EQUATIONS OF SCHRÖDER (Continued)

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Chapter III. Correspondence of the eigen values.

§ 1. Preliminaries.

In Chap. II, we have noticed that, when (2.12) holds, (2.14) always holds, however, when (2.14) holds, (2.12) does not necessarily hold. In this chapter, we study the condition that, for the given values of λ_i, there exists a set of values μ_i such that, for any sets of $(p_1, p_2, \dots, p_R)$ satisfying Chap. II (2.14), there always hold Chap. II (2.12), namely Chap. II (2.12) and (2.14) are in one-to-one correspondence.

We assume that either $0<|\lambda_i|<1$ or $|\lambda_i|>1$. Then, by the suitable arrangement of λ_i, as seen from Chap. I § 1, the relations Chap. II (2.14) which really hold are of the forms as follows:

$$\lambda = \lambda_1^{p_1}\lambda_2^{p_2}\cdots\cdots\lambda_N^{p_N} \tag{1.1}$$

where λ is an eigen value λ_i such that $i>N$. We denote by $\mathring{\mu}_i$ arbitrary value of μ_i determined by Chap. II (2.8) for given λ_i. Then, for one set $(\mathring{p}_1, \mathring{p}_2, \dots, \mathring{p}_N)$ satisfying (1.1),

$$\sum_{i=1}^{N} \mathring{\mu}_i \mathring{p}_i = \mathring{\mu} , \tag{1.2}$$

where $\mathring{\mu}$ is a suitably determined value of μ corresponding to λ. For any set of $(p_1, p_2, \dots, p_N)$ satisfying (1.1),

$$\sum_{i=1}^{N} \mathring{\mu}_i p_i = \mathring{\mu} + n\frac{2\pi}{t_0}\sqrt{-1} , \tag{1.3}$$

where n is an integer corresponding to the set of $(p_1, p_2, \dots, p_N)$. Put $\mathring{\mu}_i = A^i + B^i\sqrt{-1}$ and $\mathring{\mu} = A + B\sqrt{-1}$. Then, making use of the convention of tensor calculus, it follows that

$$(p_i - \mathring{p}_i)A^i = 0 , \quad (p_i - \mathring{p}_i)B^i = \frac{2\pi}{t_0}n . \tag{1.4}$$

The sets of $(p_1, p_2, \dots, p_N)$ satisfying (1.1) are determined by (1.4). Then, for all such sets of $(p_1, p_2, \dots, p_N)$, we seek for n^i and m such that $\sum_{i=1}^{N}(\mu_i + \frac{2\pi}{t_0}n^i\sqrt{-1})p_i = \mathring{\mu} + m\frac{2\pi}{t_0}\sqrt{-1}$, i.e.

$$p_i n^i = m - n \ . \tag{1.5}$$

Making use of the theorems on linear integral equations, we solve (1.4) and we study the condition that there exist n^i and m satisfying (1.5).

§ 2. Lemmas on integral equations.

For any given numbers a^i, we consider the integral equation as follows:

$$a^i x_i = 0 \ , \tag{2.1}$$

i summed over $1, 2, \dots, N$. We assume that, for any integral solution r_i of (2.1), $r_{M_1+1} = r_{M_1+2} = \cdots = r_N = 0$, and there exists an integral solution r_i such that $r_{M_1} \neq 0$. Among the solutions where $r_{M_1} \neq 0$, we choose a solution such that r_{M_1} is positive and smallest. Next, we consider the integral solutions such that $r_{M_1} = r_{M_1+1} = \cdots = r_N = 0$, and assume that, for any of them, $r_{M_2+1} = \cdots = r_{M_1} = \cdots = r_N = 0$ and there exists a solution such that $r_{M_2} \neq 0$. From these solutions, we choose a solution such that r_{M_2} is positive and smallest. Repeating this process, we have a set of linearly independent integral solutions, which we write as follows:

$$\begin{cases} r_i^1 = (r_1^1, r_2^1, \dots, r_{N_1}^1, 0, \dots\dots\dots\dots\dots\dots, 0), \\ r_i^2 = (r_1^2, r_2^2, \dots\dots\dots\dots, r_{N_2}^2, 0, \dots\dots\dots, 0), \\ \vdots \\ r_i^K = (r_1^K, r_2^K, \dots\dots\dots\dots\dots\dots\dots, r_{N_K}^K, 0, \dots\dots, 0), \end{cases} \tag{2.2}$$

where $N_K = M_1, N_{K-1} = M_2, \dots$, consequently $N_1 < N_2 < \cdots < N_K$. The number K of these solutions is evidently at most N. Then, we can easily prove that any integral solution of (2.1) is expressed as follows:

$$x_i = k_a r_i^a \ , \tag{2.3}$$

where k_a are integers. Namely the set of solutions (2.2) constitutes a basis of the integral solutions of (2.1).

Next, we consider the simultaneous equations as follows:

$$a_\alpha^i x_i = 0 \ , \quad (\alpha = 1, 2, \dots, m) \ . \tag{2.4}$$

Then, by induction, we can prove that there exists a basis of the integral solutions of (2.4) of the forms (2.2). For $m=1$, we have seen already. We assume that, for $m-1$ equations, there exists a basis. Then, for $a_\alpha^i x_i = 0$ $(\alpha = 1, 2, \dots, m-1)$, there exists a basis r_i^a of the forms (2.2), consequently any integral solution x_i of them is expressed as $x_i = k_a r_i^a$. Substituting these into $a_m^i x_i = 0$, we have $a_m^i r_i^a k_a = 0$. For this equation, there exists a basis s_a^λ of the forms (2.2), consequently $k_a=l_\lambda s_a^\lambda$. Then, for any integral solution of (2.4), we have: $x_i=l_\lambda s_a^\lambda r_i^a$. Therefore, $t_i^\lambda=s_a^\lambda r_i^a$ constitutes a basis for (2.4), and evidently the set of t_i^λ has the same forms and characters as (2.2). Thus we see that there exists a basis of the forms (2.2) for the simultaneous equations of any number.

Next we study the properties of the basis of the integral equations.

Lemma 1. *We assume that the m-th determinant divisor of the matrix* $A=\|a_i^\alpha\|$ *is unity, where* $\alpha=(1,2,\dots,m)$ *indicates the row and* $i=(1,2,\dots,n)$ *the column and* $m<n$. *Then, for the given* a_i *and any prime number* $p\neq 1$, *in order that* $a_i\equiv c_\alpha a_i^\alpha \pmod p$, *it is necessary and sufficient that the* $(m+1)$*-th determinant divisor of the matrix* $B=\begin{pmatrix} a_i^\alpha \\ a_i \end{pmatrix}$ *is divisible by* p.

Proof. Put

$$L\equiv\begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^m & p & 0 & \cdots & 0 \\ a_2^1 & a_2^2 & \cdots & a_2^m & 0 & p & & \vdots \\ \vdots & & & & & & \ddots & 0 \\ a_n^1 & a_n^2 & \cdots & a_n^m & 0 & \cdots & 0 & p \end{pmatrix}, \quad M\equiv\begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^m & a_1 & p & 0 & \cdots & 0 \\ a_2^1 & a_2^2 & \cdots & a_2^m & a_2 & 0 & p & & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & & \ddots & 0 \\ a_n^1 & a_n^2 & \cdots & a_n^m & a_n & 0 & \cdots & 0 & p \end{pmatrix}.$$

Then, by the theorem of algebra, in order that $a_i^\alpha c_\alpha\equiv a_i \pmod p$, it is necessary and sufficient that L and M have a common rank ρ and both ρ-th determinant divisors are equal. Now, since $p\neq 0$, L and M have a common rank n. Any determinant of n-th order of L is as follows:

$$(1)\qquad \begin{vmatrix} a_1^{i_1} & \cdots & a_1^{i_s} & 0 & 0 & \cdots & 0 \\ a_2^{i_1} & & a_2^{i_s} & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & p & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & p & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & p \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_n^{i_1} & \cdots & a_n^{i_s} & 0 & 0 & & 0 \end{vmatrix} = \pm p^{n-s} \begin{vmatrix} a_{j_1}^{i_1} & \cdots\cdots & a_{j_1}^{i_s} \\ \vdots & & \vdots \\ a_{j_s}^{i_1} & \cdots\cdots & a_{j_s}^{i_s} \end{vmatrix}.$$

Now, from $m<n$, $n-s\geqq n-m>0$, consequently any determinant of n-th order of L is divisible by p^{n-m}. When $s=m$, the right-hand side of (1) is a product of p^{n-m} and the determinant of m-th order of A. However, by the assumption, G.C.M. of all the determinants of m-th order of A is unity. Thus, G.C.M. of all the determinants of n-th order of L is p^{n-m}, namely n-th determinant divisor of L is p^{n-m}.

If $a_i^\alpha c_\alpha\equiv a_i \pmod p$, then n-th determinant divisor of M must be p^{n-m}. Then the determinants of n-th order of the following forms of M must be divisible by p^{n-m}:

$$(2)\quad \begin{vmatrix} a_1^1 & a_1^2 \cdots a_1^m & a_1 & 0 \cdots 0 \\ a_2^1 & a_2^2 \quad a_2^m & a_2 & \vdots \\ \vdots & \vdots \quad\ \vdots & \vdots\ \ p & \vdots \\ \vdots & \vdots \quad\ \vdots & \vdots & p \\ \vdots & \vdots \quad\ \vdots & \vdots & \vdots \\ a_n^1 & a_n^2 \cdots a_n^m & a_n & 0 \cdots 0 \end{vmatrix} = \pm p^{n-m-1} \begin{vmatrix} a_{j_1}^1 & a_{j_1}^2 \cdots\cdots a_{j_1}^m & a_{j_1} \\ \vdots & \vdots \qquad \vdots & \vdots \\ a_{j_{m+1}}^1 & a_{j_{m+1}}^2 \cdots a_{j_{m+1}}^m & a_{j_{m+1}} \end{vmatrix}.$$

From this, we see that any determinant of $(m+1)$-th order of B must be divisible by p, namely $(m+1)$-th determinant divisor of B is divisible by p.

Conversely, we assume that $(m+1)$-th determinant divisor of B is divisible by p. Then, since all the determinants of n-th order of M which do not contain $(m+1)$-th column of M, are the determinants of L, their G.C.M. is p^{n-m}. The determinants of n-th order of M which contains $1, 2, \dots, m$, $(m+1)$-th columns are of the form (2), consequently, by our assumption, they are divisible by p^{n-m}. The other determinants of n-th order of M which contains $(m+1)$-th column of M are evidently divisible by p^{n-m}. Thus, ultimately, we see that G.C.M. of all the determinants of n-th order of M is p^{n-m}, namely n-th determinant divisor of M is equal to that of L. Thus there exist integers c_α such that $a_i^\alpha c_\alpha \equiv a_i \pmod p$. Q.E.D.

Lemma 2. *The K-th determinant divisor of $\| r_i^\alpha \|$ of (2.2) constituted by a basis for the simultaneous equations (2.4) is unity.*

Proof. We prove this lemma by induction. From the definition, $r_1^1, r_2^1, \dots, r_{N_1}^1$ are evidently relatively prime, namely, the first determinant divisor of $R_1 = \| r_i^1 \|$ is unity. We assume that the σ-th determinant divisor of

$$R_\sigma = \begin{pmatrix} r_1^1 & r_2^1 \cdots r_{N_1}^1 & 0 \cdots\cdots & & 0 \\ r_1^2 & \cdots\cdots\cdots & r_{N_2}^2 & 0 \cdots\cdots & 0 \\ \vdots & & & & \vdots \\ r_1^\sigma & \cdots\cdots\cdots & & r_{N_\sigma}^\sigma & 0 \cdots 0 \end{pmatrix}$$

is unity. If the $(\sigma+1)$-th determinant divisor of $R_{\sigma+1} = \begin{pmatrix} R_\sigma \\ r_i^{\sigma+1} \end{pmatrix}$ is not unity, then there exists a prime number $p \neq 1$ such that all the determinants of $(\sigma+1)$-th order of $R_{\sigma+1}$ are divisible by p. Then, by Lemma 1, there exist integers c_ρ such that $r_i^{\sigma+1} \equiv c_\rho r_i^\rho \pmod p$, ρ summed over $1, 2, \dots, \sigma$. Put $r_i^{\sigma+1} - c_\rho r_i^\rho = {}'r_i$, then ${}'r_i \equiv 0 \pmod p$. Now $r_{N_{\sigma+1}}^\rho = 0$, consequently ${}'r_{N_{\sigma+1}} = r_{N_{\sigma+1}}^{\sigma+1} \neq 0$. From the definition, it is evident that $a_x^i\, {}'r_i = 0$. Then, dividing ${}'r_i$ by p, we have r_i such that $a_x^i r_i = 0$ and $0 < r_{N_{\sigma+1}} < r_{N_{\sigma+1}}^{\sigma+1}$. This contradicts the properties of r_i^σ. Therefore $(\sigma+1)$-th determinant divisor of $R_{\sigma+1}$ is unity. Thus, by induction, we see that the K-th determinant divisor of $R_K = \| r_i^\alpha \|$ is unity. Q.E.D.

§ 3. Solution of (1.5).

