THE CHARACTERIZATION OF PARTITION LATTICES

By

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The problem of giving an axiomatic characterization of a partition lattice—a lattice of all partitions of a set—can be formulated in different ways. O. Ore [1][1)] has first characterized a partition lattice as a goemetric system with points and lines with rather peculiar properties[2)] which are not familiar to us. In this paper a partition lattice is characterized in § 1 as a geometric system with points, lines, and moreover, with planes and parallel lines which satisfy the following properties:

G I. *Any line contains at most three points.*

G II. *Any two points on a line defines the same line.*

G III. *Any plane contains only three, four or six points.*

G IV. *If a line has a parallel line, then it has just two ones.*

G V. *If a line contains only two points, then it has always a parallel line.*

By applying the results obtained in § 1, we shall show in § 2 that any partition lattice is a matroid, planer lattice[3)] with a few properties of its points.

In § 3 we shall prove that such a lattice is isomorphic to the lattice of all partitions of a set if and only if it is irreducible (Theorem 3·2).

§ 1. Geometrical Characterization of Partition Lattices.

DEFINITION 1·1. A *partion* P of a set S is a decomposition of S into subsets $C_\alpha(\alpha \in \mathrm{I})$ such that every point in S belongs to one and only one set $C_\alpha(\alpha \in \mathrm{I})$. We shall call the sets $C_\alpha(\alpha \in \mathrm{I})$ the *blocks* of partition P. *Let* $P_1 \leqq P_2$ mean that P_1 is a subpartition of P_2, that is, the blocks in P_1 are obtained by subdivisions of the blocks of P_2. Then the system of all partitions of a set S forms a lattice and it is called a *partition lattice.*

DEFINITION 1·2. Let G be a set of points. With every pair of different

1) The numbers in square brackets refer to the list of references at the paper.

2) Cf. Definition 1·2 below.

3) Cf. Definition 2·3 and 2·4 below.

points p and q in G there will be associated a unique set $p+q$ of elements in G containing p and q. The set $p+q$ we shall call the *line* defined by p and q.

G is called a *general partition geometry*[4] if and only if its points and lines satisfy the following axioms:

G 1. *Any line contains at most three points.*

G 2. *Any two points on a line defines the same line.*

G 3. *Let p_1, p_2 and p_3 be distinct points on a same line and q a point not on this line such that the line $q+p_1$ contains three points. Then one and only one of the lines $q+p_2$ and $q+p_3$ contains three points.*

G 4. *Let p_1, p_2 and p_3 be distinct points on a same line and q, r points outside this line such the lines*

$$q+p_1,\ q+p_2,\ r+p_1 \text{ and } r+p_2$$

contain three points. Then the line $q+r$ contains three points.

Any general partition geometry G is called a *partition geometry* if and only if it satisfies the following axiom:

G 5. *For any two points p and q such that the line $p+q$ contains only two points, there exists a third point r so that both the lines $p+r$ and $q+r$ contain three points.*

DEFINITION 1·3. Let A be a subset of a general partition geometry G. Then the set A is called an *additively closed set* of G if and only if $p, q \in A$ implies $p+q \subset A$.

O. Ore [1][5] has geometrically characterized a partition lattice as follows:

THEOREM 1·1. *The system of all additively closed sets of a partition geometry is isomorphic to the lattice of all partitions of some set. Conversely the lattice of all partitions of a set is isomorphic to the lattice of all additively closed sets of some partition geometry.*

We shall below introduce the concepts of planes and parallel lines in a general partition geometry G and study several lemmas concerning these concepts.

DEFINITION 1·4. When the line $p+q$ contains three points, we shall say that two points p and q are *related* (in symbols $(p, q)r$). Otherwise they shall be called *unrelated* (in symbols $(p, q)\bar{r}$).

Three points p, q and r on a line shall be said to be *collinear* (in sym-

4) Cf. O. Ore [1] Chapter 4, 612-615.

