ON SOLUTIONS OF THE LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION IN THE VICINITY OF THE SINGULARITY. III.

By

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§ 1. Introduction.

In our previous paper [1], we have considered the equation

$$Xf \equiv \sum_{k=1}^{n} X_k \frac{\partial f}{\partial x_k} = 0, \tag{1.1}$$

where $X^k(x)$ are regular in the vicinity of $x_k=0$ and vanish there. When we write the expansions of $X_k(x)$ in the vicinity of $x_k=0$ as follows:

$X_k(x) = \sum_{j=1}^{n} a_{kj}x_j +$ sum of the terms of the second and higher orders,

in (I), we have assumed that the eigen values λ_i of the matrix $A=\|a_{kj}\|$ satisfy Poincaré's two conditions, namely that

I. there exists a convex domain Ω which contains all λ_i's but not the origin,

II. $\lambda_1 p_1+\lambda_2 p_2+\ldots+\lambda_n p_n-\lambda_i \neq 0$ for all non-negative integers $p_1, p_2, \ldots, p_n$, satisfying $p_1+p_2+\ldots+p_n \geqq 2$.

In this paper, we assume only the first condition, and we shall show that, in this case also, there exist solutions of the same forms as in the previous case.

From the condition I, there exists a line L passing through the origin O, in one side of which all λ_i's lie. Draw a perpendicular OH to L in the side where λ_i's lie. Let the angle between the half-line OH and the real axis be ω, and $\lambda_i=\rho_i e^{\omega_i\sqrt{-1}}$. Then $\cos(\omega_i-\omega)>0$, namely $\Re(\lambda_i e^{-\omega\sqrt{-1}})>0$.

Multiplying both sides of the equation (1.1) by $e^{-\omega\sqrt{-1}}$, we have the equation of the same form as (1.1), where all the eigen values of the matrix A have positive real parts. Thus, in (1.1), without loss of generality, we can assume that all the eigen values λ_i's of the matrix A have

(1) M. Urabe, *On Solutions of the Linear Homogenious Partial Differential Equation in the Vicinity of the Singularity, I.* This Journal, Vol. 14, No. 2, p. 115. In the following, we denote this paper by (I).

positive real parts. In the following, we shall discuss the problem under this assumption.

By lack of the assumption II, there may exist λ_ι such that

$$\lambda_\iota = \lambda_1 p_1 + \lambda_2 p_2 + \dots + \lambda_n p_n \tag{1.2}$$

for non-negative integers $p_1, p_2, \dots, p_n$ satisfying $p_1+p_2+\dots+p_n \geqq 2$. The eigen values which can not be written in the form (1.2), are denoted by λ_a ($a=1, 2, \dots, S$) and others, namely those which can be written in the form (1.2) are denoted by λ_x ($x=S+1, S+2, \dots, R$)[1]. We arrange λ_x so that

$$\Re(\lambda_{S+1}) \leqq \Re(\lambda_{S+2}) \leqq \dots \leqq \Re(\lambda_R).$$

Then, if $\lambda_x = \lambda_1 p_1 + \dots + \lambda_{x-1} p_{x-1} + \lambda_x p_x + \lambda_{x+1} p_{x+1} + \dots + \lambda_R p_R$ for $p_1+\dots+p_R \geqq 2$, taking real parts of both sides, it is readily seen that $p_x = p_{x+1} = \dots = p_R = 0$. Therefore the relations (1.2) which can really hold are of the forms as follows:

$$\lambda_x = \lambda_1 p_1 + \lambda_2 p_2 + \dots + \lambda_{x-1} p_{x-1}. \tag{1.3}$$

As stated in (I), by means of the suitable linear transformation of the variables x_k, we can transform the matrix A into one of Jordan's form. Then the equation (1.1) can be written as follows:

$$Xf \equiv \sum_{\iota=1}^{R} \sum_{l=1}^{L_\iota} \sum_{p=1}^{P_l^\iota} (\lambda_\iota x_{lp}^\iota + \delta_p x_{lp-1}^\iota + \dots) \frac{\partial f}{\partial x_{lp}^\iota} = 0, \tag{1.4}$$

where the unwritten terms in the coefficients of $\dfrac{\partial f}{\partial x_{lp}^\iota}$ are of the second and higher orders, and δ_p is 1 or 0 according as $p \geqq 2$ or $p=1$.

