ON INTEGRALS OF THE CERTAIN ORDINARY DIFFERENTIAL EQUATIONS IN THE VICINITY OF THE SINGULARITY. II.

By

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§ 1. Introduction.

In our previous paper[1], under generalized Poincaré's condition or Picard's condition, we have solved the equation as follows:

$$(1.1) \qquad \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} = 0,$$

where $X_i(x)$ are regular in the vicinity of $x_i=0$ and are expanded as follows:

$X_i(x)=\sum_{j=1}^{n} a_{ij}x_j+$ sum of the terms of the second and higher orders.

Let the eigen values of the matrix $A=\| a_{ij} \|$ be λ_i. The generalized Poincaré's condition is as follows: *all λ_i's lie in a convex domain Ω which does not contain the origin.* The generalized Picard's condition is as follows: *some of λ_i lie in a convex domain Ω which does not contain the origin.* In this paper, making use of solutions of (1.1), we are going to obtain integrals of the ordinary differential equations as follows;

$$(1.2) \qquad \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots\cdots = \frac{dx_n}{X_n}.$$

By our previous paper[2], without loss of generality, we can assume that A is of Jordan's form and that the real parts of λ_i lying in Ω are positive. As in (A), we write X_i as follows:

$$X^i_{ip}=\lambda_i x^i_{ip}+\delta_p x^i_{ip-1}+\cdots\cdots\cdots,$$

where the unwritten terms are of the second and higher orders, and δ_p is equal to 1 or zero according as $p\geqq 2$ or $p=1$.

1) M. Urabe, *On Solutions of the Linear Homogeneous Partial Differential Equation in the Vicinity of the Singularity, III.* This Journal, Vol. 15, No. 1, p. 25. In the following, we denote this paper by (A).

2) M. Urabe, *On Integrals of the Certain Ordinary Differential Equations in the Vicinity of the Singularity, I.* This Journal, Vol. 14, No. 3, p. 209. In the following, we denote this paper by (B).

§ 2. Integrals under generalized Poincaré's condition.

The integrals of (1.2) are obtained by putting $n-1$ independent solutions of (1.1) constants. By (A), as $n-1$ independent solutions, we have the following functions:

$$
(2.1)\begin{cases}
F_{l0}^{\alpha\,\frac{1}{\lambda_\alpha}}/F_{10}^{1\,\frac{1}{\lambda_1}} \text{ (except for } \alpha=l=1), \; U_{l0}^\alpha-\frac{1}{\lambda_\alpha}\log F_{l0}^\alpha,\; U_{lp}^\alpha\; (p=1, 2, \ldots, P_l^\alpha-2); \\
F_{l0}^{\gamma\,\frac{1}{\lambda_\gamma}}/F_{10}^{1\,\frac{1}{\lambda_1}},\; U_{l0}^\gamma-\frac{1}{\lambda_\gamma}\log F_{l0}^\gamma,\; U_{lp}^\gamma\; (p=1, 2, \ldots, P_l^\gamma-2); \\
\text{for } H_l^z\neq 0,\quad U_{lH_l^z+p}^z\quad (p=0, 1, \ldots, P_l^z-1); \\
\text{for } H_l^z=0,\quad U_{l0}^z-\frac{1}{\lambda_z}\log F_{l0}^z,\; U_{lp}^z\; (p=1, 2, \ldots\ldots, P_l^z-1).
\end{cases}
$$

We introduce an auxiliary variable t such that,

$$
(2.2)\qquad F_{10}^1=C_1^1t^{\lambda_1},
$$

where C_1^1 is an arbitrarily chosen constant. Then, putting the functions of (2.1) constants, we have:

$$
(2.3)\begin{cases}
F_{l0}^\alpha=C_l^\alpha t^{\lambda_\alpha},\quad U_{l0}^\alpha-\log t=C_{l0}^\alpha,\quad U_{lp}^\alpha=C_{lp}^\alpha; \\
F_{l0}^\gamma=C_l^\gamma t^{\lambda_\gamma},\quad U_{l0}^\gamma-\log t=C_{l0}^\gamma,\quad U_{lp}^\gamma=C_{lp}^\gamma; \\
\text{for } H_l^z\neq 0,\quad U_{lH_l^z+p}^z=C_{lH_l^z+p}^z, \\
\text{for } H_l^z=0,\quad U_{l0}^z-\log t=C_{l0}^z,^{(1)}\; U_{lp}^z=C_{lp}^z,
\end{cases}
$$

where C_l^i, C_{lp}^i are arbitrary constants except for C_1^1. As in (B), from the first and second rows of (2.3), we have:

$$
(2.4)\qquad F_{lp}^i=F_p(t^{\lambda_i},\; t^{\lambda_i}\log t, \ldots\ldots, t^{\lambda_i}(\log t)^p\,;\; C_l^i, C_{l0}^i, \ldots\ldots, C_{lp-1}^i),
$$

