the center of* **E**, *(γ) E has a complement $E' \in \mathbf{E}$, for which $(E, E')D$, (δ) E has a unique complement, (ϵ) E is neutral.*(4)

PROOF. Since $(\beta) \leftrightarrow (\epsilon)$ follows from Birkhoff [2] Theorem 6, we shall prove $(\alpha) \rightarrow (\beta) \rightarrow (\gamma) \rightarrow (\delta) \rightarrow (\alpha)$.

$(\alpha) \rightarrow (\beta)$. Let $E \in \bar{\mathbf{E}}$, and E' be the complement of E. We shall prove that **E** is the product of the sublattices of $F_1 \leqq E$ and $F_2 \leqq E'$. Since $EE' = E'E = E \wedge E' = 0$, we have $E \perp E'$. Let $[F_1, F_2] = F_1 \vee F_2$, then

$$[F_1, F_2] \vee [G_1, G_2] = (F_1 \vee F_2) \vee (G_1 \vee G_2) = [F_1 \vee G_1, F_2 \vee G_2].$$

It is evident that

$$\begin{aligned}[F_1, F_2] \wedge [G_1, G_2] &= (F_1 \vee F_2) \wedge (G_1 \vee G_2) \geqq (F_1 \wedge G_1) \vee (F_2 \vee G_2) \\ &= [F_1 \wedge G_1, F_2 \wedge G_2].\end{aligned}$$

Let f be any element in the range of $(F_1 \vee F_2) \wedge (G_1 \vee G_2)$. Then $f = (F_1 \vee F_2)f = (F_1 + F_2)f$ and $f = (G_1 \vee G_2)f = (G_1 + G_2)f$. Therefore $f = (F_1 + F_2)(G_1 + G_2)f = F_1 G_1 f + F_2 G_2 f$. Similarly $f = G_1 F_1 f + G_2 F_2 f$. Since $F_1 G_1, G_1 F_1 \leqq E$

(1) Murray and v. Neumann [1], 151.

(2) Cf. Birkhoff [1], 23. The *product* of two partially ordered systems X and Y is the system W whose elements are the couples $[x, y]$ with $x \in X$, $y \in Y$, $[x, y] \geqq [x', y']$ meaning that $x \geqq x'$ and $y \geqq y'$.

(3) Birkhoff [2], 702.

(4) A similar theorem holds good in reducible continuous geometry. Cf. v. Neumann [1], Theorems 5.2 and 5.3.

and F_2G_2, $G_2F_2 \leqq E'$ and $E \perp E'$, we have $F_1G_1f = G_1F_1f$, and it is contained in the range of $F_1 \wedge G_1$, and $F_2G_2f = G_2F_2f$ is contained in the range of $F_2 \wedge G_2$. Consequently f is contained in the range of $(F_1 \wedge G_1) + (F_2 \wedge G_2) = (F_1 \wedge G_1) \vee (F_2 \wedge G_2)$. That is, $(F_1 \vee F_2) \wedge (G_1 \vee G_2) \leqq (F_1 \wedge G_1) \vee (F_2 \wedge G_2)$. Thus we have $[F_1, F_2] \wedge [G_1, G_2] = [F_1 \wedge G_1, F_2 \wedge G_2]$.

$(\beta) \rightarrow (\gamma)$. Under the correspondence $F \leftrightarrow [F_1, F_2]$, where $F_1 \leqq E$, $F_2 \leqq E'$, we have $F \wedge E \leftrightarrow [F_1, 0]$ and $F \wedge E' \leftrightarrow [0, F_2]$. Hence $F = (F \wedge E) \vee (F \wedge E')$.

$(\gamma) \rightarrow (\delta)$. Let E' be a complement of E. Since $1 - E = \{(1-E) \wedge E\} \vee \{(1-E) \wedge E'\} = (1-E) \wedge E'$, we have $1 - E \leqq E'$. Therefore $(1-E)E' = E'(1-E) = 1 - E$, that is, $EE' = E'E$. Consequently $EE' = E \wedge E' = 0$ and $E' = 1 - E$. Hence E has a unique complement $1 - E$.