5) Cf. O. Ore [1] 617, Theorem 1.

bols $(pqr)C$). Otherwise they are *non-collinear.*

LEMMA 1·1. *Let $p_i(i=1, 2, 3)$, r and $r_j(j=1, 2)$ be points in a general partition geometry G such that*

$$r \notin p_1+p_2,\ (p_1p_2p_3)C,\ (rr_1p_1)C \text{ and } (rr_2p_2)C.$$

Then we have

$$(p_1,\ r_2)\bar{r},\ (p_2,\ r_1)\bar{r},\ (p_3,\ r)\bar{r} \text{ and } (r_1,\ r_2,\ p_3)C.$$

PROOF. By hypothesis $(rr_1p_1)C$, $(p_2, r)r$ and $(p_2, p_1)r$, whence $(p_2, r_1)\bar{r}$ by G 3. Since moreover $(rr_2p_2)C$ and $(r_1, r)r$ by the assumption, the line r_1+r_2 contains the third point r_3 by G 3. Therefore it is sufficient to show that p_3 is coincident with r_3. Suppose that p_3 is different from r_3.

Then since $(r_1r_2r_3)C$, $(r, r_1)r$ and $(r, r_2)r$, we have $(r, r_3)\bar{r}$ by G 3, while $(rr_2p_2)C$, $(r_2, r_3)r$ by hypothesis, whence $(p_2, r_3)r$ by G 2. Similarly we have $(p_1, r_3)r$.

Consequently two points r and r_3 not on the line p_1+p_2 are related to p_1 and p_2. Hence by G 4 r is related to r_3, contrary to $(r, r_3)\bar{r}$. And so r_3 coincides with p_3, whence $(r_1r_2p_3)C$. Therefore the rest of this proposition follows directly from G 3.

DEFINITION 1·5. Let p, q and r be non-collinear. Then the set of all points on the lines which are defined by any two points on the three lines $q+r$, $r+p$ and $p+q$, shall be called the *plane* defined by p, q and r and we shall denote it by $p+q+r$. When two lines $p+q$ and $r+s$ on the same plane have no common point, we shall say that the lines $p+q$ and $r+s$ are *parallel* to each other (in symbols $p+q \parallel r+s$).

LEMMA 1·2. *Any plane contains only three, four or six points.*

PROOF. According to Definition 1·5, there exist non-collinear three points p, q and r which define the plane. The following cases occur.

Case 1. (p, q), (q, r) and (r, p) are all unrelated pairs.

Then by Definition 1·5 the plane $p+q+r$ contains only the three points p, q and r.

Case 2. At least one of the three pairs (p, q), (q, r) and (r, p) is a related pair.

Without loss of generality we can assume that p and q are related, and s is the third point of the line $p+q$. Then the following two cases occur.

(2 *a*). r is related to none of the three points p, q and s.

5) Cf. O. Ore [1] 617, Theorem 1.

Then by Definition 1·5 obviously the plane $p+q+r$ contains only the four points p, q, r and s.

(2 *b*). r is related to at least one of the three points p, q and s.

Without loss of generality we can assume that r is related to p and t is the third point of the line $p+r$. Then according to G 3 we have

$$(r,\ q)r \text{ and } (r,\ s)\bar{r}, \text{ or } (r,\ q)\bar{r} \text{ and } (r,\ s)r.$$

In the former case, let u be the third point on the line $r+q$, then by Lemma 1·1 we have

$$(p,\ u)\bar{r},\ (r,\ s)\bar{r},\ (t,\ q)\bar{r} \text{ and } (tus)C.$$

Therefore by Definition 1·5 the plane $p+q+r$ contains only the six points p, q, r, s, t and u. Similarly in the latter case it is readily seen that the plane $p+q+r$ contains only six points.

REMARK 1·1. According to the proof of Lemma 1·2 it is readily seen that the following statements hold:

(α) Let p, q and r be non-collinear on a line, then they define the same plane.

(β) Let s be a point on the plane $p_1+p_2+p_3$, then there exists a point such that

$$s \in p_i+r \text{ and } r \in p_j+p_k \text{ for some permutation } i,\ j,\ k \text{ of } 1,\ 2,\ 3.$$

LEMMA 1·3. *Let p, q, r and s be different points in G. Then the following propositions are equivalent:*

(α) $p+q \parallel r+s$

(β) *There exist two points o and o' such that*

$$(pro)C,\ (qso)C,\ (pso')C \text{ and } (qro')C.$$

(γ) *There exists a point o with $(pro)C$ and $(qso)C$, and we have $(p,\ q)\bar{r}$ or $(r,\ s)\bar{r}$.*

PROOF. According to the proof of Lemma 1·2, one can immediately verify that there exist parallel lines on a plane only in the case (2 *b*), whence (α) implies (β). And moreover (β) implies (γ) by G 3, and (γ) implies (α) by Definition 1·5, completing the proof.