In (1.4), we denote the eigen value λ_ι by λ_{lp}^ι, which is the coefficient of x_{lp}^ι in the coefficient of $\dfrac{\partial f}{\partial x_{lp}^\iota}$. Of course, the value of λ_{lp}^ι is λ_ι. Then, the relation (1.3) can be written as follows:

$$\lambda_x = \sum_{\iota=1}^{x-1} \sum_{l=1}^{L_\iota} \sum_{p=1}^{P_l^\iota} \overset{x}{p}{}_{lp}^\iota \lambda_{lp}^\iota, \tag{1.5}$$

where $\overset{x}{p}{}_{lp}^\iota$ are non-negative integers satisfying $\sum_{\iota=1}^{x-1} \sum_{l=1}^{L_\iota} \sum_{p=1}^{P_l^\iota} \overset{x}{p}{}_{lp}^\iota \geqq 2$. After

(1) $\lambda_1, \lambda_2, \dots, \lambda_S, \lambda_{S+1}, \dots, \lambda_R$ are distinct from one another.

Horn's idea[1], we define the weights w^{ι}_{ip} of the eigen values λ^{ι}_{ip} as follows:

$$(1.6)\qquad \begin{cases} w^{a}_{i1} = 0, \quad w^{a}_{ip} = p-1; \\ w^{x}_{i1} = \max.\left(\sum_{\iota=1}^{x-1}\sum_{m=1}^{L_\iota}\sum_{p=1}^{P^{\iota}_{m}} \overset{x}{p}{}^{\iota}_{mp}\, w^{\iota}_{mp}\right)+1, \quad w^{x}_{ip} = w^{x}_{i1}+p-1; \end{cases}$$

where the symbol "max." denotes the maximum of the values which are determined for all possible sets of $\overset{x}{p}{}^{\iota}_{mp}$ satisfying (1.5) for given λ_x. From the assumption that real parts of all eigen values are positive, the number of the sets of $\overset{x}{p}{}^{\iota}_{mp}$ satisfying (1.5) is finite, therefore the finite values of w^{x}_{i1} are determined.

For any set of non-negative integers q^{ι}_{ip}, we call the number $\sum_{\iota=1}^{R}\sum_{l=1}^{L_\iota}\sum_{p=1}^{P^{\iota}_{l}} q^{\iota}_{lp} w^{\iota}_{lp}$ the weight of the set of q^{ι}_{ip}, and the number $\sum_{\iota=1}^{R}\sum_{l=1}^{L_\iota}\sum_{p=1}^{P^{\iota}_{l}} q^{\iota}_{lp}\lambda^{\iota}_{lp}$ the order of the set of q^{ι}_{lp}.

§ 2. Solution of characteristic equations.

In (I), in order to solve the equation (1.1), we have considered the equations of the forms as follows:

$$(2.1)\qquad (X-\lambda_\iota)\, f^{\iota}_{lp} = f^{\iota}_{lp-1}\,[2],$$

where $f^{\iota}_{l0}\equiv 0$ and $l=1, 2, \ldots, L_\iota$; $p=1, 2, \ldots, P^{\iota}_{l}$. When λ_ι satisfy Poincaré's two conditions, we have seen that there exist regular solutions f^{ι}_{lp} which are expanded in the vicinity of $x_k=0$ as follows:

$$(2.2)\qquad f^{\iota}_{lp} = x^{\iota}_{lp}+\ldots\ldots,$$

where the unwritten terms are of the second and higher orders.

From (2.1), it is evident that

$$(X-\lambda_\iota)^{P^{\iota}_{l}} f^{\iota}_{lp} = 0, \qquad (p=1, 2, \ldots, P^{\iota}_{l}).$$

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(1) J. Horn, *Ueber die Reihen-entwickelung der Integrale eines Systems von Differentialgleichungen in der Umgebung gewisser singulären Stellen.* Jour. f. reine u. angew. Math., Bd., 116 (1895).

(2) In (I), instead of (2.1), we have considered the equations as follows:

$$(2.1')\qquad (X-\lambda_\iota)\, f^{\iota}_{lp} = (p-1)\, f^{\iota}_{lp-1}.$$

In this paper, for simplicity of calculation, at the outset, we consider the equations of the forms of (2.1), and afterwards we shall consider the equations of the forms of (2.1'). If we write the solutions of (2.1) and (2.1') as f^{ι}_{lp} and $'f^{\iota}_{lp}$ respectively, then they are related as follows: $'f^{\iota}_{lp}=(p-1)!\, f^{\iota}_{lp}$.

