THE DECOMPOSITION OF MATROID LATTICES

By

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G. Birkhoff [1][1] and K. Menger [1] have proved the following

THEOREM I. *Any complemented modular lattice of finite length is a direct union of a finite number of simple complemented modular lattices.*

And O. Frink [1] has shown the following theorem which contains Theorem I as a special case:

THEOREM II. *Any atomic, upper continuous*[2], *complemented modular lattice is a direct union of irreducible sublattices.*

More generally F. Maeda [2] has defined the perspectivity and connectedness[3] of two points of a lattice with 0 and proved the following

THEOREM III. *Any relatively atomic*[4], *upper continuous lattice L is a direct union of sublattices $S_\alpha (\alpha \in I)$ of L. And any two points in the same S_α are connected and two points which are contained in different S_α and S_β are not connected.*

In this paper we shall prove that in a matroid lattice[5] L, the perspectivity and connectedness are equivalent (Theorem 1) and that L is irreducible if and only if any two points of L are perspective to each other (Theorem 3). Hence, when L is especially a matroid lattice in Theorem III, we have the following decomposition theorem which includes Theorem II as a special case:

THEOREM IV. *Any matroid lattice is a direct union of irreducible matroid lattices.*

DEFINITION 1. A lattice L is called *semi-modular* if and only if its elements satisfy

(ξ') *If x and y cover a, and $x \neq y$, then $x \cup y$ covers x and y.*

LEMMA 1. *Let L be a semi-modular lattice with 0 and p, q and p_i $(i=1, 2, \dots, n)$ be points of L and b be any element of L. Then the following statements hold for $n=1, 2, \cdots$.*

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1) The numbers in square brackets refer to the list of references at the end ot the paper.
2) Cf. Definition 4 below.
3) Cf. Definition 6 below.
4) Cf. Definition 3 below.
5) Cr. Definition 5 below.

(η_0'') $p \nleqq \bigvee_1^n p_i$ *implies that* $p \smile \bigvee_1^n p_i$ *covers* $\bigvee_1^n p_i$.

(η_0''') $p \nleqq b \smile \bigvee_1^n p_i$ *implies* $b \frown \bigvee_1^n p_i = b \frown (p \smile \bigvee_1^n p_i)$.

(η_0') $\bigvee_1^n p_i < q \smile \bigvee_1^n p_i \leqq p \smile \bigvee_1^n p_i$ *implies* $q \smile \bigvee_1^n p_i = p \smile \bigvee_1^n p_i$.

PROOF. We shall prove that $(\xi') \to (\eta_0'') \to (\eta_0''') \to (\eta_0')$.

$(\xi') \to (\eta_0'')$[6]. By deleting supperfluous elements from $p_1, p_2, \dots, p_n$, we may assume that $(p_1 \smile \dots \smile p_i) \frown p_{i+1} = 0$ $(i=1, 2, \dots, n-1)$, for otherwise p_{i+1} would be redundant. Then we shall prove (η_0'') by induction.

For $n=1$, $p \nleqq p_1$ implies that $p \smile p_1$ covers p by (ξ'). Next we assume that (η_0'') holds for $n=m$. Now $p \nleqq p_1 \smile \dots \smile p_{m+1}$ implies $p \nleqq p_1 \smile \dots \smile p_m$ and $p_{m+1} \nleqq p_1 \smile \dots \smile p_m$ by the above assumption, whence $p \smile \bigvee_1^m p_i$ and $\bigvee_1^{m+1} p_i$ cover $\bigvee_1^m p_i$ by the induction hypothesis and $p \smile \bigvee_1^m p_i \neq \bigvee_1^{m+1} p_i$.

Therefore by (ξ') $p \smile \bigvee_1^{m+1} p_i$ covers $\bigvee_1^{m+1} p_i$.

$(\eta_0'') \to (\eta_0''')$[7]. Since $p \nleqq b \smile \bigvee_1^n p_i$, we have $p \nleqq \bigvee_1^n p_i$, whence $p \smile \bigvee_1^n p_i$ covers $\bigvee_1^n p_i$ by (η_0''), while

$$\bigvee_1^n p_i \leqq (p \smile \bigvee_1^n p_i) \frown (b \smile \bigvee_1^n p_i) \leqq p \smile \bigvee_1^n p_i$$

and $(p \smile \bigvee_1^n p_i) \frown (b \smile \bigvee_1^n p_i) \neq p \smile \bigvee_1^n p_i$, for otherwise we should have $p \smile \bigvee_1^n p_i \leqq b \smile \bigvee_1^n p_i$, contrary to hypothesis.