Let the basis of the solutions of the forms given by § 2 of the following simultaneous equations be (t_i, t^λ)[1] $(\lambda=1, 2, \ldots, L)$:

(3.1) $$\begin{cases} A^i x_i = 0, \\ B^i x_i - \dfrac{2\pi}{t_0} x = 0. \end{cases}$$

Then, by § 2, for p_i and n satisfying (1.4), it follows that

(3.2) $$p_i = \mathring{p}_i + k_\lambda t_i^\lambda ,$$

(3.3) $$n = k_\lambda t^\lambda .$$

Substituting these into (1.5), we have:

(3.4) $$(\mathring{p}_i n^i - m) + k_\lambda (t_i^\lambda n^i + t^\lambda) = 0 .$$

This must hold for any integers k_λ such that p_i are non-negative. When $k_\lambda=0$, it must hold that

(3.5) $$\mathring{p}_i n^i = m .$$

Consequently,

(3.6) $$k_\lambda (t_i^\lambda n^i + t^\lambda) = 0 .$$

As seen from (1.1), the number of the sets of k_λ is finite. We denote them by k. Consider the equations $k_\lambda^\chi x^\lambda = 0$, and let the basis of the forms (2.2) of these equations be T_α^λ $(\alpha = 1, 2, \ldots, M)$. Then, from (3.6), $t_i^\lambda n^i + t^\lambda = l^\alpha T_\alpha^\lambda$, namely

(3.7) $$t_i n^i - T_\alpha^\lambda l^\alpha = -t^\lambda .$$

Put $\begin{pmatrix} t_1^1, & \cdots, & t_{N_1}^1, & 0, \cdots\cdots 0 & T_1^1 & T_2^1 \cdots & T_M^1 \\ t_1^2, & \cdots\cdots\cdots & , & t_{N_2}^2, \cdots 0 & T_1^2 & T_2^2 & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & \vdots \\ t_1^L & \cdots\cdots\cdots & & t_{N_L}^L & 0 & & T_M^L \end{pmatrix} \equiv \mathfrak{M}$, and $(\mathfrak{M}, t^\lambda) = \mathfrak{N}$. By Lemma 2, the L-th determinant divisor of $\mathfrak{N}$ is evidently unity. Consequently, the necessary and sufficient condition that (3.7) have integral solutions n^i and l^α, is that *the L-th determinant divisor of* $\mathfrak{M}$ *is unity.*

n^i are determined by (3.7), and for these n^i, m is determined by (3.5).

As the special cases, we have the followings:

When $k_\lambda = 0$, then we can take T_α^λ as $T_\alpha^\lambda = \delta_\alpha^\lambda$[2]. In this case, evidently

1) $t^\lambda = 0$ for $\lambda = 1, 2, \ldots, L-1$.

2) δ_α^λ is Kronecker's delta, i.e. $\delta_\alpha^\lambda = 0$ or 1 according as $\lambda \neq \alpha$ or $\lambda = \alpha$.

the L-th determinant divisor of $\mathfrak{M}$ is unity. Consequently there exist n^i satisfying (3.7).

When $t^{r}=0$, namely $t^{\lambda}=0$, $\mathfrak{M}$ and $\mathfrak{N}$ coincide, consequently it is evident that there exist n^i satisfying (3.7).

When there exist L linearly independent k_{λ}^{α}, then $T_{\alpha}^{\lambda}=0$, and the equations (3.7) become as follows:

(3.8) $$t_i^{\lambda} n^i = -t^{\lambda} .$$

We consider the case where $t^{L}\neq 0$. If there exist n^i, then all possible t_i^{L} must be relatively prime. For, if not, there exists a prime number $p\neq 1$ such that $t_i^{L}\equiv 0 \pmod p$. Then, from (3.8), t^{L} also must be divisible by p. This contradicts the definition of the basis. Next, we consider the converse, namely we assume that all possible t_i^{L} are relatively prime. If the L-th determinant divisor of $\mathfrak{M}$ is not unity, then, by Lemma 1 and 2, for an appropriate prime number $p\neq 1$, there exist integers c_{α} such that $t_i^{L}\equiv c_{\alpha} t_i^{\alpha}$ (mod p), α summed over $1, 2, \ldots, L-1$. Put $t_i^{L}-c_{\alpha}t_i^{\alpha}={}'t_i^{L}$, then ${}'t_i^{L}\equiv 0 \pmod p$ and $({}'t_i^{L}, t^{r})$ evidently satisfy (3.1). This contradicts our assumption. Thus the L-th determinant divisor of $\mathfrak{M}$ is unity, consequently there exist n^i satisfying (3.8). Thus, in the present case, *the necessary and sufficient condition for existence of n^i is that, for any set of (t_i^{r}, t^{r}) satisfying* (3.1), *t_i^{L} are relatively prime.*

Example 1. $\overset{\circ}{\mu}_1 = -1$, $\overset{\circ}{\mu}_2 = -2+\frac{2\sqrt{2}+1}{2}\pi\sqrt{-1}$, $\overset{\circ}{\mu}_3 = -3+\frac{6\sqrt{2}+1}{2}\pi\sqrt{-1}$,

$$\overset{\circ}{\mu} = -27+\frac{30\sqrt{2}+9}{2}\pi\sqrt{-1} = 6\overset{\circ}{\mu}_1+6\overset{\circ}{\mu}_2+3\overset{\circ}{\mu}_3 , \qquad t_0 = 1 .$$

The equations determining $(t_i^{\lambda}, t^{\lambda})$ are as follows:

$$\begin{cases} t_1+2t_2+3t_3 = 0 , \\ \frac{2\sqrt{2}+1}{2}\pi t_2+\frac{6\sqrt{2}+1}{2}\pi t_3 = 2\pi t . \end{cases}$$

We have: $L=1$ and $t_1=-6$, $t_2=6$, $t_3=-2$, $t=1$. Therefore $p_1=6-6k$, $p_2=6+6k$, $p_3=3-2k$, $n=k$. Consequently $k=0$ or ± 1. In this case, (3.7) becomes (3.8). Thus, there do not exist n^i, namely it is impossible to find μ_i so that the relations Chap. II (2.12) and (2.14) are in one-to-one correspondence.

Example 2. $\overset{\circ}{\mu}_i$ are all real.

$B^i=0$, consequently $t^{\lambda}=0$. Evidently there exist $n^i=l^{\alpha}=0$ satisfying (3.7), namely, if we take $\overset{\circ}{\mu}_i$ themselves as μ_i, the relations Chap II (2.12) and (2.14) are in one-to-one correspondence.

§ 4. Correspondence of the eigen values.

From our agreement on the arrangement of λ_i, there exist the sets of $\overset{\circ}{p}{}_i^x$ such that

$$\lambda_x = \prod_{i=1}^{S} \lambda_i^{\overset{\circ}{p}{}_i^x} . \tag{4.1}$$

Now, as seen from (3.1), (t_i^λ, t^λ) depend only on $\overset{\circ}{\mu}_i$ $(i=1, 2, \dots, S)$. If there exist μ_i $(i=1, 2, \dots, R)$ such that the relations Chap. II (2.12) and (2.14) are in one-to-one correspondence, then, for all possible k_λ in (3.2) for all p_i^x, there must exist integral solutions n^i of (3.6), namely the L-th determinant divisor of $\mathfrak{M}$ must be unity.

We consider conversely, namely we assume that the L-th determinant divisor of $\mathfrak{M}$ is unity. Then, there exist n^i satisfying (3.6) for all possible k_λ, consequently, when we determine each m corresponding to $\overset{\circ}{\mu}_x$ by (3.5), namely by $\overset{\circ}{p}{}_i^x n^i = m$, there exist μ_i $(i=1, 2, \dots, S)$ and μ_x $(x=S+1, \dots, R)$ such that, for any set of p_i^x satisfying $\lambda_x = \prod_{i=1}^{S} \lambda_i^{p_i^x}$, $\mu_x = \sum_{i=1}^{S} \mu_i p_i^x$. Then, for any set of p_i such that

$$\lambda_x = \lambda_1^{p_1} \lambda_2^{p_2} \cdots\cdots \lambda_S^{p_S} \lambda_{S+1}^{p_{S+1}} \cdots\cdots \lambda_{x-1}^{p_{x-1}} , \tag{4.2}$$

from (4.1), we have: $\lambda_x = \prod_{i=1}^{S} \lambda_i^{p_i + p_{S+1} \overset{\circ}{p}{}_i^{S+1} + \cdots\cdots + p_{x-1} \overset{\circ}{p}{}_i^{x-1}}$. Then, for our μ_i and μ_x,

$$\begin{aligned} \mu_x &= \sum_{i=1}^{S} \mu_i (p_i + p_{S+1} \overset{\circ}{p}{}_i^{S+1} + \cdots\cdots + p_{x-1} \overset{\circ}{p}{}_i^{x-1}) \\ &= \sum_{i=1}^{S} \mu_i p_i + p_{S+1} \sum_{i=1}^{S} \mu_i \overset{\circ}{p}{}_i^{S+1} + \cdots\cdots + p_{x-1} \sum_{i=1}^{S} \mu_i \overset{\circ}{p}{}_i^{x-1} \\ &= \sum_{i=1}^{S} \mu_i p_i + p_{S+1} \mu_{S+1} + \cdots\cdots + p_{x-1} \mu_{x-1} , \end{aligned}$$

i. e.

$$\mu_x = \sum_{i=1}^{x-1} \mu_i p_i . \tag{4.3}$$

Thus, for our μ_i $(i=1, 2, \dots, R)$, (4.3) always hold for any set of p_i satisfying (4.2).

Thus we have

Theorem IV. *For the given values of λ_i where either $0<|\lambda_i|<1$ or $|\lambda_i|>1$, if we arrange λ_i as explained in Chap. I § 1, then the necessary and sufficient condition that there may exist μ_i satisfying*

$$\lambda_i = e^{t_0 \mu_i} \tag{L}$$

such that there always hold

$$\mu_i = p_1 \mu_1 + p_2 \mu_2 + \cdots\cdots + p_{i-1} \mu_{i-1} \tag{μ}$$

for any set of $(p_1, p_2, \ldots, p_{i-1})$ *satisfying*

$$(\lambda) \qquad \lambda_i = \lambda_1^{p_1}\lambda_2^{p_2} \cdots\cdots \lambda_{i-1}^{p_{i-1}} ,$$

is that the L-th determinant divisor of $\mathfrak{M}$ *is unity.*

By this theorem, when the L-th determinant divisor of $\mathfrak{M}$ is unity, there exist μ_i such that the relations (μ) and (λ) are in one-to-one correspondence. For such μ_i, the equations (S') and their solutions (f') in Theorem II coincide with the equations (S) and their solutions (f) in Theorem I respectively.

Chapter IV. Existence of one parameter group.

§ 1. Preliminaries.

In Chap. II, for the given functions $\varphi^\mu(x) = a_\nu^\mu x^\nu + \cdots\cdots$, we have assumed that there exists a one parameter group of transformations which contains the transformation

$$(1.1) \qquad \mathfrak{T}:\ 'x^\mu = \varphi^\mu(x) = a_\nu^\mu x^\nu + \cdots\cdots , \qquad \text{det.}\,|\,a_\nu^\mu\,| \neq 0 ,$$

and we have deduced the functional equations (S'). In this chapter, from the equations of Schröder (S), we deduce the existence of group.

In Theorem I, we have seen that, when either $0 < |\lambda_i| < 1$ or $|\lambda_i| > 1$, the equations of Schröder (S) have the regular solutions. Analogously to the process of obtaining the solutions of the linear homogeneous partial differential equation from its characteristic equations, we deduce the functions U_{lp}^{i} [1] from the equations of Schröder. Making use of these functions, we prove the existence of group containing the given transformation. Next, by means of Theorem IV, we prove that, under certain conditions, there exists a group possessing the regular operator functions.

§ 2. Simpler form of the equations of Schröder.

As explained in Chap. I § 1, we arrange λ_i and, as in the previous paper[2], we attach the non-negative integers w_{lp}^i to the eigen values λ_{lp}^i as follows:

$$(2.1) \qquad \begin{cases} w_{l1}^1 = 0 , \quad w_{lp}^1 = p-1 \quad (p = 2, 3, \ldots, P_l^1) , \\ w_{l1}^x = \max.\left(\sum\limits_{i=1}^{x-1}\sum\limits_{m=1}^{L_i}\sum\limits_{p=1}^{P_m^i} p_{mp}^{i\,x}\, w_{mp}^i\right) + 1 , \quad w_{lp}^x = w_{l1}^x + p - 1 , \end{cases}$$

1) M. Urabe, This Journal, Vol. 15, No. 1 (1951), p. 25.
2) do.

where "max." denotes the maximum of the values corresponding to all sets of $\overset{x}{p}{}^{i}_{mp}$ such that

$$(2.2)\qquad \lambda_x = \prod_{i=1}^{x-1} \prod_{m=1}^{L_i} \prod_{p=1}^{P^i_m} \lambda^{i}_{mp}{}^{\overset{x}{p}{}^{i}_{mp}} .$$

We call the number w^i_{lp} the *weight of* λ^i_{lp} and $w = \sum_{i=1}^{R} \sum_{m=1}^{L_i} \sum_{p=1}^{P^i_m} w^i_{mp} p^i_{mp}$ the *weight of the set of* p^i_{mp}.

We make use of notations as follows: $\varphi f = f[\varphi(x)]$ and $(\varphi - a)f = \varphi f - af = f[\varphi(x)] - af(x)$.