$(\delta) \rightarrow (\alpha)$. Let F be any projection in $\mathbf{E}$, and write $F_1 = F - (F \wedge E)$. Since $E \wedge F_1 \leqq E \wedge F$ and $F_1 \perp E \wedge F$, we have $E \wedge F_1 = 0$. Since E has a unique complement E', by Lemma 1.1 $F_1 \leqq E'$. Then $FE' = F_1E' + (F \wedge E)E' = F_1$, and similarly $E'F = F_1$. That is, $FE' = E'F$ for every $F \in \mathbf{E}$. Hence $E' \in \overline{\mathbf{E}}$, that is, $E \in \overline{\mathbf{E}}$.

Corollary. *$\overline{\mathbf{E}}$ is a Boolean algebra.*

Proof. This follows from the property of the center. Cf. Birkhoff [2], Theorems 5 and 6. But we can easily prove directly, since, in $\overline{\mathbf{E}}$, $E \vee F = E + F - EF$ and $E \wedge F = EF$.

2. The next lemma has an interest of its own.

Lemma 2.1. *For any $A \in \mathbf{M}$ there exists $E_0 \in \overline{\mathbf{E}}$, such that*

(i) $AE_0 = E_0A = A$,

(ii) *for any $A' \in \mathbf{M}'$, $AA' = 0$ if and only if $A'(1-E_0) = (1-E_0)A' = A'$.*(1)

Proof. (i) Let $\mathfrak{M}$ be the set of all those g for which $AXg = 0$ for every $X \in \mathbf{M}$. $\mathfrak{M}$ is obviously linear and closed. Assume now $g \in \mathfrak{M}$; then

(α) if $X_0 \in \mathbf{M}$, then for every $X \in \mathbf{M}$ $AXX_0g = 0$, thus $X_0g \in \mathfrak{M}$;

(β) if $X_0' \in \mathbf{M}'$, then for every $X \in \mathbf{M}$ $AXX_0'g = X_0'AXg = 0$, thus $X_0'g \in \mathfrak{M}$.

From (α) we have $X_0P_{\mathfrak{M}}f \in \mathfrak{M}$ for every $f \in \mathfrak{H}$; hence $P_{\mathfrak{M}}X_0P_{\mathfrak{M}} = X_0P_{\mathfrak{M}}$. Replacing X_0 by X_0^*, and applying *, gives $P_{\mathfrak{M}}X_0P_{\mathfrak{M}} = P_{\mathfrak{M}}X_0$. Thus $X_0P_{\mathfrak{M}} = P_{\mathfrak{M}}X_0$ for every $X_0 \in \mathbf{M}$. Hence $P_{\mathfrak{M}} \in \mathbf{M}'$. Similarly, from ($\beta$), we have $P_{\mathfrak{M}} \in \mathbf{M}$. Consequently $P_{\mathfrak{M}} \in \mathbf{M} \cdot \mathbf{M}'$. That is, $P_{\mathfrak{M}} \in \overline{\mathbf{E}}$.

Put $E_0 = 1 - P_{\mathfrak{M}}$. Then $E_0 \in \overline{\mathbf{E}}$. Since $(A^*f, g) = (f, Ag) = 0$ for every $f \in \mathfrak{H}$, $g \in \mathfrak{M}$, $A^*f \perp \mathfrak{M}$. Thus $E_0A^* = A^*$. Consequently $AE_0 = A$.

(ii) Let $A' \in \mathbf{M}'$. If $AA' = 0$, then $AXA'f = AA'Xf = 0$ for every $X \in \mathbf{M}$. Hence $A'f \in \mathfrak{M}$. That is, $(1-E_0)A' = A'$. Conversely, if $A' = (1-E_0)A'$, then $AA' = AE_0(1-E_0)A' = 0$.

(1) When $\mathbf{M}$ is a factor, Lemma 2.1 becomes Murray and v. Neumann [1], Theorem III Corollary.