And so $\bigvee_1^n p_i = (p \smile \bigvee_1^n p_i) \frown (b \smile \bigvee_1^n p_i)$, whence we have

$$(\bigvee_1^n p_i) \frown b = (p \smile \bigvee_1^n p_i) \frown (b \smile \bigvee_1^n p_i) \frown b = (p \smile \bigvee_1^n p_i) \frown b .$$

$(\eta_0''') \to (\eta_0')$. $\bigvee_1^n p_i < q \smile \bigvee_1^n p_i \leqq p \smile \bigvee_1^n p_i$ implies $p \leqq q \smile \bigvee_1^n p_i$, for otherwise by ($\eta_0'''$) we should have $(p \smile \bigvee_1^n p_i) \frown q = q \frown \bigvee_1^n p_i = 0$, contrary to $q \leqq p \smile \bigvee_1^n p_i$. Therefore $q \smile \bigvee_1^n p_i = p \smile \bigvee_1^n p_i$.

DEFINITION 2. A finite set S of elements in any lattice L with 0 is *independent* if $\bigvee(a; a \in S_1) \frown \bigvee(a; a \in S_2) = 0$ for any two disjoint subsets S_1 and S_2 of S. When $S = \{p_1, \dots, p_n\}$ is independent, we shall denote by $(p_1, \dots, p_n)\perp$.

LEMMA 2. *Let L be a semi-modular lattice with* 0 *and* $p_i, q_i (i=1, \dots, n)$ *be points of L, then the following propositions hold for* $n=1, 2, \cdots$.

(1°) $(p_1, \dots, p_n)\perp$ *if and only if* $(p_1 \smile \dots \smile p_i) \frown p_{i+1} = 0$ *for* $i=1, \dots, n-1$.

(2°) $(q_1, \dots, q_n)\perp$ *and* $q_j \leqq \bigvee_1^n p_i (j=1, \dots, n)$ *imply* $\bigvee_1^n p_i = \bigvee_1^n q_j$.

(3°) *Let* $(p_1, \dots, p_n)\perp$ *and put* $P_i = p_1 \smile \dots \smile p_{i-1} \smile p_{i+1} \smile \dots \smile p_n$ *where* $i=1, \dots, n$, *then we have* $\bigwedge_1^r P_i = p_{r+1} \smile \dots \smile p_n$ *for* $r=1, \dots, n$.

6) We are indebted to Prof. F. Maeda about $(\xi') \to (\eta_0'')$.

7) K. Menger [1] 460.

PROOF. (1°). The foreward implication is obvious.

Suppose that $(p_1 \smile \dots \smile p_i) \frown p_{i+1} = 0$ for $i=1, \dots, n-1$. Then we shall prove by induction that $(p_1, \dots, p_n)\perp$. For $n=2$ this is obvious.

Suppose that this is true for $n=m$. In order to show that $(p_1, \dots, p_{m+1})\perp$, let $S=\{p_1, \dots, p_{m+1}\}$, then it is sufficient to show that

$\bigvee(p\,;p\in S_1)\frown\bigvee(p\,;p\in S_2)=0$ for any two disjoint subsets S_1 and S_2 of S.

Now we may assume that p_{m+1} is contained in either S_1 or S_2, since otherwise $\bigvee(p\,;p\in S_1)\frown\bigvee(p\,;p\in S_2)=0$ by the induction hypothesis. Hence let $p_{m+1}\in S_2$. Now put $S_2'=S_2-\{p_{m+1}\}$, $a=\bigvee(p\,;p\in S_2)$ and $b=\bigvee(p\,;p\in S_1)$, then $p_{m+1}\nleqq a\smile b$ by hypothesis, hence by Lemma 1 (η_0''') we have $(a\smile p_{m+1})\frown b = a\frown b$ and $a\frown b=0$ by the induction hypothesis. Consequently $\bigvee(p\,;p\in S_1)\frown \bigvee(p\,;p\in S_2)=0$. Therefore $(p_1, \dots, p_{m+1})\perp$ and the statement is true for $n=m+1$.

(2°). For $n=1$ this is obvious. We shall show that if it holds for $n=m$, then it holds for $n=m+1$. Now suppose that $(q_1, \dots, q_{m+1})\perp$ and $q_j\leqq\bigvee_1^{m+1} p_i$ for $j=1, 2, \dots, m+1$, then there exists $j_1=1, 2, \dots, m+1$ such that $q_{j_1}\nleqq\bigvee_1^m p_i$. For otherwise $\bigvee_1^m p_i=\bigvee_1^m q_j\geqq q_{j+1}$ by the induction hypothesis and this contradicts $(q_1, \dots, q_{m+1})\perp$. Moreover by the hypothesis $q_{j_1}\leqq\bigvee_1^m p_i\smile p_{m+1}$ and hence by Lemma 1 (η_0')

$$q_{j_1}\smile\bigvee_1^m p_i=\bigvee_1^{m+1} p_i\,.$$

In a similar argument, there exists $j_2=1, \dots, m+1$ such that

$$q_{j_2}\nleqq q_{j_1}\smile\bigvee_1^{m-1} p_i \quad\text{and}\quad q_{j_2}\leqq(q_{j_1}\smile\bigvee_1^{m-1} p_i)\smile p_m\,.$$