For any solution f^i_{lp} of the equations (S), put

$$(2.3)\qquad \psi = \prod_{i=1}^{x-1} \prod_{m=1}^{L_i} \prod_{q=1}^{P^i_m} f^i_{mq}(x)^{\overset{x}{p}{}^{i}_{mq}} .$$

Then we have

$$(2.4)\qquad \varphi\psi = \prod_{i=1}^{x-1} \prod_{m=1}^{L_i} \prod_{q=1}^{P^i_m} \left\{ \lambda_i f^i_{mq}(x) + f^i_{mq-1}(x) + \Psi^i_{mq}(x) \right\}^{\overset{x}{p}{}^{i}_{mq}} = \lambda_x \psi(x) + \psi'(x) .$$

As in Chap. I § 5, we associate the number λ_i to $f^i_{mq}(x)$, then $\lambda_i f^i_{mq} + f^i_{mq-1} + \Psi^i_{mq}$ is a polynominal of the order λ_i. Therefore, by the lemma in Chap. I § 5, the right-hand side of (2.4) becomes a polynomial of the order λ_x, namely a linear combination of the monomials of the form ψ. Any one monomial of $\psi'(x)$ is a monomial obtained by substituting $f^i_{mq-1}(x)$ or $\Psi^i_{mq}(x)$ for at least one $f^i_{mq}(x)$ in the product of $\psi(x)$. Now the weight w' of the indices of the monomials of Ψ^i_{mq} is as follows:

$$w' = \sum_{j=1}^{i-1} \sum_{l=1}^{L_j} \sum_{p=1}^{P^j_l} w^j_{lp} \overset{i}{p}{}^{j}_{lp} \leq w^i_{m1} - 1 \leq w^i_{mq} - 1 .$$

Thus we see that $\psi'(x)$ becomes a linear combination of the monomials, of which the weight of the indices is at most $w-1$, where w is the weight of the indices of ψ. Then, if we operate $\varphi - \lambda_x$ w-times on ψ, we have a linear combination of the monomials of the weight zero. When the weight of p^i_{mq} is zero, from the definition, p^i_{mq} are zero except for p^a_{m1}. For ψ such that $\psi = \prod_{a=1}^{S} \prod_{m=1}^{L_a} f^a_{m1}{}^{\overset{x}{p}{}^{a}_{m1}}$, it is evident that $(\varphi - \lambda_x)\psi = 0$. Thus, for ψ of the form (2.3), $(\varphi - \lambda_x)^{w+1}\psi = 0$. Now $w = \sum_{i,m,q} w^i_{mq} \overset{x}{p}{}^{i}_{mq} \leq w^x_{l1} - 1$, consequently $(\varphi - \lambda_x)^{w^x_{l1}}\psi = 0$. From this, it is evident that, for a linear combination Ψ^x_{mp} of the functions of the form ψ,

$$(2.5)\qquad (\varphi - \lambda_x)^{w^x_{l1}} \Psi^x_{mp} = 0 .$$

Then, if we operate $\varphi - \lambda_i$ successively on the solution $f^i_{lP^i_l}$ of the equations (S), then, from (2.5),

$$(\varphi-\lambda_i)^{w^i_{lP^i_l}+1} f^i_{lP^i_l}=0 \ . \tag{2.6}$$

By $(\varphi-\lambda_i)^{Q^i_l} f^i_{lP^i_l}$, we denote the first which identically vanish. Then $Q^i_l \leq w^i_{lP^i_l}+1$. Put

$$F^i_{lp}=(\varphi-\lambda_i)^{Q^i_l-p} f^i_{lP^i_l} \ , \quad (p=1, 2, \dots Q^i_l) \ . \tag{2.7}$$

Then, from the equations (S), we have:

(2.8)

$$\begin{cases} \text{for } i=a, & F^a_{lp}=f^a_{lp} \ ; \\ \text{for } i=x, & \begin{cases} F^x_{lQ^x_l}=f^x_{lP^x_l} \ , \\ F^x_{lQ^x_l-1}=f^x_{lP^x_l-1}+\Psi^x_{lP^x_l} \ , \\ \vdots \\ F^x_{lQ^x_l-P^x_l+1}=f^x_{l1}+\Psi^x_{l2}+(\varphi-\lambda_x)\Psi^x_{l3}+\cdots+(\varphi-\lambda_x)^{P^x_l-2}\Psi^x_{lP^x_l} \ , \\ F^x_{lQ^x_l-P^x_l} = \Psi^x_{l1}+(\varphi-\lambda_x)\Psi^x_{l2}+\cdots+(\varphi-\lambda_x)^{P^x_l-1}\Psi^x_{lP^x_l} \ , \\ \vdots \\ F^x_{l1}=(\varphi-\lambda_x)^{Q^x_l-P^x_l-1}\Psi^x_{l1}+(\varphi-\lambda_x)^{Q^x_l-P^x_l}\Psi^x_{l2}+\cdots+(\varphi-\lambda_x)^{Q^x_l-2}\Psi^x_{lP^x_l}. \end{cases} \end{cases}$$

Here we can prove that,

$$\begin{cases} \text{for } p=1, 2, \dots, Q^i_l-P^i_l, & F^i_{lp}=L\Big[\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{Q^j_m} F^j_{mq}{}^{'\overset{i}{p}{}^j_{mq}}\Big]^{(1)}, \\ \text{for } p=Q^i_l-P^i_l+1, \dots, Q^i_l, & F^i_{lp}=f^i_{lp-(Q^i_l-P^i_l)}+L\Big[\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{Q^j_m} F^j_{mq}{}^{'\overset{i}{p}{}^j_{mq}}\Big]. \end{cases} \tag{2.9}$$

Here $L[\dots]$ denotes a linear combination of the arguments. For $i=a$, $Q^a_l=P^a_l$ and $F^a_{lp}=f^a_{lp}$, consequently it is evident that (2.9) hold. For $i=S+1$, $\Psi^{S+1}_{lp}=L\Big[\prod_{a,m,q} f^a_{mq}{}^{\overset{S+1}{p}{}^a_{mq}}\Big]=L\Big[\prod_{a,m,q} F^a_{mq}{}^{\overset{S+1}{p}{}^a_{mq}}\Big]$ and $(\varphi-\lambda_{S+1})\Psi^{S+1}_{lp}$ are of the same forms as Ψ^{S+1}_{lp}. Therefore, for $i=S+1$, (2.9) hold. We assume that (2.9) hold for $i=1, 2, \dots, S, S+1, \dots, x-1$. From the latter of (2.9), for $i=1, 2, \dots, x-1$,

$$f^i_{lp}=F^i_{lp+(Q^i_l-P^i_l)}-L\Big[\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{Q^j_m} F^j_{mq}{}^{'\overset{i}{p}{}^j_{mq}}\Big] \ . \tag{2.10}$$

Substitute (2.10) into $\Psi^x_{lp}=L\Big[\prod_{i=1}^{x-1}\prod_{m=1}^{L_i}\prod_{q=1}^{P^i_m} f^i_{mq}{}^{\overset{x}{p}{}^i_{mq}}\Big]$. When we associate the

1) ${}'\overset{i}{p}{}^j_{mq}$ are such non-negative integers that $\lambda_i=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{Q^j_m}\lambda_j{}^{'\overset{i}{p}{}^j_{mq}}$.

number λ_i to F^i_{ip}, then, from the lemma in Chap. I § 5, f^i_{mq} is a polynomial of the order λ_i. Therefore, after substitution, Ψ^x_{ip} becomes a polynomial of the order λ_x. As $(\varphi-\lambda_x)\Psi^x_{ip}$ is of the same form as Ψ^x_{ip}, we see that (2.9) hold also for $i=x$. Thus, for any i, (2.9) hold.

For the solutions f^i_{ip} given by (f) in Theorem I, after suitable linear transformation of the variables x^μ, from (2.9), it follows that, for $p=Q^i_i-P^i_i+1, \dots, Q^i_i$, $F^i_{ip}=x^i_{ip-(Q^i_i-P^i_i)}+\cdots\cdots$. From (2.7), it is evident that

$$F^i_{ip}(\varphi)=\lambda_i F^i_{ip}(x)+F^i_{ip-1}(x)\,, \quad (p=1, 2, \dots, Q^i_i)\,. \tag{2.11}$$

Thus we have

Theorem V. *For the given functions* $\varphi^\mu(x)=a^\mu_\nu x^\nu+\cdots\cdots$, *det.* $|a^\mu_\nu|\neq 0$, *we assume that the absolute values of all the eigen values* λ_i *of* $\|a^\mu_\nu\|$ *are either greater or less than unity. We consider the functional equations as follows:*

$$F^i_{ip}(\varphi)=\lambda_i F^i_{ip}(x)+F^i_{ip-1}(x)\,. \tag{S_1}$$

When we transform the variables x^μ *and the functions* φ^μ *by* $\tilde{x}^\mu=t^\mu_\nu x^\nu$ *and* $\tilde{\varphi}^\mu=t^\mu_\nu\varphi^\nu$ *so that*

$$\tilde{\varphi}^i_{ip}=\lambda_i\tilde{x}^i_{ip}+\tilde{x}^i_{ip-1}+\cdots\cdots\,, \tag{φ}$$

then there exist the solutions F^i_{ip} *of the equations* (S_1) *such that*

$$\begin{cases} \textit{for} \quad p=Q^i_i-P^i_i+1, \dots, Q^i_i\,, & F^i_{ip}=\tilde{x}^i_{ip-(Q^i_i-P^i_i)}+\cdots\cdots\,, \\ \textit{for} \quad p=1, 2, \dots, Q^i_i-P^i_i\,, & F^i_{ip}=L\Big[\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=1}^{Q^j_m} F^j_{mq}{}^{'p^{ij}_{mq}}\Big]\,. \end{cases} \tag{F_1}$$

The equations (S_1) are of the same forms as (S_1') in Theorem III.

§ 3. *U*-functions.

In this paragraph, we make U-functions analogous to U^i_{ip} in the previous paper[1]. By means of $K'=\|'k^\mu_\nu\|$ defined in Chap. II § 4, we transform F^μ as follows: $F^\mu='k^\mu_\nu G^\nu$. Then, from (S_1), we have:

$$G^i_{ip}[\varphi(x)]=\lambda_i\Big[\frac{t_0^{p-1}}{(p-1)!}G^i_{i1}(x)+\frac{t_0^{p-2}}{(p-2)!}G^i_{i2}(x)+\cdots+t_0G^i_{ip-1}(x)+G^i_{ip}(x)\Big]\,. \tag{3.1}$$

Put $(p-1)!\,G^i_{ip}=H^i_{ip-1}$. Then (3.1) are written as follows:

$$H^i_{ip}(\varphi)=\lambda_i\sum_{r=0}^{p}\binom{p}{r}t_0^{p-r}H^i_{ir}(x)\,. \tag{3.2}$$

Put

1) Urabe, ibid.

$$(3.3)\qquad H^t_{i1}/H^t_{i0}=U^t_{i0}\ ,$$

then, $H^t_{i1}=U^t_{i0}H^t_{i0}$. Applying the formulae of Leibnitz of differentiation to these, we define successively U^t_{ip} as follows:

$$(3.4)\qquad H^t_{ip+1}=\sum_{r=0}^{p}\binom{p}{r}U^t_{ip-r}H^t_{ir}\ .$$

From (3.3), $H^t_{i1}(\varphi)=U^t_{i0}(\varphi)H^t_{i0}(\varphi)$. Substituting (3.2), we have: $\lambda_i[t_0H^t_{i0}(x)+H^t_{i1}(x)]=\lambda_iU^t_{i0}(\varphi)H^t_{i0}(x)$, namely

$$(3.5)\qquad U^t_{i0}(\varphi)=U^t_{i0}(x)+t_0\ .$$

For $p\geqq 1$, we can prove that

$$(3.6)\qquad U^t_{ip}(\varphi)=U^t_{ip}(x)\ .$$

From (3.4), $H^t_{i2}(\varphi)=U^t_{i1}(\varphi)H^t_{i0}(\varphi)+U^t_{i0}(\varphi)H^t_{i1}(\varphi)$. Substituting (3.2) and (3.5), we have:

$$t_0^2H^t_{i0}(x)+2t_0H^t_{i1}(x)+H^t_{i2}(x)=U^t_{i1}(\varphi)H^t_{i0}(x)+[U^t_{i0}(x)+t_0][t_0H^t_{i0}(x)+H^t_{i1}(x)]\ .$$

Making use of (3.4), we have: $U^t_{i1}(\varphi)=U^t_{i1}(x)$, namely, for $p=1$, (3.6) is valid. We assume (3.6) for $p=1, 2, \dots, p-1$. From (3.4), $H^t_{ip+1}(\varphi)=\sum_{r=0}^{p}\binom{p}{r}U^t_{ip-r}(\varphi)H^t_{ir}(\varphi)$. Substitute (3.2), (3.5) and (3.6) for $p=1, 2, \dots, p-1$, we have:

$$(3.7)\qquad \sum_{r=0}^{p+1}\binom{p+1}{r}t_0^{p+1-r}H^t_{ir}(x)=U^t_{ip}(\varphi)H^t_{i0}(x)+\sum_{r=1}^{p-1}\binom{p}{r}U^t_{ip-r}(x)\sum_{s=0}^{r}\binom{r}{s}t_0^{r-s}H^t_{is}(x)$$
$$+[U^t_{i0}(x)+t_0]\sum_{s=0}^{p}\binom{p}{s}t_0^{p-s}H^t_{is}(x)\ .$$

The coefficient of $t_0^r(r\geqq 1)$ in the right-hand side of the above relation is calculated as follows:

$$(3.8)\qquad \sum_{t=r}^{p}\binom{p}{t}\binom{t}{t-r}U^t_{ip-t}H^t_{it-r}+\binom{p}{p-r+1}H^t_{ip-r+1}$$
$$=\frac{p!}{r!\,(p-r)!}\sum_{t=r}^{p}\binom{p-r}{t-r}U^t_{ip-t}H^t_{it-r}+\frac{p!}{(r-1)!\,(p-r+1)!}H^t_{ip-r+1}$$
$$=\frac{p!}{r!\,(p-r+1)!}\Big[p-r+1+r\Big]H^t_{ip-r+1}$$
$$=\binom{p+1}{p+1-r}H^t_{ip+1-r}\ .$$

In the right-hand side of (3.7), the term which does not contain t_0 is calculated as follows:

$$(3.9)\qquad U^t_{ip}(\varphi)H^t_{i0}(x)+\sum_{r=1}^{p}\binom{p}{r}U^t_{ip-r}H^t_{ir}=U^t_{ip}(\varphi)H^t_{i0}(x)+H^t_{ip+1}-U^t_{ip}H^t_{i0}\ .$$

Substituting (3.8) and (3.9) into (3.7), we have (3.6) for p. Then, by induction, we see that, for any $p\geqq 1$, (3.6) is valid.