LEMMA 2.2. *For* $A \in \mathbf{M}$, E_0 *obtained in Lemma* 2.1 *is the smallest projection of all* $F \in \bar{\mathbf{E}}$ *such that* $AF=FA=A$.

PROOF. If $F \in \bar{\mathbf{E}}$ and $AF=FA=A$, then, since $1-F \in \mathbf{M}'$ and $A(1-F)=0$, we have, from Lemma 2.1, $(1-F)(1-E_0)=(1-E_0)(1-F)=1-F$; that is, $FE_0=E_0F=E_0$, Thus $E_0 \leqq F$.

DEFINITION 2.1. From Lemma 2.2, when A is a projection $E \in \mathbf{E}$, E_0 is the smallest projection of all $F \in \bar{\mathbf{E}}$ such that $E \leqq F$. We call it a *central envelope* of E, and denote it by $\bar{E}$.

LEMMA 2.3. *If* $E_1 \sim E_2$, *then* $\bar{E}_1=\bar{E}_2$.

PROOF. If $E_1 \sim E_2$, then there exists a partially isometric $U \in \mathbf{M}$, such that $U^*U=E_1$, $UU^*=E_2$, $U=UE_1$, $U^*=U^*E_2$. Since $U\bar{E}_1=UE_1\bar{E}_1=UE_1=U$, we have $E_2\bar{E}_1=UU^*\bar{E}_1=U\bar{E}_1U^*=UU^*=E_2$. That is, $E_2 \leqq \bar{E}_1$. Hence $\bar{E}_2 \leqq \bar{E}_1$. Similarly, $\bar{E}_1 \leqq \bar{E}_2$. Consequently $\bar{E}_1=\bar{E}_2$.

LEMMA 2.4. *If* $\bar{E}_1\bar{E}_2=0$, *then* $F_1 \leqq E_1$, $F_2 \leqq E_2$, $F_1 \sim F_2$ *implies* $F_1=F_2=0$.

PROOF. Since $\bar{F}_1 \leqq \bar{E}_1$, $\bar{F}_2 \leqq \bar{E}_2$, we have $\bar{F}_1\bar{F}_2=0$. But by Lemma 2.3, $\bar{F}_1=\bar{F}_2$. Hence $\bar{F}_1=\bar{F}_2=0$; that is, $F_1=F_2=0$.

LEMMA 2.5. *If* $\bar{E}_1\bar{E}_2 \neq 0$, *then there exist* F_1, F_2 *such that* $0 \neq F_1 \leqq E_1$, $0 \neq F_2 \leqq E_2$ and $F_1 \sim F_2$.

PROOF. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the ranges of E_1 and E_2 respectively. For every $f \in \mathfrak{H}$ $E_f^{\mathbf{M}} \in \mathbf{M}'$.(1) Assume that $E_2E_f^{\mathbf{M}}=0$ for every $f \in \mathfrak{M}$. By Lemma 2.1 and Definition 2.1, $E_f^{\mathbf{M}} \leqq 1-\bar{E}_2$. That is, $f \in \mathfrak{M}_f^{\mathbf{M}} \subset \mathfrak{H}-\bar{\mathfrak{N}}$ for every $f \in \mathfrak{M}$, where $\bar{\mathfrak{N}}$ is the range of $\bar{E}_2$, Therefore $\mathfrak{M} \subset \mathfrak{H}-\bar{\mathfrak{N}}$; that is, $E_1 \leqq 1-\bar{E}_2$. Hence $\bar{E}_1 \leqq 1-\bar{E}_2$, which contradicts $\bar{E}_1\bar{E}_2 \neq 0$. Consequently there exists an $f \in \mathfrak{M}$ such that $E_2E_f^{\mathbf{M}} \neq 0$. Choose $g \in \mathfrak{N}\cdot\mathfrak{M}_f^{\mathbf{M}}$ such that $\|g\|=1$. As $g \in \mathfrak{M}_f^{\mathbf{M}}$, therefore an $A \in \mathbf{M}$ with $\|g-Af\|<1$ exists. Let $\mathfrak{P}=[\text{Range } (E_2AE_1)^*]$, $\mathfrak{Q}=[\text{Range } E_2AE_1]$, Then, as in the last half of the proof of Murray and v. Neumann [1], Lemma 6.2.2, $F_1=P_{\mathfrak{P}}$, $F_2=P_{\mathfrak{Q}}$ meet our requirements.