Repeated applications of this process yield

$$\bigvee_1^{m+1} q_{j_k}=\bigvee_1^{m+1} q_j=\bigvee_1^{m+1} p_i\,.$$

(3°). This is obvious for $r=1$. Suppose that it is true for $r=m$. Then we have

$$\bigwedge_1^{m+1} P_i=(p_{m+1}\smile \dots \smile p_n)\frown P_{m+1}\,.$$

By putting $a=p_{m+2}\smile\dots\smile p_n$ and $b=P_{m+1}$, we have $p_{m+1}\nleqq a\smile b$ by the assumption, whence by (η_0''')

$$\bigwedge_1^{m+1} P_i=b\frown(a\smile p_{m+1})=b\frown a=a=p_{m+2}\smile\dots\smile p_n\,.$$

DEFINITION 3. If, for any element $a(\neq 0)$ of a lattice L with 0, there exists a point p such that $p\leqq a$, then L is called *atomic*. If $a<b$ implies $a<a\smile p\leqq b$ for some point p, then L is called *relatively atomic*.

REMARK 1. A lattice L with 0 is relatively atomic if and only if each element of L is the join of its contained points.

(Cf. F. Maeda [2] 88. Lemma $1\cdot 1$).

DEFINITION 4. Let $\{a_\delta ; \delta \in D\}$ be a directed subset of a complete lattice L. When $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$ for any $b \in L$, L is called an *upper continuous lattice.*

REMARK 2. In a relatively atomic, complete lattice L, the following propositions are equivalent:

(α) *L is upper continuous.*

(β) *Let p be a point and S a set of points in L. Then $p \leqq \bigvee(q ; q \in S)$ implies $p \leqq q_1 \smile \dots \smile q_n$ for some $q_i \in S\,(i=1, 2, \dots, n)$.*

(Cf. F. Maeda [2] 90. Lemma $1\cdot 3$).

DEFINITION 5. A relatively atomic, upper continuous, semi-modular lattice is called a *matroid* lattice.

REMARK 3. By Lemma 1, in a lattice with 0, semi-modularity implies (η_0'), and it is easily shown that in a relatively atomic, upper continuous lattice (η_0') implies the following exchange axiom

(η') If p, q are points, and $a < q \smile a \leqq p \smile a$, then $q \smile a = p \smile a$.

And so any matroid lattice L satisfies (η').

(Cf. F. Maeda [3] Theorem 3).

DEFINITION 6[8]. In a lattice L with 0, if p, q are points such that

$$q \leqq p \smile x \text{ and } q \frown x = 0 \text{ for some element } x \in L,$$

we shall say that p is *perspective* to q (in symbols $p \sim q$).

If p, q are points such that there exists a sequence $p = p_1, \dots, p_n = q$ where $p_i \sim p_{i+1}$ or $p_{i+1} \sim p_i\,(i=1, \dots, n-1)$, then we shall say that p and q are *connected.*

THEOREM 1. *Let p, q and r be points of a matroid lattice L, then the following statements hold:*

(I) *$p \sim q$ implies $q \sim p$.*

(II) *$p \sim q$ and $q \sim r$ imply $p \sim r$.*

PROOF. (I) By Remark 3 this is obvious.

(II) If any two of p, q and r are coincident, then this is obvious. Hence we may suppose that p, q and r are distinct. Since $p \sim q$, there exists an element $a \in L$ such that $q \leqq p \smile a$ and $p \frown a = q \frown a = 0$. By Remark 1 and 2 there exist points $x_i\,(i=1, \dots, m)$ such that

8) Cf. F. Maeda [2] 88, Definition 1·2.

$$q \leqq p \smile x_1 \smile \dots \smile x_m \quad \text{where} \quad x_i \leqq a \ (i=1, \dots, m), \tag{1}$$

and we may assume that redundant elements are deleted in (1).

Hence by Lemma 2 (1°) we have $(p, x_1, \dots, x_m)\perp$. Now put $X = x_1 \smile \dots \smile x_m$, then, since $q \frown X = 0$, we have by Lemma 2 (1°) $(q, x_1, \dots, x_m)\perp$. And moreover put $X_i = x_1 \smile \dots \smile x_{i-1} \smile x_{i+1} \smile \dots \smile x_m$ $(i=1, 2, \dots, m)$, then $p \frown X_i = 0$ and $p \smile X_i \ngeqq q$ by the irredundancy of x_i $(i=1, \dots, m)$.