Now, if we take K' defined in Chap. II § 4, then, from (F_1), it is easily seen that the set of functions H^l_{ip} for $p=Q^l_i-P^l_i, Q^l_i-P^l_i+1, \dots, Q^l_i-1$ is a set of n independent functions and, for $p=0, 1, \dots, Q^l_i-P^l_i-1$,

$$(3.10)\qquad H^l_{ip}=L\Big[\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=0}^{Q^j_m-1} H^j_{mq}{}^{''\overset{i}{p}{}^j_{mq}}\Big]^{(1)}.$$

By the definition, U^l_{ip} is a rational function of $H^l_{i0}, H^l_{i1}, \dots, H^l_{ip+1}$ and depends really upon H^l_{ip+1}.

Thus we see that following functions constitute a set of n independent functions:

$$(3.11)\qquad \begin{cases} \text{for } Q^l_i=P^l_i, & H^l_{i0},\ U^l_{i0},\ U^l_{ip}\ (p=1, 2, \dots, Q^l_i-2)\ , \\ \text{for } Q^l_i=P^l_i+1, & U^l_{i0},\ U^l_{ip}\ (p=1, 2, \dots, Q^l_i-2)\ , \\ \text{for } Q^l_i\geqq P^l_i+2, & U^l_{ip}\ (p=Q^l_i-P^l_i-1, Q^l_i-P^l_i, \dots, Q^l_i-2). \end{cases}$$

From (3.2), $H^l_{i0}(\varphi)=\lambda_i H^l_{i0}(x)$, consequently, taking suitable branches of logarithms, we have:

$$(3.12)\qquad \log H^l_{i0}(\varphi)=\log H^l_{i0}(x)+\mu_i t_0\ ,$$

where μ_i are determined so that

$$(3.13)\qquad \lambda_i=e^{\mu_i t_0}\ .$$

Comparing (3.5) with (3.12), we have:

$$U^l_{i0}(\varphi)-\frac{1}{\mu_i}\log H^l_{i0}(\varphi)=U^l_{i0}(x)-\frac{1}{\mu_i}\log H^l_{i0}(x)\ .$$

Comparing the formulae of (3.12) with that for $i=l=1$, we have:

$$\frac{1}{\mu_i}\log H^l_{i0}(\varphi)-\frac{1}{\mu_1}\log H^1_{10}(\varphi)=\frac{1}{\mu_i}\log H^l_{i0}(x)-\frac{1}{\mu_1}\log H^1_{10}(x)\ .$$

Thus, from these and (3.6), for the given transformation $\mathfrak{T}: {}'x^\mu=\varphi^\mu(x)$, we have the invariants as follows:

$$(3.14)\qquad H^l_{i0}{}^{\frac{1}{\mu_i}}/H^1_{10}{}^{\frac{1}{\mu_1}} \qquad (\text{except for } i=l=1)\ ,$$

$$U^l_{i0}-\frac{1}{\mu_i}\log H^l_{i0}\ , \qquad U^l_{ip} \qquad (p=1, 2, \dots, Q^l_i-2)\ .$$

From consideration of independence of the functions of (3.11), we see that, among the invariants of (3.14), the following $n-1$ functions are independent of one another:

1) $''\overset{i}{p}{}^j_{mq}$ are such non-negative integers that $\lambda_i=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=0}^{Q^j_m-1}\lambda_j{}^{''\overset{i}{p}{}^j_{mq}}$.

$$
(3.15)\quad
\begin{cases}
\text{for } Q^l_i = P^l_i , & H^l_{i0}{}^{\frac{1}{\mu_i}} / H^1_{10}{}^{\frac{1}{\mu_1}}\,{}^{(1)} ,\quad U^l_{i0} - \dfrac{1}{\mu_i} \log H^l_{i0} ,\quad U^l_{ip}\,(p=1, 2, \dots, Q^l_i - 2) , \\
\text{for } Q^l_i = P^l_i + 1 , & U^l_{i0} - \dfrac{1}{\mu_i} \log H^l_{i0} ,\quad U^l_{ip}\,(p=1, 2, \dots, Q^l_i - 2) , \\
\text{for } Q^l_i \geqq P^l_i + 2 , & U^l_{ip}\,(p = Q^l_i - P^l_i - 1, \dots, Q^l_i - 2) .
\end{cases}
$$

These functions correspond to the $n-1$ independent solutions of a linear homogeneous differential equation. Any invariant for the given transformation is expressible as a function of the above invariants. Summarizing the results, we have:

Theorem VI. *We assume the same condition as in Theorem V. Then, there exist $n-1$ independent invariants for the given transformation $\mathfrak{T} : 'x^\mu = \varphi^\mu(x)$. They are found by algebraic operation from the solutions of the equations of Schröder (S) for the given functions $\varphi^\mu(x)$. They are listed as in* (3.15).

§ 4. Existence of group.

In the table (3.11) we put

$$
(4.1)\qquad H^l_{i0} = h^\alpha , \quad U^l_{i0} = h^\sigma , \quad U^l_{ip} = h^\omega ,
$$

and transform the variables x^μ as follows:

$$
(4.2)\qquad \bar{x}^\alpha = h^\alpha(x) , \quad \bar{x}^\sigma = h^\sigma(x) , \quad \bar{x}^\omega = h^\omega(x) .
$$

If we determine μ_i so that

$$
(4.3)\qquad \lambda_i = e^{t_0 \mu_i} ,
$$

then, from (3.2), (3.5) and (3.6), we have:

$$
(4.4)\qquad
\begin{cases}
'\bar{x}^\alpha = h^\alpha('x) = H^l_{i0}(\varphi) = e^{t_0\mu_i} H^l_{i0}(x) = e^{t_0\mu_i} h^\alpha(x) = e^{t_0\mu_i} \bar{x}^\alpha , \\
'\bar{x}^\sigma = h^\sigma('x) = U^l_{i0}(\varphi) = U^l_{i0}(x) + t_0 = h^\sigma(x) + t_0 = \bar{x}^\sigma + t_0 , \\
'\bar{x}^\omega = h^\omega('x) = U^l_{ip}(\varphi) = U^l_{ip}(x) = h^\omega(x) = \bar{x}^\omega .
\end{cases}
$$

Thus, the given transformation $\mathfrak{T}$ is represented by (4.4) with regard to $\bar{x}^\mu$-system. Then it is evident that there exists a one parameter group $\mathfrak{G}$ containing $\mathfrak{T}$ such that

$$
(4.5)\qquad \mathfrak{G} : \ '\bar{x}^\alpha = e^{t\mu_i} \bar{x}^\alpha , \quad '\bar{x}^\sigma = \bar{x}^\sigma + t , \quad '\bar{x}^\omega = \bar{x}^\omega .
$$

Thus we have seen that, for the given transformation $\mathfrak{T} : 'x^\mu = \varphi^\mu(x) = a^\mu_\nu x^\nu + \cdots$, when either $0 < |\lambda_i| < 1$ or $|\lambda_i| > 1$, there exists a one parameter group of transformations containing the given transformation $\mathfrak{T}$.

1) Except for $i = l = 1$.

Next we seek for the operator functions ξ^μ of $\mathfrak{G}$. With regard to $\bar{x}^\mu$-system, the operator functions $\bar{\xi}^\mu$ are determined easily from (4.5) as follows:

$$\bar{\xi}^\alpha = \mu_i \bar{x}^\alpha, \quad \bar{\xi}^\sigma = 1, \quad \bar{\xi}^\omega = 0. \tag{4.6}$$

From the contravariant property of the operator functions for coordinate transformations, ξ^μ and $\bar{\xi}^\mu$ are related as follows:

$$X\bar{x}^\nu \equiv \xi^\mu \frac{\partial \bar{x}^\nu}{\partial x^\mu} = \bar{\xi}^\nu . \tag{4.7}$$

In the following we study (4.7).

For $Q^i_l = P^i_l$, from $X\bar{x}^\alpha = \mu_i \bar{x}^\alpha$, $XH^i_{l0} = \mu_i H^i_{l0}$. From $X\bar{x}^\sigma = 1$ and $X\bar{x}^\omega = 0$, $XU^i_{l0} = 1$ and $XU^i_{lp} = 0$. From these, we prove that

$$XH^i_{lp} = \mu_i H^i_{lp} + pH^i_{lp-1}, \quad (p = 0, 1, \dots Q^i_l - 1). \tag{4.8}$$

For $p=0$, (4.8) is valid. We assume that (4.8) is valid for $p=0, 1, 2, \dots, p$. From (3.4), $XH^i_{lp+1} = \sum_{r=0}^{p} \binom{p}{r} U^i_{lp-r} XH^i_{lr} + H^i_{lp}$. From our assumption,

$$\begin{aligned} XH^i_{lp+1} &= \sum_{r=0}^{p} \binom{p}{r} U^i_{lp-r} (\mu_i H^i_{lr} + rH^i_{lr-1}) + H^i_{lp} \\ &= \mu_i H^i_{lp+1} + p \sum_{r=1}^{p} \binom{p-1}{r-1} U^i_{lp-r} H^i_{lr-1} + H^i_{lp} \\ &= \mu_i H^i_{lp+1} + (p+1) H^i_{lp} . \end{aligned}$$

Namely, (4.8) is valid also for $p+1$. Thus, by induction, (4.8) is valid for any p.

Here we assume that, for the given λ_i, μ_i determined by (4.3) are such that the relations (μ) and (λ) in Theorem IV are in one-to-one correspondence.

We consider the case where $Q^i_l = P^i_l + 1$. In this case, from (3.10),

$$H^i_{l0} = L\left[\prod_{j=1}^{i-1} \prod_{m=1}^{L_j} \prod_{q=0}^{Q^j_m - 1} H^j_{mp}{}^{''p^{j\,i}_{mq}} \right], \tag{4.9}$$

where

$$\lambda_i = \prod_{j=1}^{i-1} \prod_{m=1}^{L_j} \prod_{q=0}^{Q^j_m - 1} \lambda_j{}^{''p^{j\,i}_{mq}} . \tag{4.10}$$

Put

$$\psi = \prod_{j=1}^{i-1} \prod_{m=1}^{L_j} \prod_{q=0}^{Q^j_m - 1} H^j_{mq}{}^{''p^{j\,i}_{mq}} . \tag{4.11}$$

We assume that, for $j \leq i-1$, (4.8) is valid. Then

$$X\psi=\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=0}^{Q^j_m-1}{}''\overset{i}{p}{}^j_{mq}\frac{\psi}{H^j_{mq}}(\mu_j H^j_{mq}+qH^j_{mq-1})$$
$$=\sum_{j,m,q}{}''\overset{i}{p}{}^j_{mq}\mu_j\cdot\psi+\bar{\psi}\ ,$$

where $\bar{\psi}$ is a linear combination of the functions of the same form as ψ. By our assumption, $\sum_{j,m,q}{}''\overset{i}{p}{}^j_{mq}\mu_j=\mu_i$. Then, for H^i_{l0} which are linear combinations of the functions of the same forms as ψ, we have:

$$XH^i_{l0}=\mu_i H^i_{l0}+\psi^i_l\ , \tag{4.12}$$

where ψ^i_l are linear combinations of the functions of the same forms as ψ. Now, from (3.10), it follows that ψ is a linear combination of the functions as follows:

$$\psi'=\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=Q^j_m-P^j_m}^{Q^j_m-1}H^j_{mq}{}^{\overset{ij}{p}_{mq}}\ . \tag{4.13}$$

In the same way as in § 2, we define the weight of the eigen values for the matrix $A'=\sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus{}'A^i_l$, where $'A^i_l$ is a matrix of the Q^i_l-th order such that $'A^i_l=\begin{pmatrix}\lambda_i & 0 & \cdots\cdots & 0\\ 1 & \lambda_i & & \\ 0 & \ddots & \ddots & \\ \vdots & & \ddots\ \ddots & 0\\ 0\cdots 0 & 1 & & \lambda_i\end{pmatrix}$. Then $w^i_{lp}\geq\sum_{j=1}^{i-1}\sum_{m=1}^{L_j}\sum_{q=0}^{Q^j_m-1}w^j_{mq}{}''\overset{i}{p}{}^j_{mq}+1$, consequently when we write ψ as a linear combination of the functions of the forms ψ', the weights of the indices of ψ' are less by at least one than the weight of the indices of ψ. We arrange the functions of the forms ψ' according to the weights of the indices, and write them as ψ^W. Now, by § 3, for $q=Q^j_m-P^j_m,\ \ldots\ldots,\ Q^j_m-1$, H^j_{mq} are independent, therefore it is evident that ψ^W are linearly independent with respect to constant coefficients. Thus we can write ψ^i_l as follows:

$$\psi^i_l=c^i_{lW}\psi^W\ , \tag{4.14}$$

then c^i_{lW} are uniquely determined.