3. Let $E \in \mathbf{M}$, and $\mathfrak{M}$ be the range of E. Consider those operators $A \in \mathbf{M}$ which are reduced by $\mathfrak{M}$ and from their parts in $\mathfrak{M}$, $A_{(\mathfrak{M})}$. These are bounded operators in $\mathfrak{M}$. Denote the set of all these $A_{(\mathfrak{M})}$ by $\mathbf{M}_{(\mathfrak{M})}$. Similarly denote the set of all $F_{(\mathfrak{M})}$ ($F \in \mathbf{E}$ and reduced by $\mathfrak{M}$) by $\mathbf{E}_{(\mathfrak{M})}$.(2)

LEMMA 3.1. *The correspondence* $X \rightleftarrows X_{(\mathfrak{M})}$ *is a one-to-one mapping*

(1) Murray and v. Neumann [1], Lemma 5.1.1.
(2) Cf. Murray and v. Neumann [1], Definition 11.3.1.

and even algebraic ring-isomorphism of the following rings on each other:

(i) *Of the ring of all* $A \in \mathbf{M}$ *with* $EA = AE = A$ *on all* $\mathbf{M}_{(\mathfrak{M})}$.

(ii) *Of the ring of all* $A' \in \mathbf{M}'$ *with* $EA' = A'E = A'$ *on all* $(\mathbf{M}_{(\mathfrak{M})})'$.

PROOF. This lemma is Murray and v. Neumann [1], Lemma 11.3.3 with a slight modification.

LEMMA 3.2. *If* F *runs over all projections* $F \in \mathbf{E}$, $F \leqq E$, *then* $F_{(\mathfrak{M})}$ *runs over all projections* $\in \mathbf{M}_{(\mathfrak{M})}$. $F_{(\mathfrak{M})} \sim G_{(\mathfrak{M})}$ $(\ldots . \mathbf{M}_{(\mathfrak{M})})$, $(F, G \in \mathbf{E}, \leqq E)$ *is equivalent to* $F \sim G$ $(\ldots . \mathbf{M})$.

PROOF. This is Murray and v. Neumann [1], Lemma 11.3.5 (i), which holds good also without the assumption that $\mathbf{M}$ is a factor.

LEMMA 3.3. *Let* $\mathfrak{M}$, E *be as above. If* $F \leqq E$, *then* $\overline{F_{(\mathfrak{M})}} = \bar{F}_{(\mathfrak{M})}$.(1)

PROOF. By Lemma 3.1, the correspondence $X \rightleftarrows X_{(\mathfrak{M})}$ is an algebraic ring-isomorphism of the ring of all $A \in \mathbf{M} \cdot \mathbf{M}'$ with $EA = AE = A$ on all $\mathbf{M}_{(\mathfrak{M})} \cdot (\mathbf{M}_{(\mathfrak{M})})'$. Since $F \leqq \bar{F}$ and $E\bar{F} = \bar{F}E$, we have $F_{(\mathfrak{M})} \leqq \bar{F}_{(\mathfrak{M})}$, where $\bar{F}_{(\mathfrak{M})} \in \mathbf{M}_{(\mathfrak{M})} \cdot (\mathbf{M}_{(\mathfrak{M})})'$. Next take $G_{(\mathfrak{M})} \in \mathbf{M}_{(\mathfrak{M})} \cdot (\mathbf{M}_{(\mathfrak{M})})'$ such that $F_{(\mathfrak{M})} \leqq G_{(\mathfrak{M})}$. This means that $F \leqq G$ for some $G \in \mathbf{M} \cdot \mathbf{M}'$ $EG = GE = G$. Therefore $\bar{F} \leqq G$, and $\bar{F}_{(\mathfrak{M})} \leqq G_{(\mathfrak{M})}$. Consequently $\overline{F_{(\mathfrak{M})}} = \bar{F}_{(\mathfrak{M})}$.