$$
(i=\alpha, \gamma;\; p=0, 1, \ldots\ldots, P_l^i-1).
$$

Here the functions F_p denote the following functions:

$$
(2.5)\qquad F_p(t^\lambda,\; t^\lambda\log t, \ldots, t^\lambda(\log t)^p\,;\; C, C_0, \ldots\ldots, C_{p-1})
$$
$$
=A_0t^\lambda+A_1t^\lambda(\log t)+\ldots+A_{p-1}t^\lambda(\log t)^{p-1}+Ct^\lambda(\log t)^p,
$$

where $A_0, A_1, \ldots, A_{p-1}$ are polynomials of $C, C_0, \ldots, C_{p-1}$ and A_0 actually depends upon C_{p-1}. Now, by § 3 of (A), the following functions are also solutions of (1.1):

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1) For, by § 3 of (A), $F_{l0}^{z\,\frac{1}{\lambda_z}}/F_{10}^{1\,\frac{1}{\lambda_1}}$ are also solutions of (1.1), consequently constant for (1.2).

$$(2.6) \qquad F_{i0}^{z\ \frac{1}{\lambda_z}}/F_{10}^{1\ \frac{1}{\lambda_1}};$$

and for $H_i^z \neq 0$, $U_{i0}^z - \frac{1}{\lambda_z}\log F_{i0}^z$, U_{ip}^z $(p=1, 2, \dots, H_i^z-1)$.

Namely, for x_i satisfying (2.3), the above functions must be constants. Then, in the same way that we have derived (2.4), we have:

$$(2.7) \qquad F_{ip}^z = F_p(t^{\lambda_z},\ t^{\lambda_z}\log t,\ \dots,\ t^{\lambda_z}(\log t)^p;\ C_i^z,\ C_{i0}^z,\ \dots,\ C_{ip-1}^z).$$

Now, by the lemma in § 3 of (A), for $p=0, 1, \dots, H_i^z$, F_{ip}^z are polynomials of F_{mq}^t where $i \leqq z-1$ and $q=H_m^t+1, \dots, H_m^t+P_m^t$. By the definition, U_{ip}^z is a rational function of F_{iq}^z $(q=0, 1, \dots, p+1)$. Then we can prove the following lemma.

Lemma. *C_i^z, C_{i0}^z, ..., $C_{lH_i^z-1}^z$ are rational functions of C_m^a, C_{m0}^a, ..., $C_{mP_m^a-2}^a$ and C_m^y, C_{m0}^y, ..., $C_{mP_m^y-2}^y$, $C_{mH_m^t}^t$, $C_{mH_m^t+1}^t$, ..., $C_{mH_m^t+P_m^t-1}^t$ for $y, t \leqq z-1$, where t denotes the index of the same property as z.*

For the smallest value of z, F_{i0}^z is a polynomial of F_{mq}^a and F_{mq}^y. Therefore, substituting (2.4) for F_{mq}^a and F_{mq}^y, F_{i0}^z becomes a polynomial of C_m^a C_{mq}^a and C_m^y, C_{mq}^y. Comparing the result with $F_{i0}^z = C_i^z t^{\lambda_z}$, it is seen that C_i^z is a polynomial of C_m^a, C_{mq}^a and C_m^y, C_{mq}^y. Next, for $p=0, 1, \dots, H_i^z-1$, U_{ip}^z are rational functions of $F_{i0}^z, \dots, F_{lH_i^z}^z$, which are polynomials of F_{mq}^a, F_{mq}^y. Therefore, substituting (2.4) for F_{mq}^a and F_{mq}^y, U_{ip}^z become rational functions of C_m^a, C_{mq}^a and C_m^y, C_{mq}^y. Comparing the results with $U_{i0}^z - \log t = C_{i0}^z$ and $U_{ip}^z = C_{ip}^z$, we see that C_{i0}^z, C_{i1}^z, ..., $C_{lH_i^z-1}^z$ are rational functions of C_m^a, C_{mq}^a and C_m^y, C_{mq}^y. Thus we have seen that the lemma is valid for the smallest value of z. In the same way, assuming that the lemma is valid for up to $z-1$, we see that the lemma is valid also for z. Thus the lemma is valid for any z.

Now, for $p=0, 1, \dots, H_i^z$, the relations (2.7) are transformed forms of the relations which are obtained by putting the functions of (2.6) constants. In the relations as follows:

$$(2.8) \qquad \begin{aligned} F_{ip}^z &= P\{F_{mq}^a,\ F_{mq}^y,\ F_{mq}^t\} \\ &= F_p(t^{\lambda_z},\ t^{\lambda_z}\log t,\ \dots,\ t^{\lambda_z}(\log t)^p;\ C_i^z,\ C_{i0}^z,\ \dots,\ C_{ip-1}^z), \end{aligned}$$

where $y, t \leqq z-1$ and $P\{\dots\}$ denotes the polynomials of the arguments, substitute (2.4) and (2.7) for F_{mq}^a, F_{mq}^y, F_{mq}^t, then, by the proof of the lemma, it is seen that the results are identities. Thus, we see that the relations (2.3) are equivalent to the relations composed of (2.4) and (2.7) for $p=H_l^z+1, \ldots, H_l^z+P_l^z$.