From our assumption, for $j\leq i-1$, (4.8) is valid, namely

$$\xi^\mu(x)\frac{\partial H^j_{mq}(x)}{\partial x^\mu}=\mu_j H^j_{mq}(x)+qH^j_{mq-1}(x)\ .$$

Substitute $'x^\mu$ for x^μ, then, making use of such property of the operator functions that $\frac{d'x^\mu}{dt}=\xi^\mu('x)$, we have

$$\frac{dH^j_{mq}('x)}{dt} = \mu_j H^j_{mq}('x) + qH^j_{mq-1}('x) \ . \tag{4.15}$$

Integrating these differential equations, we have:

$$H^j_{mq}('x) = e^{\mu_j t}\sum_{r=0}^{q}\binom{q}{r} t^{q-r} H^j_{mr}(x) \ . \tag{4.16}$$

This is easily proved by induction. Substitute $'x^\mu$ for x^μ in (4.12), then

$$\frac{dH^i_{i0}('x)}{dt} = \mu_i H^i_{i0}('x) + c^i_{iW}\psi^W('x) \ . \tag{4.17}$$

Now, $\psi^W('x)$ are of the form (4.13), and, for $H^j_{mq}('x)$, (4.16) is valid. Then, we have:

$$\begin{aligned} \psi^W('x) &= e^{\mu_i t}\prod_{j=1}^{i-1}\prod_{m=1}^{L_j}\prod_{q=Q^j_m-P^j_m}^{Q^j_m-1}\left[\sum_{r=0}^{q}\binom{q}{r} t^{q-r} H^j_{mr}(x)\right]^{p^{ij}_{mq}} \\ &= e^{\mu_i t}\,\overset{i}{\kappa}{}^W_V(t)\ \psi^V(x) \ , \end{aligned} \tag{4.18}$$

and, by the above mentioned, in the right-hand side there appear only ψ^V, in which the weights of V are less by at least one than the weights of W except for $V=W$. When the weight of V is not less than that of W, we write as follows: $V>W$. Then

$$\text{for } V>W\,,\quad \overset{i}{\kappa}{}^W_V(t)=0 \text{ except for } V=W\,,\quad \text{and } \overset{i}{\kappa}{}^W_W(t)=1 \ . \tag{4.19}$$

Integrating (4.17) after substitution of (4.18), we have:

$$H^i_{i0}('x) = e^{\mu_i t}\left[H^i_{i0}(x) + c^i_{iW}\psi^V(x)\int_0^t \overset{i}{\kappa}{}^W_V(t)\ dt\right] \ .$$

Put $t=t_0$, then

$$H^i_{i0}(\varphi) = \lambda_i\left[H^i_{i0}(x) + c^i_{iW}\psi^V(x)\int_0^{t_0} \overset{i}{\kappa}{}^W_V(t)\ dt\right] \ . \tag{4.20}$$

Now, from (3.2), $H^i_{i0}(\varphi)=\lambda_i H^i_{i0}(x)$. Comparing this with (4.20), we have: $c^i_{iW}\psi^V(x)\int_0^{t_0}\overset{i}{\kappa}{}^W_V(t)\ dt=0$. From linearly independence of $\psi^V(x)$, it follows that $c^i_{iW}\int_0^{t_0}\overset{i}{\kappa}{}^W_V(t)\ dt=0$. From (4.19), det. $\left|\int_0^{t_0}\overset{i}{\kappa}{}^W_V(t)\ dt\right| \neq 0$. Therefore $c^i_{iW}=0$, namely, from (4.14), $\psi^i_i=0$. Thus, from (4.12), we have:

$$XH^i_{i0} = \mu_i H^i_{i0} \ . \tag{4.21}$$

For the case where $Q^i_i=P^i_i+1$, $XU^i_{i0}=1$, and $XU^i_{ip}=0$. Then, analogously to the former case, we can prove (4.8). Thus, from the case where $Q^i_i=P^i_i$, consequently (4.8) is valid, by induction, we see that (4.8) is also valid for the case where $Q_i^t = P_i^t + 1$.

Last, we consider the case where $Q_i^t \geqq P_i^t + 2$. We assume that, for $j \leqq i-1$, (4.8) is valid. Then, as we have seen already, (4.16) is valid for $j \leqq i-1$. Then, in this case also, H_{i0}^t is of the same form as in the former case, therefore (4.8) is valid for $p=0$. Consequently, (4.16) is valid for $j=i$, $m=l$, $q=0$. We assume that (4.8) and (4.16) are valid for $p=0, 1, 2, \dots,$ $p-1$, where $p \leqq Q_i^t - P_i^t - 1$. Then, since H_{ip}^t and H_{ip-1}^t are of the same forms as H_{i0}^t, we have:

$$XH_{ip}^t = \mu_i H_{ip}^t + pH_{ip-1}^t + \psi_{ip}^t \ , \tag{4.22}$$

where $\psi_{ip}^t = c_{ipW}^t \psi^W$. By our assumption, for H_{ip-1}^t, (4.16) is valid. Thus, substituting $'x^\mu$ for x^μ in (4.22), by means of (4.18), we have:

$$\frac{dH_{ip}^t('x)}{dt} = \mu_i H_{ip}^t('x) + pe^{\mu_i t}\sum_{r=0}^{p-1}\binom{p-1}{r} t^{p-1-r} H_{ir}^t(x) + e^{\mu_i t} c_{ipW}^t \overset{t}{\kappa}{}_V^W(t)\ \psi^V(x) \ .$$

Integrating this differential equation, we have:

$$H_{ip}^t('x) = e^{\mu_i t}\left[\sum_{r=0}^{p}\binom{p}{r} t^{p-r} H_{ir}^t(x) + c_{ipW}^t \psi^V(x)\int_0^t \overset{t}{\kappa}{}_V^W(t)\ dt\right].$$

Put $t=t_0$, then

$$H_{ip}^t(\varphi) = \lambda_i\left[\sum_{r=0}^{p}\binom{p}{r} t_0^{p-r} H_{ir}^t(x) + c_{ipW}^t \psi^V(x)\int_0^{t_0} \overset{t}{\kappa}{}_V^W(t)\ dt\right].$$

Comparing this with (3.2), we have; $c_{ipW}^t \psi^V(x)\int_0^{t_0} \overset{t}{\kappa}{}_V^W(t)\, dt = 0$. By the same reason as in the former case, it follows that $c_{ipW}^t = 0$, namely (4.8) is valid also for p. Consequently (4.16) is valid also for p. Thus, by induction, we see that (4.8) is valid for $p=0, 1, \cdots, Q_i^t - P_i^t - 1$. Then, as in the previous paper[1], we have:

$$XU_{i0}^t = 1 \text{ and } XU_{ip}^t = 0 \text{ for } p = 1, 2, \dots, Q_i^t - P_i^t - 2 \ . \tag{4.23}$$

The proof is as follows:

First, $XH_{i1}^t = X(U_{i0}^t H_{i0}^t) = XU_{i0}^t \cdot H_{i0}^t + U_{i0}^t \cdot XH_{i0}^t$. By (4.8), this is written as follows: $\mu_i H_{i1}^t + H_{i0}^t = XU_{i0}^t \cdot H_{i0}^t + U_{i0}^t \cdot \mu_i H_{i0}^t$, from which follows: $XU_{i0}^t = 1$.

Next, $XH_{i2}^t = X(U_{i1}^t H_{i0}^t + U_{i0}^t H_{i1}^t)$, from which, by (4.8), follows:

$$\mu_i H_{i2}^t + 2H_{i1}^t = XU_{i1}^t \cdot H_{i0}^t + U_{i1}^t \cdot \mu_i H_{i0}^t + H_{i1}^t + U_{i0}^t(\mu_i H_{i1}^t + H_{i0}^t) \ .$$

Therefore we have: $XU_{i1}^t = 0$. We assume that $XU_{i1}^t = XU_{i2}^t = \dots = XU_{ip-1}^t = 0$ for $p \leqq Q_i^t - P_i^t - 2$. Then, from (4.8), since $p+1 \leqq Q_i^t - P_i^t - 1$,

1) Urabe, ibid.

$$XH^t_{ip+1}=\mu_i H^t_{ip+1}+(p+1)\,H^t_{ip}\ .$$

The left-hand side is calculated as follows:

$$\begin{aligned}&X\Big[\sum_{r=0}^{p}\binom{p}{r}U^t_{ip-r}H^t_{ir}\Big]\\&=XU^t_{ip}\cdot H^t_{i0}+U^t_{ip}\cdot\mu_i H^t_{i0}+\sum_{r=1}^{p}\binom{p}{r}U^t_{ip-r}(\mu_i H^t_{ir}+rH^t_{ir-1})+H^t_{ip}\\&=XU^t_{ip}\cdot H^t_{i0}+\mu_i\sum_{r=0}^{p}\binom{p}{r}U^t_{ip-r}H^t_{ir}+p\sum_{r=1}^{p}\binom{p-1}{r-1}U^t_{ip-r}H^t_{ir-1}+H^t_{ip}\\&=XU^t_{ip}\cdot H^t_{i0}+\mu_i H^t_{ip+1}+(p+1)H^t_{ip}\ .\end{aligned}$$

Then we see that $XU^t_{ip}=0$. Thus, by induction, we know that (4.23) is valid.

Now, in the present case, from (4.7), $XU^t_{ip}=0$ for $p=Q^t_i-P^t_i-1,\cdots,Q^t_i-2$. Thus, in the present case also, we have:

$$XH^t_{i0}=\mu_i H^t_{i0}\,,\quad XU^t_{i0}=1\,,\quad XU^t_{ip}=0\quad(p=1,2,\dots,Q^t_i-2)\ .$$

Then, analogously to the first case, we see that (4.8) is valid. Now, we have known that, in the first and second cases, (4.8) is valid. Therefore, by induction, in the present case, (4.8) is also valid for any i.

Collecting these cases, if μ_i are such that the relations (μ) and (λ) in Theorem IV are in one-to-one correspondence, then (4.8) is always valid. Now H^t_{ip} ($p=Q^t_i-P^t_i,\dots,Q^t_i-1$) are independent of one another and moreover, for H^t_{ip} made from the solutions (F_1) in Theorem V, the Jacobian of H^t_{ip} with respect to x^μ is not zero for $x^\mu=0$. Then, for these H^t_{ip}, the operator functions ξ^μ are determined by (4.8), and moreover ξ^μ are regular in the vicinity of $x^\mu=0$ and vanish there.

Summarizing the results we have

Theorem VII. *For the given transformation* $\mathfrak{T}$: $'x^\mu=\varphi^\mu(x)=a^\mu_\nu x^\nu+\dots$, *where either* $0<|\lambda_i|<1$ *or* $|\lambda_i|>1$, *there exists a one parameter group* $\mathfrak{G}$ *of transformations containing the given transformation* $\mathfrak{T}$. *Besides, when the L-th determinant divisor of* $\mathfrak{M}$ *in Theorem IV is unity, there exists a group, of which the operator functions* ξ^μ *are expanded as follows:* $\xi^\mu=c^\mu_\nu x^\nu+\dots$.

When the relations (μ) and (λ) are not in one-to-one correspondence, (4.8) is not necessarily valid, consequently the operator functions of the group obtained above are not necessarily regular.

For example, let the given transformation be

$$\mathfrak{T}:\begin{cases} 'x^1 = \varphi^1(x) = \lambda_1 x^1 \ , \\ 'x^2 = \varphi^2(x) = \lambda_2 x^2 \ , \\ 'x^3 = \varphi^3(x) = \lambda_3 x^3 + (x^1)^3 x^2 + (x^2)^3 \ , \end{cases}$$

where $\lambda_1 = e^{-1}$, $\lambda_2 = e^{-\frac{3}{2}}$, $\lambda_3 = e^{-\frac{9}{2}} = \lambda_1^3 \lambda_2 = \lambda_2^3$. We assume that $t_0 = 1$. The transformation of the variables is as follows: $\bar{x}^1 = x^1$, $\bar{x}^2 = x^2$, $\bar{x}^3 = \frac{\lambda_3 x^3}{(x^1)^3 x^2 + (x^2)^3}$. Then the given transformation $\mathfrak{T}$ is represented with regard to $\bar{x}^\mu$-system as follows: $'\bar{x}^1 = \lambda_1 \bar{x}^1$, $'\bar{x}^2 = \lambda_2 \bar{x}^2$, $'\bar{x}^3 = \bar{x}^3 + 1$. Consequently, the group containing $\mathfrak{T}$ is as follows: $'\bar{x}^1 = e^{\mu_1 t} \bar{x}^1$, $'\bar{x}^2 = e^{\mu_2 t} \bar{x}^2$, $'\bar{x}^3 = \bar{x}^3 + t$. The operator functions ξ^μ of the group become as follows:

(i) For $\mu_1 = -1, \quad \mu_2 = -\frac{3}{2} + 2\pi i, \quad \mu_3 = -\frac{9}{2} + 2\pi i = 3\mu_1 + \mu_2 \neq 3\mu_2$,

$$\xi^1 = \mu_1 x^1, \quad \xi^2 = \mu_2 x^2, \quad \xi^3 = \mu_3 x^3 + \frac{1}{\lambda_3}[(x^1)^3 x^2 + (x^2)^3] + 4\pi i \frac{(x^2)^3 x^3}{(x^1)^3 x^2 + (x^2)^3} \ ,$$

(ii) For $\mu_1 = -1, \quad \mu_2 = -\frac{3}{2} \quad \mu_3 = -\frac{9}{2} = 3\mu_1 + \mu_2 = 3\mu_2$,

$$\xi^1 = \mu_1 x^1, \quad \xi^2 = \mu_2 x^2, \quad \xi^3 = \mu_3 x^3 + \frac{1}{\lambda_3}[(x^1)^3 x^2 + (x^2)^3] \ .$$

§ 5. Reduction of (4. 8).