LEMMA 3.4. $(E_1, E_2)D$ *is equivalent to* $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$,(2) $\mathfrak{M}$ *being the range of* $E_1 \vee E_2$.

PROOF. $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$ means that

$$F_{(\mathfrak{M})} = (F_{(\mathfrak{M})} \wedge E_{1(\mathfrak{M})}) \vee (F_{(\mathfrak{M})} \wedge E_{2(\mathfrak{M})}) \quad \text{for every} \quad F_{(\mathfrak{M})} \in \mathbf{E}_{(\mathfrak{M})},$$

that is, $$F = (F \wedge E_1) \vee (F \wedge E_2) \quad \text{for every} \quad F \in \mathbf{E},\ F \leqq E. \tag{1}$$

Therefore $(E_1, E_2)D$ implies $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$.

Next assume that $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$ and take any $F \in \mathbf{E}$. Then, from (1), $F \wedge E = (F \wedge E \wedge E_1) \vee (F \wedge E \wedge E_2) = (F \wedge E_1) \vee (F \wedge E_2)$. Therefore $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$ implies $(E_1, E_2)D$.

THEOREM II. *For any* $E_1, E_2 \in \mathbf{E}$, *the following three assertions are equivalent:* (α) $\bar{E}_1\bar{E}_2 = 0$; (β) $E_1 \wedge E_2 = 0$, $(E_1, E_2)D$; (γ) $F_1 \leqq E_1$, $F_2 \leqq E_2$, $F_1 \sim F_2$ *implies* $F_1 = F_2 = 0$.(3)

PROOF. We shall prove that $(\alpha) \rightarrow (\beta) \rightarrow (\gamma) \rightarrow (\alpha)$. Assume $\bar{E}_1\bar{E}_2 = 0$. Since $\bar{E}_1 \perp \bar{E}_2$, we have $\bar{E}_{1(\mathfrak{M})} \perp \bar{E}_{2(\mathfrak{M})}$, where $\mathfrak{M}$ is the range of $E_1 \vee E_2$.

(1) $\overline{F_{(\mathfrak{M})}}$ means the central envelope of $F_{(\mathfrak{M})}$ in $\mathbf{E}_{(\mathfrak{M})}$.

(2) $D_{(\mathfrak{M})}$ means distributivity in $\mathbf{E}_{(\mathfrak{M})}$.

(3) This theorem holds good also in reducible continuous geometry where (α) means $\bar{E}_1 \wedge \bar{E}_2 = 0$. $(\gamma) \rightarrow (\beta)$ and $(\beta) \rightarrow (\alpha)$ are provided in v. Neumann [1], Theorem 5.7, and v. Neumann [2], Lemma 1.1 respectively. $(\alpha) \rightarrow (\gamma)$ is proved as follows: Let $F_1 \leqq E_1$, $F_2 \leqq E_2$, $F_1 \sim F_2$. Then, by v. Neumann [2], Theorem 1.4, $\bar{F}_1 = \bar{F}_2$. Hence $\bar{F}_1 \leqq \bar{E}_1 \wedge \bar{E}_2 = 0$; that is, $F_1 = 0$. Similarly $F_2 = 0$.

On the other hand, from $E_1 \vee E_2 = E$, we have $E_{1(\mathfrak{M})} \vee E_{2(\mathfrak{M})} = E_{(\mathfrak{M})}$. Hence $\overline{E_{1(\mathfrak{M})}} \vee \overline{E_{2(\mathfrak{M})}} = E_{(\mathfrak{M})}$. Since, by Lemma 3.3, $E_{1(\mathfrak{M})} \leqq \overline{E_{1(\mathfrak{M})}} = \bar{E}_{1(\mathfrak{M})}$, $E_{2(\mathfrak{M})} \leqq \overline{E_{2(\mathfrak{M})}} = \bar{E}_{2(\mathfrak{M})}$, it must follow that $E_{1(\mathfrak{M})} = \overline{E_{1(\mathfrak{M})}}$, $E_{2(\mathfrak{M})} = \overline{E_{2(\mathfrak{M})}}$. Hence, from Theorem I, $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$. Consequently, by Lemma 3.4, $(E_1, E_2)D$. Since $E_1 \perp E_2$, we have $E_1 \wedge E_2 = 0$.