LEMMA 1·4. *If a line has a parallel line, then it has just two ones.*

PROOF. Let $p+q$ and $r+s$ be parallel lines. Then by Lemma 1·3 there exist two points o and o' such that

$$(pro)C,\ (qso)C,\ (pso')C \text{ and } (qro')C,$$

and so $p+q \| o+o'$. Consequently the line $p+q$ has at least two parallel lines. Suppose that the line $p+q$ has a parallel line $x+y$ except $r+s$ and $o+o'$. Then by Lemma 1·3, there exist as above, two points t and t' such that $(pxt)C$, $(qyt)C$, $(pyt')C$ and $(qxt')C$.

Now $(p, q)\bar{r}$ yields $(o, y)\bar{r}$ and $(o, t')\bar{r}$ by G 4. But G 3 shows that this is impossible, since $(pyt')C$ and $(o, p)r$. Therefore the line $p+q$ has just two parallel lines and the proof is complete.

In the following, we shall assume that a general partition geometry G is especially a partition geometry, that is, it satisfies the axiom G 5.

Then the following lemma holds.

LEMMA 1·5. *If the line $p+q$ contains only two points, then it has always a parallel line.*

PROOF. Since the line $p+q$ contains only two points, by G 5 there exists a third point r such that the lines $r+p$ and $r+q$ contain the third points s and t respectively. Hence by Lemma 1·3 the line $s+t$ is parallel to the line $p+q$.

We now prove the principal results of this section.

THEOREM 1·2. *Let G be a general partition geometry. If the planes and parallel lines are defined in G by Definition* 1·2 *and* 1·5, *then G satisfies the following axioms:*

G I. *Any line contains at most three points.*

G II. *Any two points on a line define the same line.*

G III. *Any plane contains only three, four or six points.*

G IV. *If a line has a parallel line, then it has just two ones.*

Conversely, suppose that G is a geometric system with points, lines, planes and parallel lines which are defined by Definition 1·2 and 1·5 *and that it satisfies the above axioms* G I-G IV. *Then G is a general partition geometry.*

Any general partition geometry is a partition geometry if and only if it satisfies the following axiom:

G V. *If a line contains only two points, then it has a parallel line.*

PROOF. We shall show that any general partition geometry G satisfies the axioms G I-G IV. By Definition 1·2, G satisfies G I and G II. And moreover it satisfies G III and G IV by Lemma 1·2 and 1·4 respectively.

Conversely in order to prove that any geometry G which satisfies the G I-G IV, is a general partition geometry, that is, it satisfies the axioms G 1-G 4, we shall show first that the following lemma holds in such a geometric system G.

LEMMA 1·6. *Suppose that* G *is a geometric system which satisfies the above axioms* G I-G IV. *Let* $p_i(i=1, 2, 3)$, $q_j(j=1, 2)$ *and* q *be points in* G *such that*

$$(p_1p_2p_3)C,\ q \notin p_1+p_2,\ (qq_1p_1)C \text{ and } (qq_2p_2)C,$$

then we shall have

$$(q_1q_2p_3)C,\ (q,\ p_3)\bar{r},\ (p_1,\ q_2)\bar{r} \text{ and } (p_2,\ q_1)\bar{r}.$$

PROOF. Suppose that the three points q_1, q_2 and p_3 be non-collinear, then by Definition 1·5 the plane p_1+p_2+q contains the six points p_1, p_2, p_3, q, q_1 and q_2. Hence it is impossible that the plane p_1+p_2+q contains no more points by G III and so we have

$$(q_1,\ q_2)\bar{r},\ (q_2,\ p_3)\bar{r},\ (q_1,\ p_3)\bar{r},\ (q_1,\ p_2)\bar{r},\ (p_1,\ q_2)\bar{r} \text{ and } (q,\ p_3)\bar{r}.$$

Therefore the line p_1+q_2 has three parallel lines q_1+p_2, q_1+p_3, and $q+p_3$, contrary to G IV. Hence we have $(q_1u_2p_3)C$.

