On Topological Operations Determined by Local Characters

By

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Given a property P of sets in a topological space, let P denote the family of sets having this property. C. Kuratowski defined set A^*, by a local character determined by P and studied the operation A^* imposing to P two conditions called hereditary and additive[1]. Here we concern with P more or less restricted and investigate the formal properties of operation A^*. We say P to be maximal when $A=\bigvee A_\iota$, A_ι open in A, and $A_\iota \in P$ imply $A \in P$. Our results are related to this property, e.g., if P is hereditary, additive and maximal, then every A^* is a regular closed set if and only if P contains a family consisting of non-dense sets. And then $A^*=A^{*c*c*}$ holds.

§1. Preliminary Definitions and their Immediate Consequences.

1° Let R be a topological space[2]. Unless otherwise stated, G, H, with or without index, stand for open subsets of R, and F closed subsets of R. Let P_ϕ be any family of sets $\subseteqq R$ such that $0 \in P_\phi$. For any set $A \subseteqq R$ we define

(i) $A^\phi=[\bigvee\{G\,;\; G\bigwedge A \in P_\phi\}]^c$

(ii) $A^{(\phi)}=A\bigwedge A^\phi$

(iii) $A^{\phi*}=A\bigwedge A^{\phi c}$

and we call $A^{(\phi)}$ and $A^{\phi*}$, ϕ-coherence and ϕ-adherence of A respectively. For example, let P_α consist of only a null set, then A^α means the closure of A. It is easy to see that A^ϕ is closed and $A^\phi \subseteqq A^\alpha$.

THEOREM 1·1. (1) If $G\bigwedge X \neq 0$ implies $G\bigwedge A \notin P_\phi$ for arbitrary G, then we have $X \subseteqq A^\phi$

(2) If $X \subseteqq A^\phi$, then $G\bigwedge X \neq 0$ implies $G\bigwedge A \notin P_\phi$ for arbitrary G.

(3) A^ϕ is the maximal set X satisfying the condition of (1).

(4) The condition $p \in A^\phi$ is equivalent to that $G\bigwedge A \notin P_\phi$ for any neighbourhood G of p.

1) Cf. [2], 29. Numbers in brackets refer to the bibliography at the ent of the paper.
2) Cf. [1], 37.

Proof. (1): Assume that $X \subsetneqq A^{\phi}$, then $A^{\phi c} \frown X \neq 0$. Since by definition $A^{\phi c} = \bigvee \{G;\ G \frown A \in P_{\phi}\}$, there must be a G such that $G \frown X \neq 0$ and $G \frown A \in P_{\phi}$. This is a contradiction.

(2): Let $X \subseteqq A^{\phi}$. If $G \frown A \in P_{\phi}$, then $G \subseteqq A^{\phi c} \subseteqq X^{c}$, that is, $G \frown X = 0$.

(3), (4) follows immediately from (1), (2).

This theorem shows that definition of A^{ϕ} is equivalent to that due to C. Kuratowski[3].

THEOREM 1·2.

(1) $0^{\phi} = 0$.

(2) $A \in P_{\phi}$ implies $A^{\phi} = 0$.

(3) $A^{\phi\phi} \subseteqq A^{\phi}$.

(4) $(G \frown A)^{\phi} \supseteqq G \frown A^{\phi}$.

(5) $G \subseteqq R^{\phi}$ implies $G^{\phi} = G^{a}$.

Proof; (1), (2) follow from the definition of A^{ϕ}.

(3): $A^{\phi\phi} \subseteqq A^{\phi a} = A^{\phi}$.

(4): Let $H \frown (G \frown A^{\phi}) \neq 0$, that is, $(H \frown G) \frown A^{\phi} \neq 0$, then it follows from Theorem 1·1 that $H \frown G \frown A \notin P_{\phi}$, hence $(G \frown A)^{\phi} \supseteqq G \frown A^{\phi}$.

(5): $G^{\phi} = (G \frown R)^{\phi} \supseteqq G \frown R^{\phi} = G$ by (4), hence $G^{\phi} = G^{a}$.

2° Given two P_{ϕ}, P_{ψ}, we write $\psi \geqq \phi$ when $P_{\phi} \supseteqq P_{\psi}$.

THEOREM 1·3. $\psi \geqq \phi$ implies $A^{\psi} \supseteqq A^{\phi}$.

Proof. $A^{\psi c} = \bigvee \{G; G \frown A \in P_{\psi}\} \subseteqq \bigvee \{G; G \frown A \in P_{\phi}\} = A^{\phi c}$, that is, $A^{\psi} \supseteqq A^{\phi}$.

Now we define $P_{\bar{\phi}} = \{A;\ A^{\phi} = 0\}$. Then we see that $\bar{\phi} \leqq \phi$.