Let the characteristic polynomial of the matrix A be $D(\lambda)=\det.|\lambda I-A|$. Then $D(\lambda)=\prod_{i=1}^{R}(\lambda-\lambda_i)^{\sum_{l=1}^{L_i}P_l^i}$, therefore evidently $D(X)f_{lp}^i=0$. Thus we can suppose the equation (2.1) as the resolvent of the equation $D(X)f=0$. In this sense, we call the equations (2.1) the characteristic equations of the given equation (1.1).

When the second condition of Poincaré's conditions is not assumed, should we consider what forms of the characteristic equations? In the following we shall attack this problem.

When $i=a$, the relations of the forms (1.2) do not hold, therefore the equations (2.1) have solutions of the forms (2.2), and from the definition of w_{lp}^a in §1, it is evident that

$$(2.3)\qquad (X-\lambda_a)^{w_{lp}^a+1}f_{lp}^a=0\,.$$

Next, we consider the case where $i=x$. First, we consider the case where $i=S+1$. For simplicity of notation, in the following, we omit the index which indicates $S+1$. For an arbitrary set of p_{lp}^i satisfying (1.5), we consider a function as follows:

$$(2.4)\qquad \varphi=\prod_{a=1}^{S}\prod_{l=1}^{L_a}\prod_{p=1}^{P_l^a}f_{lp}^{a\;p_{lp}^a}\,.$$

We define the weight and the order of the function of such a form by the weight and the order of the set of exponents p_{lp}^a respectively. If we operate X on φ, then we have:

$$(2.5)\qquad \begin{cases} X\varphi=\sum_{a=1}^{S}\sum_{l=1}^{L_a}\sum_{p=1}^{P_l^a}p_{lp}^a\dfrac{\varphi}{f_{lp}^a}(\lambda_a f_{lp}^a+f_{lp-1}^a) \\ \quad =\lambda\varphi+\sum_{a=1}^{S}\sum_{l=1}^{L_a}\sum_{p=1}^{P_l^a}p_{lp}^a\dfrac{\varphi}{f_{lp}^a}f_{lp-1}^a\,. \end{cases}$$

The second term in the right-hand side is a linear combination with constant coefficients of the functions of the same forms as φ. Take an arbitrary function from them and let its exponents be $'p_{lp}^a$. Then, for a certain (a, l, p), $'p_{lp-1}^a=p_{lp-1}^a+1$, $'p_{lp}^a=p_{lp}^a-1$ and for other (a, l, p), $'p_{lp}^a=p_{lp}^a$. Therefore the orders of $'p_{lp}^a$ and p_{lp}^a are equal to each other. However, if we write the weights of p_{lp}^a and $'p_{lp}^a$ as w and w' respectively, then

$$w' = \sum_{a=1}^{S}\sum_{l=1}^{L_a}\sum_{p=1}^{P_l^a}{}'p_{lp}^a w_{lp}^a = \sum_{a=1}^{S}\sum_{l=1}^{L_a}\sum_{p=1}^{P_l^a} p_{lp}^a w_{lp}^a + w_{lp-1}^a - w_{lp}^a = w-1.$$

Hence $(X-\lambda)\varphi$ becomes a linear combination of the functions of the same forms as φ with the weight $w-1$ and with the same order. Therefore, after we have operated $X-\lambda$ on φ w-times, we have a linear combination of the functions of the same forms as φ with the weight zero. However, when the weight of φ is zero, the exponents p_{lp}^a are zero for $p\geqq 2$, therefore $\varphi = \prod_{a=1}^{S}\prod_{l=1}^{L_a} f_{l1}^{a\,p_{l1}^a}$. For such φ, from (2.5), it is evident that $(X-\lambda)\varphi=0$. Thus, for φ of (2.4), we have: $(X-\lambda)^{w+1}\varphi=0$. Now, from the definition, $w_{11}\geqq w+1$, therefore $(X-\lambda)^{w_{11}}\varphi=0$. Then we see that, for any linear combination Φ with constant coefficients of the functions of the same forms as φ,

$$(X-\lambda)^{w_{11}}\Phi = 0. \tag{2.6}$$

Now we consider the equation

$$(X-\lambda)f = \Phi, \tag{2.7}$$

i.e. $$\sum_{i=1}^{R}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}(\lambda_i x_{lp}^i + \delta_p x_{lp-1}^i)\frac{\partial f}{\partial x_{lp}^i} - \lambda f = \Phi + \sum_{i=1}^{R}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i} v_{lp}^i \frac{\partial f}{\partial x_{lp}^i},$$