Since the plane p_1+p_2+q contains obviously six points, the rest of the lemma follows directly from G III.

By making use of this lemma we shall prove that G satisfies the axioms G 3 and G 4.

First to prove that G satisfies G 3, let us suppose that

$$(p_1p_2p_3)C,\ q \notin p_1+p_2 \text{ and } (p_1q_1q)C,$$

then the plane p_1+p_2+q contains the five points p_1, p_2, p_3, q and q_1. Hence by G III it must contain one more point and so at least one of $(q,\ p_2)$, $(q,\ p_3)$, $(q_1,\ p_2)$ and $(q_1,\ p_3)$ must be related. For instance let q and p_2 be related and q_2 be the third point of the line p_2+q. Then by Lemma 1·6 we can prove $(p_3,\ q)\bar{r}$.

Similarly in the other case we have also

$$(p_2,\ q)r \text{ and } (p_3,\ q)\bar{r}, \text{ or } (p_2,\ q)\bar{r} \text{ and } (p_3,\ q)r.$$

and G 3 is satisfied.

Next in order to prove that G satisfies G 4, let $p_i(i=1, 2, 3)$, $q_j(j=1, 2)$, q and r be points in G such that

$$(p_1p_2p_3)C,\ q,\ r \notin p_1+p_2,\ q \neq r,\ (p_1qq_1)C \text{ and } (p_2qq_2)C.$$

Now suppose that q and r are unrelated. Then by G 3 and Lemma 1·6 there exist points o, o', r_1 and r_2 such that

$$(oq_1r)C,\ (oqr_1)C,\ (o'qr_2)C \text{ and } (o'q_2r)C,$$

and moreover we have

$$(q_1,\ r_1)\bar{r},\ (o,\ p_1)\bar{r},\ (o',\ p_2)\bar{r} \text{ and } (q_2,\ r_2)\bar{r}.$$

This contradicts G IV, since the line $q+r$ has four parallel lines p_1+o, p_2+o', q_1+r_1 and q_2+r_2. Hence q and r are related and thus G 4 is satisfied. And the first part of the theorem is proved.

Finally we shall prove that any general partition geometry G is a partition geometry if and only if it satisfies G V.

The foreward implication follows directly from Lemma 1·5. Conversely, suppose that a general partition geometry G satisfies G V. Now let p and q are unrelated points, then by G V there exist two points r and s with $p+q \parallel r+s$. Hence by Lemma 1·3 there exists a point o with $(opr)C$ and $(oqs)\ C$, and thus G 5 is satisfied, that is, G is a general partition geometry and the theorem is completely proved.

§ 2. The Lattice of All Additively Closed Sets of General Partition Geometries.

In this section, let G be a general partition geometry, and hence according to the results obtained in § 1, G satisfies the axioms G 1-G 4 or G I-G IV.

DEFINITION 2·1. Let A be an additively closed set of G and p be a point in G. Then we define that

$$p+A=\bigcup_{q,\ r\in A}(p+q+r) \text{ where } p+p=p, \text{ and } p+q+r=p+q \text{ if } r\in p+q.$$

REMARK 2·1. Let p, q and r be points of G, then both the lines $p+q$ and the plane $p+q+r$ are additively closed sets of G and we have

$$p+q+r=p+(q+r)=q+(r+p)=r+(p+q).$$

Any subset A of G is an additively closed set if and only if

$$p,\ q,\ r\in A \text{ implies } p+q+r\subset A.$$

LEMMA 2·1. *Let A be an additively closed set of G. Then a point s belongs to $p+A$ if and only if either $s\in p+r$ for some $r\in A$ or $p+s\parallel q+r$ for some $q,\ r\in A$.*

PROOF. Necessity. Let s be a point of $p+A$, by Definition 2·1 there exist two points q and r such that $s\in p+q+r$ where $q,\ r\in A$. Without loss of generality we may assume that p, q and r are non-collinear. Now by

Remark 2·1 we have $p+s\subset p+q+r$. Hence two lines $p+s$ and $q+r$ on the plane $p+q+r$ have a common point, or they are parallel to each other. Therefore we have

$$s\in p+t,\ t\in q+r\subset A \text{ or } p+s \parallel q+r \text{ for some } q,\ r\in A.$$

Thus the necessity has been proved.