THEOREM 1·4. $A^{\phi} = A^{\bar{\phi}}$.

Proof. Since $\phi \geqq \bar{\phi}$, hence it follows from Theorem 1·3 that $A^{\phi} \supseteqq A^{\bar{\phi}}$. Making use of Theorem 1·2 (5) we see that $A^{\phi c} = \bigvee \{G;\ G \frown A^{\phi} = 0\} \supseteqq \bigvee \{G;\ G \frown A \in P_{\bar{\phi}}\} = A^{\bar{\phi} c}$, that is, $A^{\phi} \subseteqq A^{\bar{\phi}}$. Hence we have $A^{\phi} = A^{\bar{\phi}}$.

THEOREM 1·5. $A^{\psi} \supseteqq A^{\phi}$ holds for any A if and only if $A^{\psi} = 0$ implies $A^{\phi} = 0$.

Proof. The necessity is trivially true, so we prove the sufficiency. $A^{\psi c} = \bigvee \{G;\ (G \frown A)^{\psi} = 0\} \subseteqq \bigvee \{G;\ (G \frown A)^{\phi} = 0\} = A^{\phi c}$, that is, $A^{\psi} \supseteqq A^{\phi}$.

A family of sets P is said to be maximal when

(M) If $A = \bigvee A_{\iota}$, A_{ι} open in A, and $A_{\iota} \in P$, then $A \in P$.

If P_{ϕ} is maximal, we shall also say that ϕ is maximal.

THEOREM 1·6. $P_{\phi} = P_{\bar{\phi}}$ for maximal ϕ, that is, $A \in P_{\phi}$ is equivalent to $A^{\phi} = 0$.

3) Cf. [2], 29.

Proof. Let $A^{\phi}=0$. Since by difinition $\bigcup\{G;\ G\cap A\in P_{\phi}\}=A^{\phi c}=R$, it follows that $A=\bigcup\{G\cap A;\ G\cap A\in P_{\phi}\}\in P_{\phi}$.

THEOREM 1·7. Let ϕ be maximal, then $A^{\psi}\supseteqq A^{\phi}$ holds for any A if and only if $\psi\geqq\phi$.

Proof. By theorem 1·3 it suffices only to show the necessity. It follows from Theorem 1·5 that $\psi\geqq\bar{\psi}\geqq\bar{\phi}=\phi$.

THEOREM 1·8. ϕ is maximal if and only if $A^{\phi *}\in P_{\phi}$ holds for any A.

Proof. Let ϕ be maximal, then $A^{\phi *}=\bigcup\{G\cap A;\ G\cap A\in P_{\phi}\}\in P_{\phi}$. Conversely assume that $A^{\phi *}\in P_{\phi}$ holds for any A. If $A=\bigcup A_{\iota}$, A_{ι} open in A, that is, $A_{\iota}=A\cap G_{\iota}$, and $A_{\iota}\in P_{\phi}$, then $A^{\phi *}=\bigcup\{G\cap A;\ G\cap A\in P_{\phi}\}\supseteqq\bigcup(G_{\iota}\cap A)=A$. Hence $A=A^{\phi *}\in P_{\phi}$.

COROLLARY. If ϕ is maximal, then $A^{\phi *\phi}=0$·

3° A family of sets P is said to be hereditary, weakly hereditary, additive and weakly additive respectively when

(H) $A\in P$, $B\subseteqq A$ implies $B\in P$,

$(H)_W$ If $A\in P$ and B is open in A, then $B\in P$,

(A) $A, B\in P$ imply $A\cup B\in P$,

and

$(A)_W$ If $A, B\in P$ and one of them is closed in $A\cup B$, then $A\cup B\in P$.

If P_{ϕ} satisfies one of these conditions, say (H), we say that ϕ is hereditary, or ϕ has the property (H)

THEOREM 1·9. Let ϕ be weakly hereditary. Then $G\cap A^{\phi}=0$ implies $G\cap A\in P_{\phi}$ for arbitrary G if and only if $A^{\phi *}\in P_{\phi}$.

Proof. It is esay to see that the necessity holds, since $A^{\phi c}$ is open. Assume that $A^{\phi *}\in P_{\phi}$. If $G\cap A^{\phi}=0$, then $G\cap A=G\cap A^{\phi *}\in P_{\phi}$.

THEOREM 1·10. The following two conditions are equivalent

(1) $G\cap A^{\phi}=0$ implies $G\cap A\in P_{\phi}$ for any A.

(2) ϕ is weakly hereditary and maximal.

Proof. (1) → (2): Let $A\in P_{\phi}$, then (1) implies that $G\cap A\in P_{\phi}$. Hence ϕ has the property $(H)_W$. It follows from Theorems 1·8 and 1·9 that ϕ is also maximal.