In this paragraph, we express (4. 8) in terms of the solutions of the equations of Schröder.

Substituting $H^i_{lp} = p!\, G^i_{lp+1}$ into (4. 8), we have:

(5. 1) $$XG^i_{lp} = \mu_i G^i_{lp} + G^i_{lp-1} \ .$$

Let $C' = \| 'c^\mu_\nu \| = \sum_{i=1}^{R} \sum_{l=1}^{L_i} \oplus\, 'C^i_l$, where $'C^i_l$ is a matrix of Q^i_l-th order such that $'C^i_l = \begin{pmatrix} \mu_i & 0 & \cdots & 0 \\ 1 & \mu_i & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 1 & \mu_i \end{pmatrix}$. Then we write (5. 1) briefly as follows:

(5. 2) $$XG^\mu = 'c^\mu_\nu G^\nu \ .$$

Substitute $F^\mu = 'k^\mu_\nu G^\nu$, then we have;

(5. 3) $$XF^\mu = 'd^\mu_\nu F^\nu \ ,$$

where $\| 'd^\mu_\nu \| = D' = K'C'K'^{-1} = \sum_{i=1}^{R} \sum_{l=1}^{L_i} \oplus\, 'D^i_l$ and $'D^i_l = 'K^{i\prime}_l C^i_l ('K^i_l)^{-1}$. When $i = a$, $F^a_{lp} = f^a_{lp}$, consequently (5. 3) are written as follows:

(5. 4) $$Xf^a_{lp} = \sum_{r=1}^{p} \overset{a}{'d}{}^p_r f^a_{lr} \ ,$$

where $\| \overset{a}{'d}{}^p_q \| = D^a_l$.[1] We consider a function of the form $\psi = \prod_{a=1}^{S} \prod_{l=1}^{L_a} \prod_{p=1}^{P^a_l} f^{a\ \overset{S+1}{p}{}^a_{lp}}_{lp}$.

1) For $i = a$, we write D^a_l insead of $'D^a_l$.

Then, from (5.4), $X\psi$ is a linear combination of the functions of the form ψ, consequently, for Ψ_{ip}^{S+1} a linear combination of the functions of the form ψ, $X\Psi_{ip}^{S+1}$ also becomes a linear combination of the functions of the form ψ. Thus, substituting (2.8) into (5.3), we have:

$$(5.5) \qquad Xf_{ip}^{S+1}=\sum_{r=1}^{p}{}'d^{S+1}{}_{Q_i^{S+1}-P_i^{S+1}+r}^{Q_i^{S+1}-P_i^{S+1}+p} f_{ir}^{S+1}+\Phi_{ip}^{S+1} ,$$

where $\Phi_{ip}^{S+1}=L\Big[\prod_{a=1}^{S}\prod_{m=1}^{L_a}\prod_{q=1}^{P_m^a} f_{mq}^{a}{}^{\overset{S+1}{p}{}^{a}_{mq}}\Big]$. Namely the following formulae are valid for $x=S+1$:

$$(5.6) \qquad Xf_{ip}^{x}=\sum_{r=1}^{p}{}'d^{x}{}_{Q_i^x-P_i^x+r}^{Q_i^x-P_i^x+p} f_{ir}^{x}+\Phi_{ip}^{x} ,$$

where $\Phi_{ip}^{x}=L\Big[\prod_{i=1}^{x-1}\prod_{m=1}^{L_i}\prod_{q=1}^{P_m^i} f_{mq}^{i}{}^{\overset{x}{p}{}^{i}_{mq}}\Big]$. We assume that (5.6) is valid for $x=S+1, S+2, \ldots, x-1$. When we associate the number λ_i to f_{ip}^{i}, then, by our assumption, for $i=1, 2, \ldots, x-1$, Xf_{ip}^{i} is a polynomial of the order λ_i. Then, by the same reasoning as on (5.5), we see that (5.6) is valid also for x. Thus, by induction, we see that (5.6) is valid for any x. As in Chap. II § 4, we write as follows:

$$'K_l^i=\begin{pmatrix}\overset{1}{K}{}_l^i & 0\\ \overset{2}{K}{}_l^i & K_l^i\end{pmatrix}, \quad 'C_l^i=\begin{pmatrix}\overset{1}{C}{}_l^i & 0\\ \overset{2}{C}{}_l^i & C_l^i\end{pmatrix}, \quad 'D_l^i=\begin{pmatrix}\overset{1}{D}{}_l^i & 0\\ \overset{2}{D}{}_l^i & D_l^i\end{pmatrix},$$

where K_l^i, C_l^i and D_l^i are of P_l^i-th order. Then it is evident that $D_l^i=K_l^iC_l^i(K_l^i)^{-1}$. Put $\sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus D_l^i=D=\|d_\nu^\mu\|$, then (5.4) and (5.6) are written briefly as follows;

$$(5.7) \qquad Xf^\mu=d_\nu^\mu f^\nu+\Phi^\mu(f) .$$

If we put $\sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus K_l^i=K=\|k_\nu^\mu\|$, then $K\overset{\circ}{C}K^{-1}=D$, where $\overset{\circ}{C}=\sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus C_l^i$. Put $f^\mu=k_\nu^\mu g^\nu$, then (5.7) become as follows:

$$(5.8) \qquad Xg^\mu=\overset{\circ}{c}{}_\nu^\mu g^\nu+\Phi^\mu(g) ,$$

where $\|\overset{\circ}{c}{}_\nu^\mu\|=\overset{\circ}{C}$ and $\Phi^\mu(g)$ are of the same forms as $\Phi^\mu(f)$ with regard to the arguments. In non-abbreviate form, (5.8) are written down as follows:

$$(5.9) \qquad Xg_{ip}^{i}=\mu_i g_{ip}^{i}+g_{ip-1}^{i}+\Phi_{ip}^{i} .$$

These are the characteristic equations of the linear homogeneous partial differential equation.

If we take solutions f^μ of the equations of Schröder given by (f) in Theorem I, then $f^\mu = \tilde{x}^\mu + \cdots\cdots$, where $\tilde{x}^\mu = t^\mu_\nu x^\nu$ and $\| t^\mu_\nu \| = T$ is such that $TAT^{-1} = \overset{\circ}{A}$. Then $g^\mu = K^\mu_\nu f^\nu = K^\mu_\nu \tilde{x}^\nu + \cdots\cdots$, where $\| K^\mu_\nu \| = K^{-1}$. Put $K^{-1}T = S = \| s^\mu_\nu \|$, and $\bar{x}^\mu = s^\mu_\nu x^\nu = K^\mu_\nu \tilde{x}^\nu$, then $g^\mu = \bar{x}^\mu + \cdots\cdots$, namely $g^i_{l\nu} = \bar{x}^i_{l\nu} + \cdots\cdots$, and all the coefficients of the terms $\prod_{j=1}^{i-1} \prod_{m=1}^{L_j} \prod_{q=1}^{P^j_m} \bar{x}^{j}_{mq}{}^{\overset{i}{p}{}^{j}_{mq}}$ vanish. Then, with regard to $\bar{x}^\mu$-system, (5.9) are written as follows: $\bar{\xi}^\mu \dfrac{\partial g^i_{l\nu}}{\partial \bar{x}^\mu} = \mu_i g^i_{l\nu} + g^i_{l\nu-1} + \Phi^i_{l\nu}$, consequently,

$$\bar{\xi}^i_{l\nu} = \mu_i \bar{x}^i_{l\nu} + \bar{x}^i_{l\nu-1} + \cdots\cdots , \tag{5.10}$$

and, from the contravariance of ξ^μ, $\bar{\xi}^\mu = s^\mu_\nu \xi^\nu$. Therefore, if we put $\xi^\mu = c^\mu_\nu x^\nu + \cdots\cdots$, then $C = \| c^\mu_\nu \| = S^{-1} \overset{\circ}{C} S$, namely the Jordan's form of C is $\overset{\circ}{C}$.

Summarizing the results we have

Theorem VIII. *For the given transformation* $\mathfrak{T}$: $'x^\mu = \varphi^\mu(x) = a^\mu_\nu x^\nu + \cdots\cdots$, *we assume that the absolute values of all the eigen values of* $\| a^\mu_\nu \|$ *are either greater or less than unity. When the L-th determinant divisor of* $\mathfrak{M}$ *in Theorem IV is unity, there exists a group containing the given transformation* $\mathfrak{T}$, *of which the operator functions* ξ^μ *satisfy the equations as follows:*

$$Xg^i_{l\nu} = \mu_i g^i_{l\nu} + g^i_{l\nu-1} + \Phi^i_{l\nu} , \tag{C}$$

where $\Phi^i_{l\nu} = L\left[\prod_{j=1}^{i-1} \prod_{m=1}^{L_j} \prod_{q=1}^{P^j_m} g^{j}_{mq}{}^{\overset{i}{p}{}^{j}_{mq}} \right]$, *and* g^μ *are related to the solutions* f^μ *of the equations of Schröder*

$$f^i_{l\nu}(\varphi) = \lambda_i f^i_{l\nu}(x) + f^i_{l\nu-1}(x) + \Psi^i_{l\nu}(x) , \tag{S}$$

in such a way as $f^\mu = k^\mu_\nu g^\nu$. *Here* $\Psi^i_{l\nu}$ *are of the same forms as* $\Phi^i_{l\nu}$ *and* $\| k^\mu_\nu \| = K$ *is a matrix such that* $K = \sum_{i=1}^{R} \sum_{l=1}^{L_i} \oplus K^i_l$, *where* K^i_l *are of the forms*

$$\begin{pmatrix} \times & 0 & \cdots\cdots & 0 \\ \times & \times & \ddots & \vdots \\ \vdots & & \ddots & \ddots \quad \vdots \\ \vdots & & \ddots & 0 \\ \times & \cdots\cdots & & \times \end{pmatrix} \quad \text{and} \quad K^i_l \begin{pmatrix} \lambda_i & \cdots\cdots\cdots\cdots & 0 & \cdots\cdots\cdots & 0 \\ \lambda_i t_0 & & \lambda_i & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & \lambda_i & 0 \\ \lambda_i \dfrac{t_0^{P^i_l - 1}}{(P^i_l - 1)!} & & \cdots\cdots & \lambda_i t_0 & \lambda_i \end{pmatrix} (K^i_l)^{-1} = \begin{pmatrix} \lambda_i & 0 & \cdots\cdots & \\ 1 & \lambda_i & & \vdots \\ 0 & & \ddots & \vdots \\ \vdots & & \lambda_i & 0 \\ 0 & \cdots & 0 \; 1 & \lambda_i \end{pmatrix} .$$

If we write the Jordan's form of $A = \| a^\mu_\nu \|$ *as* $\overset{\circ}{A} = TAT^{-1}$ *and put* $K^{-1}T = S = \| s \|$, *then, for the solutions of* (S) *given by* (f) *in Theorem I, the operator functions* ξ^μ *satisfying* (C) *are expanded as follows:*

$$\xi^\mu = c^\mu_\nu x^\nu + \cdots\cdots , \tag{ξ}$$

where $S\,\|\,c^{u}_{v}\,\|\,S^{-1}=\overset{\circ}{C}=\sum_{i=1}^{R}\sum_{l=1}^{L_i}\oplus C^i_l$. *Here* C^i_l *is a matrix of* P^i_l*-th order such that* $C^i_l=\begin{pmatrix}\mu_i & 0 & \cdots & 0\\ 1 & \mu_i & & \vdots\\ 0 & \ddots & & \\ \vdots & & \ddots & 0\\ 0 & \cdots\; 0 & 1 & \mu_i\end{pmatrix}$, *where* μ_i *are such that the relations* (μ) *and* (λ) *in Theorem IV are in one-to-one correspondence.*

Writing Chap. III (1.1) as follows:

$$\lambda_x=\prod_{a=1}^{S}\prod_{l=1}^{L_a}\lambda_l^{a\,p_l^{a}{}^{x}},$$

where λ^a_l denotes the common value of λ^a_{lp}, we see that, when and only when the L-th determinant divisor of $\mathfrak{M}$ constructed for λ^a_l considered seperately is unity, it is possible to determine μ^i_l for λ^i_l (common value of λ^i_{lp}) by means of (4.3), namely by $\lambda^i_l=e^{t_0\mu^i_l}$, so that the relations (μ) and (λ) in Theorem IV are in one-to-one correspondance. In this case, for $i=x$, all μ^i_l ($l=1, 2, \dots L_i$) are equal to one another, however, for $i=a$, μ^i_l ($l=1, 2, \dots L_i$) are not necessarily equal to one another. Now, from (4.4),

$$H^i_{l0}(\varphi)=\lambda^i_l H^i_{lc}(x)\ .$$

Consequently, with regard to $\bar{x}^\mu$-system, the transformation $\mathfrak{T}$ is represented as follows:

$$\mathfrak{T}:\qquad '\bar{x}^\alpha=e^{t_0\mu^i_l}\bar{x}^\alpha,\quad '\bar{x}^\sigma=\bar{x}^\sigma+t_0\,,\quad '\bar{x}^\omega=\bar{x}^\omega\ .$$