The sufficiency follows directly from Definition 2·1.

In the following, to prove that $p+A$ is an additively closed set, we need the following

LEMMA 2·2. *Let x, y, a, b, p and q be distinct points of G such that*

$$x+y \parallel a+b,\ (xpq)C$$

then there exist two points a' and b' such that either

(i) $y+p \parallel a+b'$, $(bb'q)C$, $y+q \parallel a'+b$ *and* $(aa'p)C$

or (ii) $y+p \parallel a'+b$, $(aa'q)C$, $y+q \parallel a+b'$ *and* $(bb'p)C$.

PROOF. Since $x+y \parallel a+b$, by Lemma 1·3 there exist two points c and d such that

$$(xad)C,\ (ybd)C,\ (xbc)C \text{ and } (yac)C.$$

Now since $(xad)C$, $(xpq)C$ and $a\neq p,\ q$,

(i) there exists a point a' with $(paa')C$ and $(qa'd)C$,

or (ii) $a+p \parallel d+q$.

In the former case, both a' and y are related to a and d, whence by G 4 there exists a point b' with $(ya'b')C$.

Since $(ya'b')C$, $d\notin y+a'$, $(dqa')C$, $(dby)C$, we have $(bb'q)C$ by Lemma 1·6. In a similar way, we have $(cb'p)C$. Hence we have by Lemma 1·3

$$y+p \parallel a+b',\ (bb'q)C \text{ and } y+q \parallel a'+b,\ (aa'p)C.$$

In the latter case, by Lemma 1·3 there exists a point a' with $(aa'q)C$ and $(a'dp)C$, hence interchanging p and q we have in the same manner as in (i)

$$y+p \parallel a'+b,\ (aa'q)C \text{ and } y+q \parallel a+b',\ (bb'p)C,$$

and the proof is complete.

LEMMA 2·3. *Let A be an additively closed set and p be a point in G. Then $p+A$ is also an additively closed set.*

PROOF. When p belongs to A, this is obvious, hence let us suppose that p does not belong to A. Now to prove $p+A$ to be an additively closed set, we need only to show that $x, y \in p+A$ implies $x+y \subset p+A$, hence, that $x, y \in p+A$ and $(xyz)C$ imply $z \in p+A$.

By Lemma 2·1 the following three cases occur.

Case 1. $x \in p+a$, $y \in p+b$ where $a, b \in A$.

By Remark 2·1 we have obviously

$$z \in x+y \subset p+a+b \subset p+A.$$

Case 2. $x+p \parallel a+a'$, $y \in p+b$ where $a, a', b \in A$.

Since $x+p \parallel a+a'$, $(xyz)C$ and we may assume that the four points a, a', y and z are distinct (since otherwise the statement trivialy holds), by Lemma 2·2 there exist two points c and d such that

(i) $p+y \parallel a+c$, $(ca'z)C$, $p+z \parallel a'+z$ and $(day)C$,

or (ii) $p+y \parallel a'+d$, $(daz)C$, $p+z \parallel a+c$ and $(ca'y)C$.

In the former case, since $(p, y)\bar{r}$, $y \in p+b$ yields $y=b$. Hence $d \in a+b \subset A$ and so we have $p+z \parallel a'+d$ where $a', d \in A$, whence $z \in p+A$ by Lemma 2·1. In the latter case, similarly we have $z \in p+A$.

Case 3. $x+p \parallel a+a'$, $y+p \parallel b+b'$ where $a, a', b, b' \in A$.

We may assume that the four points a, a', y and z are distinct, since otherwise the statement is obvious. And moreover $x+p \parallel a+a'$ and $(xyz)C$ by assumption, hence by Lemma 2·2 there exist two points c and d such that

(i) $p+y \parallel a+c$, $(ca'z)C$ and $p+z \parallel a'+d$, $(day)C$

or (ii) $p+y \parallel a'+d$, $(daz)C$ and $p+z \parallel a+c$, $(ca'y)C$.