(2) → (1): This follows from Theorems 1·8 and 1·9.

THEOREM 1·11. Let ϕ has the property $(H)_W$, (M), then the following two conditions are equivalent for a given set A.

(1) $A^{\phi\phi}=A^{\phi}$.

(2) $G\cap A^{\phi}\in P_{\phi}$ implies $G\cap A^{\phi}=0$.

Proof.

(1) → (2): $0=(G\frown A^{\phi})^{\phi}\supseteqq G\frown A^{\phi\phi}=G\frown A^{\phi}$ by Theorem 1·1.

(2) → (1): $A^{\phi\phi c}=\bigcup\{G;\ G\frown A^{\phi}\in P_{\phi}\}=\bigcup\{G;\ G\frown A^{\phi}=0\}=A^{\phi c}$, that is, $A^{\phi\phi}=A^{\phi}$.

4° Given operation A^{*} such that A^{*} is a closed set, and $0^{*}=0$.

Put $P_{\phi}=\{A;\ A^{*}=0\}$. Then

THEOREM 1·12. If $G\frown A^{*}=0$ is equivalent to $(G\frown A)^{*}=0$ for any A, then $A^{*}=A^{\phi}$ and ϕ has the properties $(H)_{W}$, (M).

Proof. $A^{\phi c}=\bigcup\{G;(G\frown A)^{*}=0\}=\bigcup\{G;G\frown A^{*}=0\}=A^{*c}$, that is $A^{\phi}=A^{*}$. Theorem 1·10 shows that ϕ has the properties $(H)_{W}$, (M).

5° **THEOREM 1·13.** Let ϕ, ψ have the property $(H)_{W}$, and if assume that $R^{\phi}=R^{\psi}$, $R^{\phi c\phi}=R^{\psi c\psi}=0$, then $G^{\phi}=G^{\psi}$ for any G.

Proof. $(H\frown G)^{\phi}=0$ means that $H\frown G\subseteqq R^{\phi c}=R^{\psi c}$. Hence $(H\frown G)^{\phi}=0$ equivalent to $(H\frown G)^{\psi}=0$, so we have $G^{\phi}=G^{\psi}$.

§2. Properties of Operation A^{ϕ}.

1° **THEOREM 2·1.** Let ϕ have the pyoperty $(H)_{W}$. Then

(1) $A^{\phi}\supseteqq(G\frown A)^{\phi}$,

(2) $G\frown A^{\phi}=G\frown(G\frown A)^{\phi}$,

(3) $(G\frown A)^{\phi\phi}\supseteqq(G\frown A^{\phi})^{\phi}$,

(4) $(G\frown A)^{(\phi)}=G\frown A^{(\phi)}$,

(5) $(G\frown A)^{\phi *}\supseteqq G\frown A^{\phi *}$,

(6) $A^{\phi *(\phi)}=0$,

(7) $A^{\phi *\phi *}=A^{\phi *}$, and

(8) If $A=\bigcup A_{\iota}$, A_{ι} open in A, then $A^{(\phi)}=\bigcup A_{\iota}^{(\phi)}$.

Proof.

(1): Let $H\frown A\in P_{\phi}$, then $H\frown G\frown A\in P_{\phi}$ since ϕ has the property $(H)_{W}$. Hence $H\frown(G\frown A)^{\phi}=0$ by Theorem 1·2 (4). Theorem 1·1 shows that $A^{\phi}\supseteqq(G\frown A)^{\phi}$.

(2): $G\frown A^{\phi}\supseteqq G\frown(G\frown A)^{\phi}$ by (1). On the other hand $G\frown(G\frown A)^{\phi}\supseteqq G\frown(G\frown A^{\phi})=G\frown A^{\phi}$ by Theorem 1·2. Hence $G\frown A^{\phi}=G\frown(G\frown A)^{\phi}$.

(3): $G\frown A^{\phi}$ is open in $(G\frown A)^{\phi}$ by (2), hence $(G\frown A^{\phi})^{\phi}\subseteqq(G\frown A)^{\phi\phi}$ by (1).

(4): $(G\frown A)^{(\phi)}=G\frown A\frown(G\frown A)^{\phi}=G\frown A\frown A^{\phi}=G\frown A^{(\phi)}$ by (2).

(5): $A^{\phi c}\subseteqq(G\frown A)^{\phi c}$ by (1), hence $(G\frown A)^{\phi *}\supseteqq G\frown A^{\phi *}$.

(6): $A^{\phi *(\phi)}=(A\frown A^{\phi c})^{(\phi)}=A^{(\phi)}\frown A^{c\phi}=A\frown A^{\phi}\frown A^{\phi c}=0$ by (4).