Hence by Lemma 2 (1°) $(p, q, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)\perp$ for $i=1, \dots, m$. Moreover $p \smile X$ contains p, q and x_i by (1). Therefore we have by Lemma 2 (2°)

$$p \smile X = q \smile X = p \smile q \smile X_i \quad \text{for} \quad i = 1, \dots, m. \tag{2}$$

Since $q \sim r$, there exist points y_i $(i=1, \dots, n)$ such that

$$r \leqq q \smile Y, \ q \frown Y = r \frown Y = 0 \quad \text{and} \quad Y = y_1 \smile \dots \smile y_n. \tag{3}$$

By (1) and (3) $r \leqq p \smile X \smile Y$. Hence if $r \frown (X \smile Y) = 0$, then $p \sim r$ and (II) is proved. On the other hand if $r \leqq X \smile Y$, then we have by (η') $q \leqq r \smile X \leqq X \smile Y$ and $p \leqq q \smile X \leqq X \smile Y$. Hence we have

$$X \smile Y = p \smile X \smile Y = p \smile x_1 \smile \dots \smile x_m \smile y_1 \smile \dots \smile y_n.$$

Since $(p, x_1, \dots, x_m)\perp$, there exist $y_{i_j} \in \{y_1, \dots, y_n\}$ for $j=1, \dots, k$ such that

$$X \smile Y = p \smile x_1 \smile \dots \smile x_m \smile y_{i_1} \smile \dots \smile y_{i_k} \text{ and } (p, x_1, \dots, x_m, y_{i_1}, \dots, y_{i_k})\perp.$$

Now put $Y' = y_{i_1} \smile \dots \smile y_{i_k}$, then by (2)

$$X \smile Y = p \smile X \smile Y' = p \smile q \smile X_i \smile Y' \quad \text{for} \quad i = 1, \dots, m. \tag{4}$$

Since $(p, x_1, \dots, x_m, y_{i_1}, \dots, y_{i_k})\perp$, $(q, x_1, \dots, x_m)\perp$ and $p \smile X = q \smile X$, we have by Lemma 2 (1°) $(q, x_1, \dots, x_m, y_{i_1}, \dots, y_{i_k})\perp$. Hence by Lemma 2 (3°) $(X \smile Y') \frown \bigwedge_1^m (q \smile X_i \smile Y') = Y'$.

Since $r \frown Y' \leqq r \frown Y = 0$, at least one of $X \smile Y'$ and $q \smile X_i \smile Y'$ $(i=1, \dots, m)$ does not contain r. And moreover we have by (4)

$$r \leqq X \smile Y = p \smile (X \smile Y') = p \smile (q \smile X_i \smile Y').$$

Hence p is perspective to r. Thus the proof is complete.

THEOREM 2. *A matroid lattice L is a direct union of sublattices L_α $(\alpha \in I)$ of L, that is, $L = \sum(\oplus L_\alpha; \alpha \in I)$. And any two points in the same L_α are perspective and two points which are contained in different L_α and L_β are not perspective.*

PROOF. By virtue of Theorem 1, in a matroid lattice L the perspectivity and connectedness are equivalent and therefore the theorem follows immediately from F. Maeda [2] Theorem 1·1.

THEOREM 3. *In a matroid lattice L, the following statements are equivalent:*

(α) *L is irreducible.*

(β) *Any two points of L are perspective to each other.*

PROOF. $(\alpha)\rightarrow(\beta)$. Suppose the statement to be false, then there exist at least two points of L which are not perspective. Then by Theorem 2, L is a direct union of more than two sublattices $L_\alpha (\alpha \in I)$ and so L is reducible, since L has 0[9]. This contradicts (α).

$(\beta)\rightarrow(\alpha)$. Suppose L is reducible. Then, since L has 0, L is a direct union of sublattices L_α of L, that is, $L=\sum(\oplus L_\alpha\,;\alpha\in I)$.

Let $p\in L_\alpha$ and $q\in L_\beta\,(\alpha\neq\beta)$ be two points. Then, by (β) $p\sim q$ and so there exists an element x of L such that $p\cup x=q\cup x$ and $p\cap x=q\cap x=0$.

On the other hand $x=\bigvee(\oplus x_\alpha\,;\alpha\in I)$ where $x_\alpha\in L_\alpha\,(\alpha\in I)$. By the uniqueness of the representation $p\cup x_\alpha=x_\alpha$, whence $0<p\leq x_\alpha$, this contradicts $p\cap x_\alpha\leq p\cap x=0$. Therefore L is irreducible.

THEOREM 4. *Any matroid lattice is a direct union of irreducible matroid lattices.*

PROOF. This follows directly from Theorem 2 and Theorem 3.

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9) Cf. F. Maeda [1] 86. Theorem 1·1.