In the case (i) the line $p+a$ and $y+c$ have a common point e and $p+y \parallel d+e$ by Lemma 1·3. While $p+y \parallel b+b'$, hence by G IV we have $b+b'=a+c$ or $d+e$.

If $b+b'=a+c$, then $c \in A$ since $b+b' \subset A$ and so $(ca'z)C$ yields $z \in a'+c \subset A \subset p+A$. If $b+b'=d+e$, then $d \in A$ since $b+b' \subset A$ and so we have $p+z \parallel a'+d$ where $a', d \in A$. Hence $z \in p+A$ by Lemma 2·1.

In the case (ii), similarly it is readily seen that $z \in p+A$. Therefore the set $p+A$ is an additively closed set of G.

DEFINITION 2·3. *A* lattice L with 0 is called a *planer* lattice when for any two points p and q with $q \leq p \smile a (a \neq 0)$, there exist two points r and s such that $q \leq p \smile r \smile s$ where $r, s \leq a$.

DEFINITION 2·4.[6] A relatively atomic, upper continuous, semi-modular

6) Cf. U. Sasaki & S. Fujiwara [1] Definition 1, 3 and 4.

lattice is called a *matroid* lattice.

A matroid and planer lattice is called a *general partition lattice* if and only if its points satisfy:

L 1. *If p and q are distinct points, then $p \smile q$ contains at most three points.*

L 2. *If p, q and r are points such that $p \nleqq q \smile r$ and $q \neq r$, then $p \smile q \smile r$ contains only three, four or six points.*

L 3. *If $p_i (i=1, 2, 3)$ and q be points with $q \leqq p_1 \smile p_2 \smile p_3$, then there exists a point r such that*

$$q \leqq p_i \smile r \text{ and } r \leqq p_j \smile p_k \text{ for some permutation } i, j, k \text{ of } 1, 2, 3.$$

REMARK 2·2. In the definition of a general partition lattice one can replace semi-modularity by the following weaker condition:

(η_1') *If p, q and r be points and $p \leqq q \smile r (p \neq r)$, then $q \leqq p \smile r$.*

For from (η_1') and L 3 we can easily deduce

(η_2') *Let $p_i (i=1, 2, 3)$ and q be points, then $q \leqq p_1 \smile p_2 \smile p_3$ and $q \nleqq p_2 \smile p_3$ imply $p_1 \leqq q \smile p_2 \smile p_3$.*

Since moreover the lattice is planer, it satisfies the exchange axiom

(η') *If p, q are points and $a < p \smile a \leqq q \smile a$, then $p \smile a = q \smile a$.*

In a relatively atomic, upper continuous lattice (η') is equivalent to[7]

(ξ') *If x and y cover a, and $x \neq y$, then $x \smile y$ covers x and y.*

Therefore L is a semi-modular lattice.

THEOREM 2·1. *The lattice $L(G)$ of all additively closed sets of a general partition geometry G is a general partition lattice.*

PROOF. (i) One can easily verify that the lattice $L(G)$ is a relatively atomic, upper continuous lattice as F. Maeda [1] 93 Theorem 2·1.

(ii) By Definition 2·3 and Lemma 2·3 $L(G)$ is a planer lattice.

(iii) By axioms G I, G II and Remark 1·1, $L(G)$ satisfies L 1, L 2 and L 3.

(iv) By G II, $L(G)$ satisfies (η_1'), while it is relatively atomic, upper continuous and planer lattice by (i), (ii), and satisfies L 3 by (iii). Hence it satisfies (ξ') by Remark 2·2. And so $L(G)$ is a matroid lattice.

Thus the lattice $L(G)$ is a general partition lattice.

§ 3. Lattice Theoretic Characterization of Partition Lattices.

In this section, let the lattice L be a general partition lattice.

7) Cf. F. Maeda [2] Theorem 3.

DEFINITION 3·1. Let p and q be different points of L. If $p \smile q$ contains the third point r, then we shall say that p and q are *related* (in symbols $(p, q)r$), and that p, q and r are *collinear* (in symbols $(pqr)C$). Otherwise they are *unrelated* (in symbols $(p, q)\bar{r}$).