(7): $A^{\phi *}=A^{\phi *\phi *}\bigcup A^{\phi *(\phi)}=A^{\phi *\phi *}$ by (6).

(8): We can write A_ι as $G_\iota\frown A$, hence $A_\iota^{(\phi)}=(G_\iota\frown A)^{(\phi)}=G_\iota\frown A^{(\phi)}$ by (4). Therefor $A^{(\phi)}=\bigcup(A^{(\phi)}\frown G_\iota)=\bigcup A_\iota^{(\phi)}$.

THEOREM 2·2. Let ϕ has the property (H). Then

(1) $A\supseteqq B$ implies $A^\phi\supseteqq B^\phi$, $A^{(\phi)}\supseteqq B^{(\phi)}$ and $A^{\phi *}\frown B\subseteqq B^{\phi *}$, and (2) $(A\smile B)^\phi\supseteqq A^\phi\smile B^\phi$.

Proof.

(1): Let $G\frown A\in P_\phi$, then $G\frown B\in P_\phi$ since ϕ has the property (H). It follows from the definition of A^ϕ that $A^\phi\supseteqq B^\phi$, hence $A^{(\phi)}=A\frown A^\phi \supseteqq B\frown B^\phi=B^{(\phi)}$, and $A^{\phi *}\frown B=A^{\phi c}\frown B\subseteqq B^{\phi *}$.

(2) follows from (1).

THEOREM 2·3. Let ϕ have the properties (H) and (M). Then $A^\phi=A^{x\phi}$ holds for any A if and only if $A\in P_\phi$ implies $A^x\in P_\phi$.

Proof. It suffices only to show the sufficiency. Let $G\frown A\in P_\phi$, then $G\frown A\subseteqq G\frown A^x\subseteqq(G\frown A)^x\in P_\phi$, hence $G\frown A\in P_\phi$ is equivalent to $G\frown A^x\in P_\phi$. Therefore we conclude that $A^\phi=A^{x\phi}$.

THEOREM 2·4.

(1) If ϕ has the properties (H), $(\mathrm{A})_\mathrm{W}$, then $(A\smile B)^\phi=A^\phi\smile B^\phi$ if one of A, B is closed or open in $A\smile B$.

(2) If ϕ has the properties (H), (A), then $(A\smile B)^\phi=A^\phi\smile B^\phi$.

Proof (1): It needs only to show that $(A\smile B)^\phi\subseteqq A^\phi\smile B^\phi$. First consider the case A is closed in $A\smile B$.

$$\begin{aligned} A^{\phi c}\frown B^{\phi c} &= [\bigcup\{G;\ G\frown A\in P_\phi\}]\frown[\bigcup\{H;\ H\frown B\in P_\phi\}] \\ &= \bigcup\{G\frown H;\ G\frown A,\ H\frown B\in P_\phi\} \\ &\subseteqq \bigcup\{G\frown H;\ G\frown H\frown A,\ G\frown H\frown B\in P_\phi\} \quad \text{by } (\mathrm{H})_\mathrm{W} \\ &\subseteqq \bigcup\{G\frown H;\ G\frown H\frown(A\smile B)\in P_\phi\} \quad \text{by } (\mathrm{A})_\mathrm{W} \\ &= (A\smile B)^{\phi c}, \end{aligned}$$

that is, $A^\phi\smile B^\phi\supseteqq(A\smile B)^\phi$.

Next assume that A is open in $A\smile B$. Then $B_1=B\frown A^c$ is closed in $A\smile B_1=A\smile B$. Therefore $(A\smile B)^\phi=(A\smile B_1)^\phi\subseteqq A^\phi\smile B_1^\phi\subseteqq A^\phi\smile B^\phi$ by Theorm 2·2.

(2) can be proved as (1).

2° **THEOREM 2·5.** Let ϕ have the properties $(\mathrm{H})_\mathrm{W}$, $(\mathrm{A})_\mathrm{W}$, and (M). Then (1) $G\frown A\in P_\phi$, $G\frown A^{(\phi)}=0$ and $G\frown A^{(\phi)}\in P_\phi$ are all equivalent to one another.

(2) $A^\phi=A^{(\phi)\phi}=A^{(\phi)\psi}$ for any $\psi\geqq\phi$.

(3) $(G\frown A)^{\phi}=(G\frown A^{(\phi)})^{\phi}=(G\frown A^{(\phi)})^{\psi}$ for any $\psi\geqq\phi$.

(4) $F^{\phi\phi}=F^{\phi}$.

(5) $G^{\phi\phi}=G^{\phi}$.

(6) $A^{\phi\phi\phi}=A^{\phi\phi}$.