Next assume (β). Then, from Lemmas 3.1 and 3.4, we have $E_{1(\mathfrak{M})} \wedge E_{2(\mathfrak{M})} = 0_{(\mathfrak{M})}$, $(E_{1(\mathfrak{M})}, E_{2(\mathfrak{M})})D_{(\mathfrak{M})}$. Therefore, from Theorem I, $E_{1(\mathfrak{M})} = \overline{E_{1(\mathfrak{M})}}$, $E_{2(\mathfrak{M})} = \overline{E_{2(\mathfrak{M})}}$. Hence $\overline{E_{1(\mathfrak{M})}}\,\overline{E_{2(\mathfrak{M})}} = 0_{(\mathfrak{M})}$. When $F_1 \leqq E_1$, $F_2 \leqq F_2$, $F_1 \sim F_2$ $(\ldots.\mathbf{M})$, from Lemmas 3.1 and 3.2, $F_{1(\mathfrak{M})} \leqq E_{1(\mathfrak{M})}$, $F_{2(\mathfrak{M})} \leqq E_{2(\mathfrak{M})}$, $F_{1(\mathfrak{M})} \sim F_{2(\mathfrak{M})}$ $(\ldots.\mathbf{M}_{(\mathfrak{M})})$. Therefore, by Lemma 2.4, $F_{1(\mathfrak{M})} = F_{2(\mathfrak{M})} = 0_{(\mathfrak{M})}$. Consequently $F_1 = F_2 = 0$.

$(\gamma) \rightarrow (\alpha)$ follows from Lemma 2.5.

THEOREM III. *For every* $E_1, E_2 \in \mathbf{E}$, *there exist* $E_1', E_1'', E_2', E_2'' \in \mathbf{E}$ *such that*

(i) $E_1' \vee E_1'' = E_1$, $E_1' \wedge E_1'' = 0$ *and* $E_2' \vee E_2'' = E_2$, $E_2' \wedge E_2'' = 0$,

(ii) $E_1' \sim E_2'$ *and* $\overline{E_1''}\,\overline{E_2''} = 0$.(1)

PROOF. This theorem can be proved by the same methods as v. Neumann [2], Lemma 2.1, and Murray and v. Neumann [1], Lemma 6.2.3.

THEOREM IV. $\mathbf{M}$ *is a factor if and only if any two* $E_1, E_2 \in \mathbf{E}$ *are comparative, that is, either* $E_1 \sim E_2' \leqq E_2$ *or* $E_2 \sim E_1' \leqq E_1$.(2)

PROOF. When $\mathbf{M}$ is a factor, in Theorem III $\overline{E_1''} = 0$ or $\overline{E_2''} = 0$; that is, $E_1'' = 0$ or $E_2'' = 0$. Consequently $E_1 \sim E_2' \leqq E_2$ or $E_2 \sim E_1' \leqq E_1$.

When $\mathbf{M}$ is not a factor, any two $E_1, E_2 \in \bar{\mathbf{E}}$ $(\neq 0, 1)$ with $E_1 E_2 = 0$ are not comparative by Theorem II.

References.

G. Birkhoff [1], *Lattice Theory*, New York, 1940.

[2], *Neutral Elements in General Lattices*, Bull. Amer. Math. Soc., **46** (1940), 702–705.

F. J. Murray and J. v. Neumann [1], *On Rings of Operators*, Annals of Math., **37** (1936), 116–229.

J. v. Neumann [1], *Continuous Geometry* (mimeographed note of lecture given at Princeton), Part I, 1935–1936.

[2], Do., Part III, 1936–1937.

HIROSIMA UNIVERSITY.

(1) Corresponding theorem in continuous geometry, cf. v. Neumann [2], Theorem 2.1.

(2) Similarly we can say that continuous geometry is irreducible if and only if any two elements are comparative.