LEMMA 3·1. *Let $p_i(i=1, 2, 3)$, $q_j(j=1, 2)$ and q be points in L such that*

$$(p_1p_2p_3)C, \ (qq_1p_1)C, \ (qq_2p_2)C \text{ and } q \not\leq p_1 \smile p_2,$$

then we have

$$(p_2, q_1)\bar{r}, \ (p_1, q_2)\bar{r}, \ (p_3, q)\bar{r} \text{ and } (q_1q_2p_3)C.$$

PROOF. Obviously $q \smile p_1 \smile p_2 \geq p_1, p_2, p_3, q_1, q_2, q$ and any two of them are distinct. Hence by L 2 $q \smile p_1 \smile p_2$ contains no more points. And so we have

$$(p_2, q_1)\bar{r}, \ (p_1, q_2)\bar{r}, \ (p_3, q)\bar{r}.$$

Suppose that q_1, q_2 and p_3 are non-collinear points. Then we have $q_1 \not\leq q_2 \smile p_3$, $q_2 \neq p_3$ and $q_1, q_2, p_3 \leq q \smile p_1 \smile p_2$, hence by U. Sasaki & S. Fujiwara [1] Lemma 2 we have $q_1 \smile q_2 \smile p_3 = q \smile p_1 \smile p_2$. While $q \smile p_1 \smile p_2$ contains the six points p_1, p_2, p_3, q, q_1 and q_2, hence by L 2 we have $(q_1, q_2)\bar{r}$, $(q_2, p_3)\bar{r}$ and $(q_1, p_3)\bar{r}$.

Therefore by L 3 we have $p_1 \not\leq q_1 \smile q_2 \smile p_3$, this contradicts $q_1 \smile q_2 \smile p_3 = q \smile p_1 \smile p_2$. Hence we have $(q_1q_2p_3)C$.

COROLLARY 3·1. *If $p_i(i=1, 2, 3)$, $q_j(j=1, 2)$ and q be points in L and $(p_1p_2p_3)C$ and $q \not\leq p_1 \smile p_2$, then q is related to one and only one of p_2 and p_3.*

PROOF. By Lemma 3·1 this is proved as Theorem 1·3, G 3.

Now we shall prove the principal theorem in this section.

THEOREM 3·1. *Let L be a general partition lattice and denote by $G(L)$ the set of all points of L. In $G(L)$, if we define $p+q$ by the set of all points contained in $p \smile q$, then the following statements hold:*

(1) *$G(L)$ is a general partition geometry.*

(2) *L is isomorphic to the lattice of all additively closed sets of $G(L)$.*

(3) *L is isomorphic to the lattice of all partitions of some set if and only if L is irreducible.*

PROOF. (1) We shall show that $G(L)$ satisfies the axioms G 1-G 4. Obviously G 1, 2, 3 are implied by L 1, semi-modularity and Corollary 3·1 respectively. And hence we need only to prove that $G(L)$ satisfies G 4.

Let $(p_1p_2p_3)C$, q, $r \notin p_1+p_2$, $q \neq r$, $(qq_1p_1)C$, $(qq_2p_2)C$, $(rr_1p_1)C$ and $(rr_2p_3)C$ and suppose the conclusion of G 4 to be false, that is $(q, r)\bar{r}$.

By Lemma 3·1 we have $(q_1q_2p_3)C$, $(r_1r_2p_3)C$, $(q, p_3)\bar{r}$, $(r, p_3)\bar{r}$.

And moreover by Lemma 3·1 and G 3 there exist two points s and t such that

$$(qsr_1)C,\ (q_1sr)C,\ (q_1tr_2)C \text{ and } (q_2tr_1)C,$$

and such that

$$t\leqq p_2\smile s\leqq p_2\smile q\smile r_1\leqq p_2\smile q\smile p_1\smile r\leqq p_2\smile(q\smile r\smile p_3).$$

Since L is a planer lattice, we have $t\leqq p_2\smile u\smile v$ where u, $v\leqq q\smile r\smile p_3$. While $(q, r)\bar{r}$, $(r, p_3)\bar{r}$ and $(p_3, q)\bar{r}$, hence by L 3 the set $\{u, v\}$ is equal to one of the sets $\{q, r\}$, $\{r, p_3\}$ and $\{p_3, q\}$. For instance let $\{u, v\}=\{p_3, q\}$, then we have $p_2\smile u\smile v=p_2\smile p_3\smile q\geqq p_1, p_2, p_3, q, q_1, q_2$, whence by L 2 $p_2\smile p_3\smile q$ contains no more point, contrary to $p_2\smile u\smile v\geqq t$. Therefore we have $(q, r)\, r$ and thus G 4 is satisfied.