Proof. (1): $G\frown A\in P_{\phi}$ implies $G\frown A^{(\phi)}=(G\frown A)^{(\phi)}=0$ by Theorem 2·1, hence also $G\frown A^{(\phi)}\in P_{\phi}$, and in turn the latter implies $G\frown A=(G\frown A^{(\phi)})\smile(G\frown A^{\phi*})\in P_{\phi}$ by Theorem 1·9 and by $(\mathrm{H})_{\mathrm{W}}$, $(\mathrm{A})_{\mathrm{W}}$.

(2): $A^{\phi c}=\bigcup\{G;\ G\frown A\in P_{\phi}\}=\bigcup\{G;\ G\frown A^{(\phi)}\in P_{\phi}\}=A^{(\phi)\phi c}$ by (1), that is, $A^{\phi}=A^{(\phi)\phi}$. Since $A^{\phi}=A^{(\phi)\phi}\subseteqq A^{(\phi)\psi}\subseteqq A^{(\phi)\alpha}\subseteqq A^{\phi\alpha}=A^{\phi}$, we have $A^{\phi}=A^{(\phi)\phi}=A^{(\phi)\psi}$.

(3): By making use of (2) and Theorem 2·1 we show that $(G\frown A)^{\phi}=(G\frown A)^{(\phi)\phi}=(G\frown A^{(\phi)})^{\phi}$, $(G\frown A)^{\phi}=(G\frown A)^{(\phi)\psi}=(G\frown A^{(\phi)})^{\psi}$.

(4): Since $F^{\phi}=F^{(\phi)}$, $F^{\phi\phi}=F^{(\phi)\phi}=F^{\phi}$ by (2).

(5): $G^{(\phi)}$ is open in G^{ϕ}, hence $G^{\phi}\supseteqq G^{\phi\phi}\supseteqq G^{(\phi)\phi}=G^{\phi}$ by (2).

(6): A^{ϕ} is closed, hence $A^{\phi\phi\phi}=A^{\phi\phi}$ by (4).

THEOREM 2·6. Let ϕ have the properties (H), $(\mathrm{A})_{\mathrm{W}}$ and (M). And let $\psi\geqq\phi$. Then

(1) $A^{\phi}=A^{\phi\phi}=A^{\phi\psi}$,

(2) $A^{\phi}=A^{(\phi)\psi}=A^{(\psi)\phi}$,

(3) $(G\frown A)^{\phi}=(G\frown A^{\phi})^{\phi}=(G\frown A^{\phi})^{\psi}$,

(4) $G\frown A\in P_{\phi}$, $G\frown A^{\phi}\in P_{\phi}$ and $G\frown A^{\phi}=0$ are all equivalent to one another,

(5) If $A=\bigcup A_{\iota}$, A_{ι} open in A, then $A^{\phi}=(\bigcup A_{\iota}^{\phi})^{\psi}$, and

(6) If P_{ϕ} is completely additive, and if $A=\bigcup_{n=1}^{\infty} A_{n}$, then

$$A^{\phi}=\Big(\bigcup_{n=1}^{\infty} A^{\phi}\Big)^{\psi}.$$

Proof. (1): $A^{\phi}\supseteqq A^{\phi\psi}\supseteqq A^{\phi\phi}\supseteqq A^{(\phi)\phi}=A^{\phi}$ by Theorem 2·5.

(2): $A^{\phi}\supseteqq(A\frown A^{\psi})^{\phi}\supseteqq(A\frown A^{\phi})^{\phi}=A^{\phi}=A^{(\phi)\psi}$ by Theorem 2·5, hence we have $A^{\phi}=A^{(\psi)\phi}=A^{(\phi)\psi}$.

(3): $(G\frown A)^{\phi}=(G\frown A^{(\phi)})^{\phi}\subseteqq(G\frown A^{\phi})^{\phi}\subseteqq(G\frown A^{\phi})^{\psi}\subseteqq(G\frown A^{\phi})^{\alpha}\subseteqq(G\frown A)^{\phi\alpha}=(G\frown A)^{\phi}$ by (2). Hence $(G\frown A)^{\phi}=(G\frown A^{\phi})^{\phi}=(G\frown A^{\phi})^{\psi}$.

(4) follows from (3).

(5): By making use of Theoren 2·1 and (1), (2) we show that $A^{\phi}=A^{(\phi)\phi}=\{\bigcup A_{\iota}^{(\phi)}\}^{\phi}\subseteqq\{\bigcup A_{\iota}^{\phi}\}^{\phi}\subseteqq\{\bigcup A_{\iota}^{\phi}\}^{\psi}\subseteqq\{\bigcup A_{\iota}^{\phi}\}^{\alpha}\subseteqq A^{\phi\alpha}=A^{\phi}$. Hence we have $A^{\phi}=(\bigcup A_{\iota}^{\phi})^{\psi}$.