Therefore, there exists a group $\mathfrak{G}'$ containing $\mathfrak{T}$ such that

$$\mathfrak{G}':\qquad '\bar{x}^\alpha=e^{t\mu^i_l}\bar{x}^\alpha,\quad '\bar{x}^\sigma=\bar{x}^\sigma+t\,,\quad '\bar{x}^\omega=\bar{x}^\omega\ .$$

In this case, the relations (μ) and (λ) in Theorem IV are in one-to-one correspondence for λ^i_{lp} and μ^i_{lp} such that

$$\lambda^i_{lp}=\lambda^i_l=\lambda_i\ ;$$

$$\mu^a_{lp}=\mu^a_l\,,\quad \mu^x_{lp}=\mu^x_l=\mu_x\,,$$

and the reasonings in § 4 and § 5 are also valid. Thus we see that *Theorem VIII is valid by substitution as follows:*

for $\mathfrak{M}$ *constructed for* λ_i, $\mathfrak{M}$ *constructed for* λ^a_l;

for (C), (C′) *as follows:*

$$(C')\qquad Xg^i_{lp}=\mu^i_l g^i_{lp}+g^i_{lp-1}+\Phi^i_{lp}\ ;$$

for $C^i_l=\begin{pmatrix}\mu_i & 0 & \cdots & 0\\ 1 & \mu_i & & \vdots\\ 0 & \ddots & \ddots & \\ \vdots & & \ddots\;\ddots & 0\\ 0 & \cdots\; 0 & 1 & \mu_i\end{pmatrix}$, C^i_l *as follows:* $C^i_l=\begin{pmatrix}\mu^i_l & 0 & \cdots & 0\\ 1 & \mu^i_l & & \vdots\\ 0 & \ddots & \ddots & \\ \vdots & & \ddots\;\ddots & 0\\ 0 & \cdots\; 0 & 1 & \mu^i_l\end{pmatrix}$.

We call the theorem obtained in such a way *Theorem VIII′*.

When the L-th determinant divisor of $\mathfrak{M}$ constructed for λ_t is unity, there exist μ_t such that $\mu_l^t = \mu_t$, consequently the L-th determinant divisor of $\mathfrak{M}$ constructed for λ_l^a is unity. Thus the condition in Theorem VIII′ is weaker than that in Theorem VIII.

§ 6. Remarks.

When the L-th determinant divisor of $\mathfrak{M}$ in Theorem IV is unity, by Theorem VIII′, we see that, when either $0 < |\lambda_t| < 1$, or $|\lambda_t| > 1$, there exists a group containing the given transformation $\mathfrak{T}$, of which the operator functions ξ^μ are expanded as $\xi^\mu = c_\nu^\mu x^\nu + \cdots\cdots$. In this case, the eigen values μ_l^t of $\| c_\nu^\mu \|$ are so related to λ_t that $\lambda_t = e^{t_0 \mu_l^t}$, consequently real parts of all μ_l^t are negative or positive, therefore all μ_l^t lie in a convex domain which does not contain the origin. Thus, in this case, the conditions in Theorem II are all satisfied. However, it is evident that, even when the conditions in Theorem II are all satisfied, the absolute values of all the eigen values λ_t are not necessarily either greater or less than unity. In this sense, when the L-th determinant divisor of $\mathfrak{M}$ is unity, the conditions in Theorem II, although they are apparently complicated, are weaker than the conditions in Theorem I. However, when the L-th determinant divisor of $\mathfrak{M}$ is not unity, by Theorem IV, there does not exist μ_l^t so that the relations (μ) and (λ) are in one-to-one correspondence, consequently the equations of Schröder (S) and the equations (S′) in Theorem II do not coindide with each other. In this case, in Theorem I and II, the conditions themselves slip out each other, and the results also do, although they are resembled closely.

Chapter V. Totality of groups.

§ 1. Preliminaries.

By Theorem VIII′, when the L-th determinant divisor of $\mathfrak{M}$ is unity, we have seen that, when either $0 < |\lambda_t| < 1$ or $|\lambda_t| > 1$, for any μ_l^t such that the relations (μ) and (λ) in Theorem IV are in one-to-one correspondence, there exists a group, of which the operator functions are expanded as follows:

$$\xi^\mu = c_\nu^\mu x^\nu + \ldots\ldots , \tag{1.1}$$

where the Jordan's form $\overset{\circ}{C}$ of $C = \| c_\nu^\mu \|$ is of the form $\overset{\circ}{C} = \sum_{t=1}^{R} \sum_{l=1}^{L_t} \oplus C_l^t$,

$C_l^i = \begin{pmatrix} \mu_l^i & 0 & \cdots\cdots & 0 \\ 1 & \mu_l^i & & \vdots \\ 0 & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \mu_l^i \end{pmatrix}$. In this chapter, we shall investigate the totality of the groups containing the given transformation $\mathfrak{T}$.

For subsequent discussions, we make some preparations.

Let the operator functions ξ^μ of any group containing $\mathfrak{T}$ be (1.1). Then, by Chap. II (2.5),

$$(1.2) \qquad A = e^{t_0 C} .$$

Let the Jordan's form of A be $\overset{\circ}{A} = TAT^{-1} = \sum_{i=1}^{R}\sum_{l=1}^{L_i} \oplus A_l^i$. Put $\sum_{l=1}^{L_i} \oplus A_l^i = A_i$, then $A_i = \lambda_i I_i + N_i$, where I_i are unit matrices and $N_i = \sum_{l=1}^{L_i} \oplus N_l^i$, $N_l^i = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 \\ 1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$. Put $\overset{\circ}{L}(A_i) = \operatorname{Log} \lambda_i \cdot I_i + \sum_{r=1}^{m_i - 1} (-1)^{r-1} \frac{(N_i)^r}{r(\lambda_i)^r}$,

where $\operatorname{Log} \lambda_i$ denotes the principal value of $\log \lambda_i$ and $m_i = \sum_{l=1}^{L_i} P_l^i$. Put

$$(1.3) \qquad L(A) = T^{-1}\Big[\sum_{i=1}^{R} \oplus \overset{\circ}{L}(A_i)\Big] T ,$$

then, evidently $e^{L(A)} = A$. By K. Morinaga and T. Nōno,[1] the general solutions of the matrix equation $e^X = A$ are as follows:

$$(1.4) \qquad X = L(A) + P^{-1}FP ,$$

where $F = T^{-1}[\sum_{i=1}^{R} \oplus F_i]T$, $F_i = 2\pi\sqrt{-1}\sum_{l=1}^{L_i} \oplus n_l^i I_l^i$ (n_l^i: arbitrary integers) and P is an arbitrary matrix commuting with A. Then P is of the from $P = T^{-1}[\sum_{i=1}^{R} \oplus P_i]T$, where P_i are of the m_i-th order, and it is easily seen that $L(A)$ commutes with P. Then, from (1.2), it is seen that

$$(1.5) \qquad C = \frac{1}{t_0}\Big[L(A) + P^{-1}FP\Big] = P^{-1}\Big[\frac{1}{t_0}\Big\{L(A) + F\Big\}\Big] P .$$

Consequently the Jordan's form $\overset{\circ}{C}$ of C must be of the form as follows: $\overset{\circ}{C} = \sum_{i=1}^{R}\sum_{l=1}^{L_i} \oplus C_l^i$, where $C_l^i = \begin{pmatrix} \mu_l^i & 0 & \cdots\cdots & 0 \\ 1 & \mu_l^i & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \mu_l^i \end{pmatrix}$ and $\mu_l^i = \frac{1}{t_0}(\operatorname{Log} \lambda_i + 2n_l^i \pi\sqrt{-1})$.

1) This Journal, Vol. 14, No. 2 (1950), p. 111.

Then, from the last part of Chap. IV § 5, *the necessary and sufficient condition that, for such* λ_i *and* μ_l^i, *the relations* (μ) *and* (λ) *in Theorem IV may be in one-to-one correspondence, is that, for* $i=x$, *all* μ_l^i *are equal to one another and the L-th determinant divisor of* $\mathfrak{M}$ *constructed for* λ_l^a *is unity, namely that this* $\overset{\circ}{C}$ *coincides with the Jordan's form of* C *in Theorem VIII'.*

§ 2. Non-existence of group.

Let the operator functions ξ^μ of any group containing $\mathfrak{X}$ be (1.1). Then we have (1.2). Therefore, by our assumption that either $0<|\lambda_i|<1$ or $|\lambda_i|>1$, the real parts of all μ_l^i are either negative or positive. Then, by Theorem II, there exist regular solutions of the equations (S'). However, when the Jordan's form $\overset{\circ}{C}$ of C of ξ^μ does not coincide with the Jordan's form of C in Theorem VIII', from the result of § 1, the relations (μ) and (λ) in Theorem IV are not in one-to-one correspondence, consequently from the remarks in Chap. II § 3, the equations (S') have not in general regular solutions. Thus it is seen that, in general, there does not exist any group, of which the Jordan's form $\overset{\circ}{C}$ of C does not coincide with that of C in Theorem VIII'.

Even when $\overset{\circ}{C}$ does not coincide with the Jordan's form in Theorem VIII', if the equations (S') have regular solutions of the form (f') for $\overset{\circ}{C}$, then, by the analogous discussions as those in Chap. IV, we see that there exists a group. From this result and Theorem II, we have

Theorem IX. *When the Jordan's form* $\overset{\circ}{C}$ *of* C *such that* $A=e^{t_0 C}$ *does not coincide with that of* C *in Theorem VIII', the necessary and sufficient condition that, for that* $\overset{\circ}{C}$, *there may exist the group possessing the regular operator functions, is that the equations (S') in Theorem II have regular solutions of the form* (f') *for the eigen values* μ_i *of* $\overset{\circ}{C}$.

§ 3. Uniqueness of group.

Let the operator functions of the group $\mathfrak{G}$ obtained in Theorem VIII be as follows:

$$\xi^\mu = c_\nu^\mu x^\nu + \cdots\cdots . \tag{3.1}$$

Let the operator functions of any other group $\mathfrak{G}'$ be as follows:

$$'\xi^\mu = {}'c_\nu^\mu x^\nu + \cdots\cdots . \tag{3.2}$$

We assume that the Jordan's forms of $C=\| c_\nu^\mu \|$ and $C'=\| {}'c_\nu^\mu \|$ coincide with each other, and let common Jordan's form be $\overset{\circ}{C}=\sum_{i=1}^{R}\sum_{l=1}^{L_i} \oplus C_l^i$, where

$C_i^t=\begin{pmatrix}\mu_i & 0 & \cdots\cdots & 0\\ 1 & \mu_i & & \vdots\\ 0 & \ddots & \ddots & \vdots\\ \vdots & & \ddots\ \ddots & 0\\ 0 & \cdots & 0\ 1 & \mu_i\end{pmatrix}$. In this case, $\mu_i^t=\mu_i$, therefore, in (1.4), $F_i=2n_i\pi\sqrt{-1}\,I_i$

where n_i are integers, consequently $P^{-1}FP=F$ for any P. Thus

$$C=C'=\frac{1}{t_0}\Big[L(A)+F\Big]\,. \tag{3.3}$$

For ξ^μ of (3.1), by Theorem VIII, it is valid that

$$Xg_{ip}^t=\mu_i g_{ip}^t+g_{ip-1}^t+\Phi_{ip}^t \tag{3.4}$$

for g^ν such that $f^\mu=k_\nu^\mu g^\nu$, where f^μ are the solutions of the equations as follows:

$$f_{ip}^t(\varphi)=\lambda_i f_{ip}^t(x)+f_{ip-1}^t(x)+\Psi_{ip}^t(x)\ . \tag{3.5}$$

Moreover, by the same theorem, if $T=\|\,t_\nu^\mu\,\|$ and $K=\|\,k_\nu^\mu\,\|$ are the matrices such that $TAT^{-1}=\overset{\circ}{A}$, $Ke^{t_0\overset{\circ}{C}}K^{-1}=\overset{\circ}{A}$, and $S=\|\,s_\nu^\mu\,\|=K^{-1}T$, then $f^\mu=\tilde{x}^\mu+\cdots\cdots$, $g^\mu=\bar{x}^\mu+\cdots\cdots$, and $SCS^{-1}=\overset{c}{C}$. Now, because of (3.3), if we consider the characteristic equations for $'\xi^\mu$ as follows:

$$'X'g_{ip}^t=\mu_i{}'g_{ip}^t+{}'g_{ip-1}^t+{}'\Phi_{ip}^t\ , \tag{3.4$'$}$$

then these equations have regular solutions as follows:

$$'g^\mu=\bar{x}^\mu+\cdots\cdots\ .$$

Put $'f^\mu=k_\nu^\mu{}'g^\nu$, then, by Chap. II (3.9), we have:

$$'f_{ip}^t(\varphi)=\lambda_i{}'f_{ip}^t(x)+{}'f_{ip-1}^t(x)+{}'\Psi_{ip}^t(x)\ . \tag{3.5$'$}$$

and $'f^\mu=\tilde{x}^\mu+\cdots\cdots$. Here both f^μ and $'f^\mu$ are of the form (f) in Theorem I. Therefore, by Theorem I, $'f^\mu=f^\mu$ and $'\Psi_{ip}^t=\Psi_{ip}^t$, consequently $'g^\mu=g^\mu$. Thus we have:

$$\begin{cases} Xg_{ip}^t=\mu_i g_{ip}^t+g_{ip-1}^t+\Phi_{ip}^t\ ,\\ 'Xg_{ip}^t=\mu_i g_{ip}^t+g_{ip-1}^t+{}'\Phi_{ip}^t\ ,\end{cases} \tag{3.6}$$

and the results obtained by integrating both equations are the same equations (3.5).