(2) Put $S(a)=\{p\,;\ p\leqq a\}$ for $a\in L$ and $a(S)=\bigvee\{p\,;\ p\in S\}$ for an additively closed set of $G(L)$. Then we have

$$a(S(a))=\bigvee(p\,;\ p\in S(a))=\bigvee(p\,;\ p\leqq a)=a,$$

and $$S(a(S))=\{p\,;\ p\leqq a(S)\}=\{p\,;\ p\leqq\bigvee(q\,;\ q\in S)\}\supset S.$$

In order to prove $S(a(S))=S$, we shall show that $p\in S(a(S))$ implies $p\in S$.

Suppose that $p\in S(a(S))$, that is $p\leqq\bigvee(q\,;\ q\in S)$.

Then since L is relatively atomic, upper continuous, we have

$$p\leqq q_1\smile q_2\smile\ldots\ldots\smile q_n \text{ where } q_i\in S(i=1,\ 2, \ldots\ldots, n)$$ [8]

Now we shall prove that p belongs to S by induction.

For $n=1$ this is obvious. Suppose that this is true for $n=m$. Since L is a planer lattice, if $p\leqq q_1\smile q_2\smile\ldots\ldots\smile q_{m+1}$, then there exist two points r and s such that

$$p\leqq q_1\smile r\smile s \text{ and } r,\ s\leqq q_2\smile\ldots\ldots\smile q_{m+1} \text{ where } q_i\in S(i=1,\ 2, \ldots\ldots, m+1).$$

By the induction hypothesis r, $s\in S$ and hence $p\in S$. Hence this is true for $n=m+1$.

Therefore there exists a one-to-one correspondence $a\leftrightarrow S(a)$ between L and the lattice of all additively closed sets of $G(L)$, which preserves the inclusion relation. Hence L is isomorphic to the lattice of all additively closed sets of $G(L)$.

8) Cf. F. Maeda [1] 90 Lemma 1·3.

(3) Let L be a irreducible general partition lattice, then by U. Sasaki and S. Fujiwara [1] Theorem 3, any two points are perspective to each other. Hence for any two points p and q there exist r and s with $q \leqq p \smile r \smile s$ and $q \frown (r \smile s) = 0$, since L is planer.

If p and q are unrelated, then by L 3 there exist points x and y such that

$$p \leqq x \smile y \text{ and } x \leqq q \smile s, \text{ or } p \leqq s \smile y \text{ and } y \leqq q \smile r.$$

Hence a general partition geometry $G(L)$ satisfies G 5 and so $L(G)$ is a partition geometry. Combining Theorem 1·1 and (2), it is readily seen that L is isomorphic to the lattice of all partitions of some set.

Conversely, let L be a partition lattice, then there exists a partition geometry G such that L is isomorphic to the lattice of all additively closed sets of G by Theorem 1·1, whence L is a general partition lattice by Theorem 2·1.

If p and q are unrelated, then by G 5 there exists a point r such that $(pxr)C$ and $(pyr)C$. Hence there exists a point z with $(yzp)C$ by Corollary 3·1 and we have $(x; y)\bar{r}$ by Lemma 3·1 and so $q \leqq p \smile x \smile y$ and $q \frown (x \smile y) = 0$. This shows that p is perspective to q (of course if p and q are related, p is perspective to q). Hence any two points of L are perspective to each other and so L is irreducible by U. Sasaki and S. Fujiwara [1] Theorem 3. Thus the theorem is completely proved.

THEOREM 3·2. *Any general partition lattice is a direct union of partition lattices.*

PROOF. This follows directly from U. Sasaki and S. Fujiwara [1] Theorem 4 and the above Theorem 3·1.

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