(6): $G\frown A \in P_\phi$ implies and is implied by that $G\frown A_n \in P_\phi$ for any n, hence by (4) $G\frown A_n^\phi=0$ or $\in P_\phi$, that is, $G\frown(\smile A_n^\phi)=0$ or $\in P_\phi$. Hence $A^\phi=(\smile A_n^\phi)^\phi=(\smile A_n^\phi)^\alpha$. Since $\phi\leqq\psi\leqq\alpha$, we have $A^\phi=(\smile A_n^\phi)^\psi$.

THEOREM 2·7. Let ϕ have the properties (H), $(A)_W$ and (M), Then the following conditions are equivalent.

(1) $A^\phi=A^{\psi\phi}$ holds for any A.

(2) $(G\frown A)^\phi=(G\frown A^\psi)^\phi$ holds for any A, G.

(3) $\psi\geqq\phi$, and $A\in P_\phi$ implies $A^\psi\in P_\phi$.

Proof.

(1) → (2): $(G\frown A^\psi)^\phi=(G\frown A^{\psi\phi})^\phi=(G\frown A^\phi)^\phi=(G\frown A)^\phi$.

(2) → (1): (1) is a special case of (2).

(3) → (1): Let $G\frown A\in P_\phi$, then $0=(G\frown A)^{\psi\phi}\supseteqq(G\frown A^\psi)^\phi\supseteqq G\frown A^{\psi\phi}$, hence $A^\phi\supseteqq A^{\psi\phi}$. Since $A^{\psi\phi}\supseteqq A^{\phi\phi}=A^\phi$, we have $A^\phi=A^{\psi\phi}$.

(1) → (3): This is obvious.

§3. Operation $\phi c\phi c\phi$

1° In this §, unless otherwise stated, we assume that ϕ has the properties (H), $(A)_W$ and (M).

THEOREM 3·1. Given ϕ, there exists a ρ such that ρ has the properties (H), $(A)_W$ and (M) and $A^\rho=A^{c\phi c\phi}$ for any A.

Proof. Put $P_\rho=\{A\,;\,A^{c\phi c\phi}=0\}$. Since $0^{c\phi c\phi}=0$, $0\in P_\rho$. $A\in P_\rho$ is characterized by $A^{c\phi}=R^\phi$, or $A^{c\phi}\supseteqq R^\phi$.

(i) Suppose that $A\in P_\rho$ and $A\supseteqq B$. Then $B^{c\phi}\supseteqq A^{c\phi}\supseteqq R^\phi$. Hence P_ρ has the property (H).

(ii) Suppose $A, B\in P_\phi$ and B is closed in $A\smile B$. If we write $C=A\smile B$, then there exists an F such that $B=C\frown F$. Then

$$C^{c\phi}=(A^c\frown B^c)^\phi\supseteqq(A^c\frown F^c)^\phi=(A^{c\phi}\frown F^c)^\phi=(R^\phi\frown F^c)^\phi=F^{c\phi}.$$

Using this relation we see that $C^{c\phi}=C^{c\phi}\smile F^{c\phi}=(C\frown F)^{c\phi}=B^{c\phi}=R^\phi$. Thus P_ρ has the property $(H)_W$.

(iii) $(G\frown A)^{c\phi c\phi}=(G^{c\phi c}\frown A^{c\phi c\phi})^\phi=(G\frown A^{c\phi c\phi})^\phi\smile(G^{c\phi*}\frown A^{c\phi c\phi})^\phi$ $=(G\frown A^{c\phi c\phi})^\phi$. Therefor $G\frown A\in P_\rho$ is equivalent to $G\frown A^{c\phi c\phi}=0$.

Theorem 1·2 shows that ρ has the property (M) and $A^\rho=A^{c\phi c\phi}$.

Remark. In the sequel we identify ρ with $c\phi c\phi$. Then $\rho\rho=\rho$ implies $\phi c\phi=\phi c\phi c\phi$, that is, $A^{\phi c\phi}=A^{\phi c\phi c\phi}$ for any A. If ϕ has the property (A), this is a special case of a result obtained by S. Matsushita[4]. If ψ has the property $(H)_W$, and if $R^\phi=R^\psi$, $R^{\psi c\psi}=0$ holds, then we see that $\phi c\phi=\phi c\psi$, hence $A^{c\phi c\phi}=A^{c\phi c\psi}$.

THEOREM 3·2. $\phi c\phi c\phi$ has the properties (H), $(A)_W$ and (M). If ϕ has the property (A), then $\phi c\phi c\phi$ has it.