We consider the functions $\overset{x}{\varphi}{}^W=\prod_{t=1}^{x-1}\prod_{l=1}^{L_t}\prod_{p=1}^{P_l^t} g_{ip}^{t\ \overset{x}{p}{}_{ip}^t}$ and arrange them according to the weights of the indices. Since g_{ip}^t are independent, $\overset{x}{\varphi}{}^W$ are linearly independent with regard to constant coefficients. Then $\Phi_{ip}^x=c_{ipW}^x\overset{x}{\varphi}{}^W$ and c_{ipW}^x are uniquely determined. Integrating $Xg_{ip}^a=\mu_a g_{ip}^a+g_{ip-1}^a$, by Chap. II (3.4), we have:

$$g_{ip}^a('x)=e^{\mu_a t}\left[\frac{t^{p-1}}{(p-1)!}g_{i1}^a(x)+\frac{t^{p-2}}{(p-2)!}g_{i2}^a(x)+\cdots+tg_{ip-1}^a(x)+g_{ip}^a(x)\right]. \tag{3.7}$$

Then we have:

$$(3.8)\qquad \overset{S+1}{\varphi}{}^{W}('x)=\prod_{a,i,p} g^a_{ip}('x)^{\overset{S+1}{p}{}^a_{ip}}$$

$$=e^{\mu_{S+1}t}\prod_{a,i,p}\left[\frac{t^{p-1}}{(p-1)!}g^a_{i1}(x)+\cdots+tg^a_{ip-1}(x)+g^a_{ip}(x)\right]^{\overset{S+1}{p}{}^a_{ip}}$$

$$=e^{\mu_{S+1}t}\,\overset{S+1}{\kappa}{}^W_V(t)\,\overset{S+1}{\varphi}{}^V(x)\ ,$$

here

$$(3.9)\qquad \text{for } V>W^{(1)},\ \overset{S+1}{\kappa}{}^W_V(t)=0 \text{ except for } V=W,\quad \text{and } \overset{S+1}{\kappa}{}^W_W(t)=1\ .$$

Then, after the substitution of $'x^\mu$ for x^μ, the equations $Xg^{S+1}_{i1}=\mu_{S+1}g^{S+1}_{i1}+\Phi^{S+1}_{i1}$ become as follows:

$$\frac{dg^{S+1}_{i1}('x)}{dt}=\mu_{S+1}g^{S+1}_{i1}('x)+e^{\mu_{S+1}t}\,c^{S+1}_{i1W}\,\overset{S+1}{\kappa}{}^W_V(t)\,\overset{S+1}{\varphi}{}^V(x)\ .$$

Integrating these equations, we have:

$$g^{S+1}_{i1}('x)=e^{\mu_{S+1}t}\left[\,g^{S+1}_{i1}(x)+\,c^{S+1}_{i1W}\,\overset{S+1}{\varphi}{}^V(x)\int_0^t \overset{S+1}{\kappa}{}^W_V(t)\ dt\right]\ .$$

By induction on p, we can easily prove that, for $x=S+1$, the following formulae are valid:

$$(3.10)\quad g^x_{ip}('x)=e^{\mu_x t}\left[\frac{t^{p-1}}{(p-1)!}g^x_{i1}(x)+\cdots+tg^x_{ip-1}(x)+g^x_{ip}(x)+\Psi^x_{ip}(x,t)\right]\ ,$$

where

$$(3.11)\qquad \Psi^x_{ip}(x,t)=\overset{x}{\varphi}{}^V(x)\Big[\,c^x_{i1W}\underbrace{\int_0^t dt\cdots\int_0^t dt}_{p\text{ times}}+c^x_{i2W}\underbrace{\int_0^t dt\ldots\int_0^t dt}_{p-1\text{ times}}$$

$$+\cdots+c^x_{ipW}\int_0^t dt\Big]\,\overset{x}{\kappa}{}^W_V(t)\ .$$

We assume that (3.10) are valid for $x=S+1,\cdots,x-1$. Then,

$$\overset{x}{\varphi}{}^W('x)=\prod_{t=1}^{x-1}\prod_{i=1}^{L_t}\prod_{p=1}^{P^t_i} g^t_{ip}('x)^{\overset{x}{p}{}^t_{ip}}$$

$$=e^{\mu_x t}\prod_{t=1}^{x-1}\prod_{i=1}^{L_t}\prod_{p=1}^{P^t_i}\left[\frac{t^{p-1}}{(p-1)!}g^t_{i1}(x)+\cdots+tg^t_{ip-1}(x)+g^t_{ip}(x)+\Psi^t_{ip}(x,t)\right]^{\overset{x}{p}{}^t_{ip}}$$

$$=e^{\mu_x t}\,\overset{x}{\kappa}{}^W_V(t)\ \overset{x}{\varphi}{}^V(x)\ .$$

Now, from the definition of the weights, the weights of indices of $\overset{t}{\varphi}{}^V(x)$ are less by at least one than w^t_{i1}. Consequently

1) For the meaning of the symbol $>$, cf. Chap. IV §4.

(3.12) for $V > W$, $\overset{x}{\kappa}{}_V^W(t)=0$ except for $V=W$, and $\overset{x}{\kappa}{}_W^W(t)=1$.

Then, in the same way as in the case $x=S+1$, we can prove that (3.10) are valid also for x. Thus, for any x, (3.10) are valid, and (3.12) also do.

Then, putting $t=t_0$ in (3.7) and (3.10), we have:

$$g_{ip}^i(\varphi)=\lambda_i\left[\frac{t_0^{\,p-1}}{(p-1)!}g_{i1}^i(x)+\cdots+t_0 g_{ip-1}^i(x)+g_{ip}^i(x)+\Psi_{ip}^i(x)\right], \tag{3.13}$$

where

$$\Psi_{ip}^i(x)=\overset{i}{\varphi}{}^V(x)\Big[c_{i1W}^i\underbrace{\int_0^{t_0}dt\ldots\int_0^t dt}_{p\text{ times}}+c_{i2W}^i\underbrace{\int_0^{t_0}dt\ldots\int_0^t dt}_{p-1\text{ times}} \tag{3.14}$$
$$+\cdots+c_{ipW}^i\int_0^{t_0}dt\Big]\overset{i}{\kappa}{}_V^W(t) .$$

From (3.9) and (3.12), it is readily seen that det. $\left|\int_0^{t_0}\overset{i}{\kappa}{}_V^W(t)\,dt\right|\neq 0$. Then, from (3.14), if Ψ_{ip}^i are given, because of independence of $\overset{i}{\varphi}{}^V(x)$, c_{i1W}^i, c_{i2W}^i, ... c_{ipW}^i are uniquely determined successively, namely Φ_{ip}^i are uniquely determined.

Now, g_{ip}^i of (3.13) are related to the solutions f_{ip}^i of the equations of Schröder in such a way as $f^\mu=k_\nu^\mu g^\nu$. Therefore, when the equations of Schröder are given, Ψ_{ip}^i in (3.13) are uniquely determined, consequently Φ_{ip}^i are uniquely determined.

Then, since the same equations of Schröder are obtained from both of (3.6), Φ_{ip}^i and $'\Phi_{ip}^i$ must coincide. Then $'Xg_{ip}^i=Xg_{ip}^i$, consequently, $'\xi^\mu$ and ξ^μ must coincide. Namely the group $\mathfrak{G}'$ must coincide with the group $\mathfrak{G}$.

Thus we have

Theorem X. *The group, of which the operator functions ξ^μ are expanded as $\xi^\mu=c_\nu^\mu x^\nu+\cdots\cdots$, and where the Jordan's form of $\|c_\nu^\mu\|=C$ coincides with that of $\overset{\circ}{C}$ of the group obtained in Theorem VIII, is uniquely determined when the Jordan's form is fixed, and there does not exist such group other than that obtained in Theorem VIII.*

§ 4. Total groups. (i).

Let the operator functions ξ^μ of any group be

$$\xi^\mu=c_\nu^\mu x^\nu+\cdots\cdots . \tag{4.1}$$

We assume that the Jordan's form $\overset{\circ}{C}$ of $C=\|c_\nu^\mu\|$ coincides with that of C in Theorem VIII'. Given the Jordan's form $\overset{\circ}{C}$, we seek for general solutions C of the matrix equation $e^{t_0 C}=A$. Let C be any solution, and $SCS^{-1}=\overset{\circ}{C}$.

Then, making use of any fixed K such that $Ke^{t_0\mathring{C}}K^{-1}=\mathring{A}$, we have:

$$S^{-1}K^{-1}\mathring{A}KS=A\ .$$

Namely $KS=T$ is a matrix which transforms A to its Jordan's form in such a way as $TAT^{-1}=\mathring{A}$. Thus we have:

$$C=S^{-1}\mathring{C}S\ , \tag{4.2}$$

where $S=K^{-1}T$. Conversely, if T is any matrix transforming A to its Jordan's form, then, for C defined by (4.2), we have:

$$e^{t_0C}=S^{-1}e^{t_0\mathring{C}}S=S^{-1}K^{-1}\mathring{A}KS=T^{-1}\mathring{A}T=A\ .$$

Thus, for arbitrary T, (4.2) gives the general solutions.

Now, by Theorem VIII′, for given $\mathring{C}$, C of the group obtained there is as follows: $C=S^{-1}\mathring{C}S$. Thus, if we vary T, then we get all possible C [1]. However, by the same reasonings as in § 3, we see that all operator functions having common C coincide with one another. Thus it is seen that all possible groups are obtained by varying T from the groups obtained in Theorem VIII′. Thus we have

Theorem XI. *When the Jordan's form of C of the operator functions of the group $\mathfrak{G}$ coincides with that of the group obtained in Theorem VIII′, $\mathfrak{G}$ is a group obtained for a suitable T in Theorem VIII′, namely there exists no other such group than that obtained in Theorem VIII′.*

§ 5. Total groups. (ii).

In Theorem IX, if the equations (S′) have formal solutions f^μ of the form (f'), then, (S′) being the equations of Schröder, f^μ converge, namely f^μ become regular functions. Thus the condition in Theorem IX can be replaced by the weaker condition that (S′) have formal solutions of the form (f') for the eigen values μ_i of $\mathring{C}$. In the case where this condition is satisfied, the solutions of the form (f') of (S′) are not unique, but the coefficients of $\prod_{j,m,q}\tilde{x}^{j\ \ \overset{ij}{p}_{mq}}_{mq}$ for $\overset{i}{p}{}^{j}_{mq}$ such that $\lambda_i=\prod_{j,m,q}\lambda^{j\ \ \overset{ij}{p}_{mq}}_{mq}$ and $\mu_i\neq\sum_{j,m,q}\mu^j_{mq}\overset{\circ}{p}{}^j_{mq}$, are arbitrary. Then, when the solutions f^μ and $'f^\mu$ of the form (f') have different coefficients in the terms stated above, the corresponding ξ^μ and $'\xi^\mu$ which are determined by the equations of the forms as follows:

$$Xg^i_{ip}=\mu_i g^i_{ip}+g^i_{ip-1}+\Phi^i_{ip} \tag{C}$$

1) In the case of Theorem VIII, C is unique, namely C is unaltered even when T is varied. On the contrary, in the case of Theorem VIII′, C may be altered when T is varied.

for g^μ related as $f^\mu = k^\mu_\nu g^\nu$, have the same coefficients in the terms of the first order, but they do not coincide with each other. For, g^μ and $'g^\mu$ are of the form (f') in terms of $\bar{x}^\mu$, therefore, by the previous paper[1], if ξ^μ and $'\xi^\mu$ coincide with each other, then it must be $g^\mu = 'g^\mu$, consequently $f^\mu = 'f^\mu$. This is a contradiction. Now, let $\xi^\mu = c^\mu_\nu x^\nu + \cdots\cdots$ be the operator functions of any group containing $\mathfrak{T}$ and assume that the Jordan's form of $C = \| c^\mu_\nu \|$ is $\overset{\circ}{C}$. Then, by Theorem II, the equations of the form (C) have regular solutions g^μ of the form (f') in terms of $\bar{x}^\mu$, consequently $f^\mu = k^\mu_\nu g^\nu$ satisfy the equations (S′) and f^μ are of the form (f') in terms of $\tilde{x}^\mu$. Namely ξ^μ of any group are determined by the equations of (C). Thus, by the same reasonings as in § 4, we have

Theorem XII. *When the equations (S′) have formal solutions of the form (f') for the eigen values μ_i of C which does not coincide with that of the group obtained in Theorem VIII′, the operator functions of all the groups are obtained from all the solutions f^μ of the form (f') of the equations (S′) by varying T such that $TAT^{-1} = A$.*

Summarizing the results, we see that all the groups having regular operator functions are obtained in the following way:

Jordan's form $\overset{\circ}{C}$	Solutions f^μ of the form (f')	T
Th. VIII.	any one fixed	any one fixed
Th. VIII′.	any one fixed	all varied
not Th. VIII′.	all varied	all varied

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1) M. Urabe, ibid.