Now, by § 3 of (A), the value of the Jacobian of F_{lp}^a, F_{lp}^y and $F_{lH_l^z+p}^z$ $(p\geqq 1)$ with respect to x_ι is not zero for $x_\iota=0$. Therefore, for sufficiently small value of $|t|$ (Arg. t=finite), we can solve the relations (2.4) and (2.7) for $p=H_l^z+1, \ldots, H_l^z+P_l^z$, with regard to x_ι, and there the solutions $x_\iota=x_\iota(t)$ are expressed as the regular functions of the following functions of t:

$$t^{\lambda_\iota},\ t^{\lambda_\iota}(\log t),\ \ldots\ldots,\ t^{\lambda_\iota}(\log t)^{p_\iota}. \tag{2.9}$$

Besides, by the lemma, these $x_\iota=x_\iota(t)$ actually contain arbitrary constants C_l^a (except for C_1^1), C_{lp}^a, C_l^y, C_{lp}^y and $C_{lH_l^z}^z$, $C_{lH_l^z+1}^z$, ..., $C_{lH_l^z+P_l^z-1}^z$, and evidently, when $t\to 0$ (Arg. t=finite), $x_\iota\to 0$, therefore there arises no restriction upon these constants, namely $x_\iota=x_\iota(t)$ actually contain these arbitrary constants. The number of these constants is evidently $n-1$.

Now, from § 1 of (A), $P_l^a-1=w_{lP_l^a}^a$, $P_l^y-1<w_{lP_l^y}^y$ and, from § 3 of (A), $H_l^z+P_l^z\leqq w_{l1}^z-1+P_l^z=w_{lP_l^z}^z$. Therefore, the exponent p_ι of (2.9) does not exceed the maximum weight of the eigen value λ_ι.

Thus we have the following theorem.

Theorem. *When all the eigen values λ_ι's of the matrix A lie in a convex domain which does not contain the origin, there exist integrals of (1.2) which, in the vicinity of $x_\iota=0$, are expressed as regular functions of the functions of (2.9) for sufficiently small value of $|t|$ (Arg. t=finite) and contain $n-1$ arbitrary constants. Here p_ι does not exceed the maximum weight of the eigen value λ_ι. Conversely, any integrals of (1.2) are necessarily transformed into the functions of this form.*

Thus we see that, for solving the equation (1.2), the Poincaré's second condition[1] is not necessary, and yet the results is unaltered when the second condition is omitted.

§ 3. Integrals under generalized Picard's condition.

As stated in (A), without loss of generality, we can assume that, for some λ_ι, for example λ_a $(a=1, 2, \ldots, m)$, $\Re(\lambda_a)>0$ and for others which we denote by λ_x, $\Re(\lambda_x)\leqq 0$.

By § 4 of (A), there exist regular integrals $x_o=x_o(x_a)$ satisfying the

1) (A).

the equations as follows:

$$\sum_{\alpha} X_{\alpha}\frac{\partial x_{\sigma}}{\partial x_{\alpha}} = X_{\sigma}. \tag{3.1}$$

Put $X_{\alpha}(x_{\beta}, x_{\sigma}(x_{\alpha}))=\overline{X}_{\alpha}(x_{\beta})$, then, by § 4 of (A), there exist $m'-1$ independent integrals $g_{\beta}(x_{\alpha})$ of the forms of (2.1) satisfying the equation as follows:

$$\sum_{\alpha} \overline{X}_{\alpha}\frac{\partial g}{\partial x_{\alpha}} =0. \tag{3.2}$$

Here m' denotes the number of the variables x_{α}. Then, by (B), x_{ι} satisfying the relations as follows:

$$g_{\beta}(x_{\alpha})=\text{const.}, \quad \text{and} \quad x_{\sigma}=x_{\sigma}(x_{\alpha}), \tag{3.3}$$

are integrals of (1.2). Since $g_{\beta}(x_{\alpha})$ are of the same forms as (2.1), from $g_{\beta}(x_{\alpha})=\text{const.}$, we see that x_{α} are expressed as regular functions of the same forms as (2.9) and they contain $m'-1$ arbitrary constants. In this case, the exponent p_{ι} of (2.9) does not exceed the maximum weight of the eigen value λ_{ι} for which $\Re(\lambda_{\iota})>0$. Substituting these $x_{\alpha}=x_{\alpha}(t)$ into $x_{\sigma}=x_{\sigma}(x_{\alpha})$, we see that x_{σ} become also regular functions of the arguments of $x_{\alpha}(t)$. Thus, it is seen that *there exist integrals of the equation (1.2), which are expressed as regular functions of the functions of the forms of (2.9) and contain $m'-1$ arbitrary constants.*

Thus the results are the same as stated in (B).

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