Proof. Put $\rho=c\phi c\phi$, then $c\rho c\rho=\phi c\phi c\phi$, hence by Theorem 3·1 $\phi c\phi c\phi$ has the properties (H), $(A)_W$ and (M). If ϕ has the property (A), then $(A\cup B)^{\phi c\phi c\phi}=(A^\phi\cup B^\phi)^{c\phi c\phi}=A^{\phi c\phi c\phi}\cup B^{\phi c\phi c\phi}$, so that $\phi c\phi c\phi$ has the property (A).

THEOREM 3·3. $\phi=\phi c\phi c\phi$ if and only if $(G^\phi\cap G^c)^\phi=0$ for any G.

Proof. $\phi=\phi c\phi c\phi$ implies that $(G^\phi\cap G^c)^\phi=(G^\phi\cap G^c)^{\phi c\phi c\phi}\subseteqq(G^\phi\cap G^c)^{c\phi c\phi}$ $=(G^{\phi c\phi c}\cap G^{\phi c})^\phi=G^{\phi c\phi *\phi}=0$. Conversely we assume $(G^\phi\cap G^c)^\phi=0$. Since $G^c\cup G^{\phi *}=G^{\phi c}\cup(G^\phi\cap G^c)$ holds, $G^{c\phi}=G^{\phi c\phi}$. If we put $G=A^{\phi c}$, then we have $A^\phi=A^{\phi c\phi c\phi}$.

THEOREM 3·4. When $\phi=\phi c\phi c\phi$ is satisfied, we have

(1) $(A\cap B)^\phi=(A^\phi\cap B)^\phi$ if one of A, B is closed or open,

(2) $(A^\phi\cap B)^\phi=(A\cap B^\phi)^\phi=(A^\phi\cap B^\phi)^\phi$,

and

(3) $(A\cap B^\psi)^\phi=(A\cap B^\phi)^\phi$ if $\phi=\psi\phi$ is satisfied.

Proof. (1): One of A^c, B^c is closed or open, hence $(A^c\cup B^c)^{c\phi c\phi}$ $=A^{\phi c\phi}\cup B^{\phi c\phi}$ by Theorem 3·1, that is, $(A\cap B)^{\phi c\phi}=A^{\phi c\phi}\cup B^{\phi c\phi}$. From this we have $(A\cap B)^\phi=(A\cap B)^{\phi c\phi c\phi}=(A^{\phi c\phi}\cup B^{\phi c\phi})^{c\phi}$. Since A^ϕ is closed, by the same argument we see that $(A^\phi\cap B)^\phi=(A^{\phi c\phi}\cup B^{\phi c\phi})^{c\phi}$. Therfore we have $(A\cap B)^\phi=(A^\phi\cap B)^\phi$.

(2) follows from (1).

(3): $(A\cap B^\psi)^\phi=(A\cap B^{\psi\phi})^\phi=(A\cap B^\phi)^\phi$.

THEOREM 3·5. Suppose that ϕ has the properties (H), (A) and (M). Then $\phi=c\phi c\phi$ if and only if $(A^\phi\cap A^c)^\phi=0$ holds for any A.

Proof. $\phi=c\phi c\phi$ implies that $(A^\phi\cap A^c)^\phi=(A^\phi\cap A^c)^{c\phi c\phi}=(A^{\phi c\phi}\cup A^\phi)^{c\phi}$ $=(A^{\phi c}\cup A^\phi)^{\phi c\phi}=R^{\phi c\phi}=0$. The converse follows from the fact that the identity $A\cup(A^{c\phi c}\cap A^c)=A^{c\phi c}\cup(A\cap A^{c\phi})$ implies $A^\phi=A^{c\phi c\phi}$.

THEOREM 3·6. Suppose that ϕ has the properties (H), (A) and (M), and that $\phi=\phi c\phi c\phi$. Then $P=\{A\,;\,(A^\phi\cap A^c)^\phi=0\}$ is a set field containing every G and F and has the properties (H) and (M). And

4) Cf. [3], 30.

(1) $A^{\phi}=A^{c\phi c\phi}$ for $A\in P$,

(2) $(A\frown B)^{\phi}=(A^{\phi}\frown B)^{\phi}=(A\frown B^{\phi})^{\phi}$ if $A\in P$,

(3) $(A\smile B)^{c\phi c\phi}=A^{c\phi c\phi}\frown B^{c\phi c\phi}$ if $A\in P$,

(4) If P_{ϕ} is completely additive, then P is also completely additive and is invariant under A-operator[5].

Proof. By making use of Theorem 1·12 and Theorm 3·4 we can show that P satisfies the statement in the Theorem.

(1): It follows from the identity $A\smile A^{c\phi *}=A^{c\phi c}\smile(A^{c\phi}\frown A)$ that $A^{\phi}=A^{c\phi c\phi}$.

(2): $(A\frown A^{\phi c})^{\phi}=0$ by the Corollary to Theorem 1·8. And by the assumpition $(A^{\phi}\frown A^{c})^{\phi}=0$. Hence there exists a set $Q\in P_{\phi}$ such that $A\smile Q=A^{\phi}\smile Q$, so that $(A\frown B)\smile Q=(A^{\phi}\smile B)\smile Q$, hence $(A\frown B)^{\phi}=(A^{\phi}\frown B)^{\phi}$.

(3): $A\in P$ implies $A^{c}\in P$. Hence $(A\smile B)^{c\phi}=(A^{c}\frown B^{c})^{\phi}=(A^{c\phi}\frown B^{c\phi})^{\phi}$ by (2), so that $(A\smile B)^{c\phi c\phi}=(A^{c\phi}\frown B^{c\phi})^{\phi c\phi}=(A^{c\phi c}\smile B^{c\phi c})^{c\phi c\phi}=A^{c\phi c\phi}\smile B^{c\phi c\phi}$.

(4): Let $A_{n}\in P$, then $A_{n}\smile Q_{n}=A_{n}^{\phi}\smile Q_{n}$ for some $Q_{n}\in P_{\phi}$, hence $A\smile Q=(\smile A_{n}^{\phi})\smile Q$ for some $Q\in P_{\phi}$. Therefor we have $((\smile A_{n}^{\phi})\frown A^{c})^{\phi}=0$. And then $(A^{\phi}\frown A^{c})^{\phi}=(A\frown A^{c\phi})^{\phi}=(\smile(A_{n}\frown A^{c\phi}))^{\phi}=(\smile(A_{n}\frown A^{c\phi})^{\phi})^{\phi}=(\smile(A_{n}^{\phi}\frown A^{c})^{\phi})^{\phi}=((\smile A_{n}^{\phi})\frown A^{c})^{\phi}=0$. Thus we conclude that $A\in P_{\phi}$. Then we can show that the rest of the statement is true[6].

Remark. If $\phi=\alpha o\alpha$, then $c\phi c\phi=o\alpha$.

2° **THEOREM 3·7.** Every A^{ϕ} is a regular closed set if and only if every non-dense set belongs to P_{ϕ}. Then $\phi=\phi c\phi c\phi$.

Proof. Assume that every A^{ϕ} is a regular closed set, that is, $A^{\phi}=A^{\phi o\alpha}$. Then $A^{\alpha o\alpha}=0$ implies that $A^{\phi}=A^{\phi o\alpha}\subseteqq A^{\alpha o\alpha}=0$, hence $A\in P_{\phi}$. The converse is true by Theorem 2·6. And $\alpha o\alpha\geqq\phi$ implies that $(G^{\phi}\frown G^{c})^{\phi}\subseteqq(G^{\alpha}\frown G^{c})^{\alpha o\alpha}=0$, hence we conclude that $\phi=\phi c\phi c\phi$ by Theorem 3·3.

THEOREM 3·8. If A^{ϕ} is a regular closed set, then $A^{\phi}=(A\frown A^{\phi o})^{\phi}$. Conversely if $A^{\phi}=(A\frown A^{\phi o})^{\phi}$ holds for any A, then P_{ϕ} contains every non-dense set.

Proof. The first part of the Theorem follows from $(A\frown A^{\phi o})^{\phi}=(A^{\phi}\frown A^{\phi o})^{\phi}=A^{\phi o\phi}\subseteqq A^{\phi o\alpha}\subseteqq A^{\phi}$.

Next assume that P_{ϕ} contains every non-dense set, then $A^{\phi}=(A\frown A^{\phi})^{\phi}=(A\frown A^{\phi o})^{\phi}$, since $A^{\phi}\frown A^{\phi c}$ is a non-dense set.

5) Cf. [2], 56.
6) Cf. [2], 56.

THEOREM 3·9. If every A^ϕ is a regular closed set, then

(1) $(A^\phi \frown B)^\phi = (A^\phi \frown B)^\phi = (A^\phi \frown B)^{\circ\alpha}$ if B is closed or open, and

(2) $(A^\phi \frown B^\phi)^\phi = (A^\phi \frown B^\phi)^{\circ\alpha}$.

Proof. (1): $(A^\phi \frown B)^{\circ\alpha} = (A^{\phi\circ\alpha} \frown B^\circ)^\alpha = (A^\phi \frown B^\circ)^\alpha = (A^\phi \frown B^\circ)^\alpha = (A^\phi \frown B^\circ)^\phi \subseteq (A^\phi \frown B)^\phi$. Hence if B is closed or open, we have the statement.

(2) follows from (1).

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