where v_{lp}^i are sums of the terms of the second and higher orders. By $F(p_{lp}^i)$, we denote the value of the derivative of the function $F(x_{mq}^i)$ for $x_{mq}^i=0$, which is obtained by putting $x_{mq}^i=0$ after having differentiated F p_{lp}^i-times with respect to x_{lp}^i respectively. Then, after having differentiated both sides of (2.7) p_{lp}^i-times with respect to x_{lp}^i respectively, putting $x_{lp}^i=0$, we have the relations between the derivatives of f as follows:

$$\left(\sum_{i=1}^{R}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i} p_{lp}^i\lambda_{lp}^i - \lambda\right) f(p_{lp}^i) + \sum_{i=1}^{R}\sum_{l=1}^{L_i}\sum_{p=2}^{P_l^i}\delta_p p_{lp-1}^i f('p_{lp}^i) = \Phi(p_{lp}^i) + L, \tag{2.8}$$

where L is a linear homogeneous expression of the derivatives of f of the order $\sum_{i=1}^{R}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i} p_{lp}^i - 1$ at most and the relation between p_{lp}^i and $'p_{lp}^i$ is as follows: for a certain (i, l, p), $'p_{lp-1}^i = p_{lp-1}^i - 1$ and $'p_{lp}^i = p_{lp}^i + 1$, for other (i, l, p), $'p_{lp}^i = p_{lp}^i$.

When p_{lp}^i do not satisfy (1.5), $\sum_{i=1}^{R}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i} p_{lp}^i\lambda_{lp}^i - \lambda \neq 0$. When p_{lp}^i

satisfy (1.5), $\Phi(p^t_{ip})$ contains an arbitrary constant. For, the term of the lowest order in the expansion of φ is $\prod_{a=1}^{S}\prod_{i=1}^{L_a}\prod_{p=1}^{F^a_i} x^{a\ p^a_{ip}}_{ip}$ and Φ is a linear combination of these φ's with arbitrary constant coefficients. Thus, $f(p^t_{ip})$ and arbitrary constants in Φ are successively determined, except for $f(p^t_{ip})$ corresponding to p^t_{ip} satisfying (1.5). By calculation of the values of the derivatives of the first order, putting $f(p^t_{ip})$ corresponding to p^t_{ip} satisfying (1.5) zero, we obtain the following series satisfying (2.7) formally:

(2.9) $$f_{l1} = x_{l1} + \dots, \quad (l = 1, 2, \dots, L)$$

where the unwritten terms are of the second and higher orders. The convergency of the series of (2.9) is easily proved as follows:

There exists a positive number ε such that, for all p^a_{ip} not satisfying (1.5),

(2.10) $$\left| \frac{\sum_{a=1}^{S}\sum_{i=1}^{L_a}\sum_{p=1}^{P^a_i} p^a_{ip}\lambda^a_{ip} - \lambda}{\sum_{a=1}^{S}\sum_{i=1}^{L_a}\sum_{p=1}^{P^a_i} p^a_{ip} - 1} \right| > \varepsilon .$$

As in (I), we consider the equation

(2.11)
$$\varepsilon\Big(\sum_{t=1}^{R}\sum_{i=1}^{L_t}\sum_{p=1}^{P^t_i} x^t_{ip}\frac{\partial F}{\partial x^t_{ip}} - F\Big) = \sum_{t=1}^{R}\sum_{i=1}^{L_t}\sum_{p=2}^{P^t_i} \delta_p x^t_{ip-1}\frac{\partial F}{\partial x^t_{ip}} + V\sum_{t=1}^{R}\sum_{i=1}^{L_t}\sum_{p=1}^{P^t_i}\frac{\partial F}{\partial x^t_{ip}} + W,$$

where $v^t_{ip} \ll V$ and $\Phi \ll W$. Then, by (I), there exist regular solutions of (2.11) which are expanded as follows:

$$F^t_{i1} = x^t_{i1} + \dots\dots,$$

where the unwritten terms are of the second and higher orders. However, by (2.10), $f_{l1} \ll F^{S+1}_{l1}$. Therefore the series of (2.9) are convergent for sufficiently small values of $|x^t_{ip}|$. We denote the functions Φ corresponding to f_{l1} by Φ_{l1}. Then we have:

(2.12) $$(X-\lambda) f_{l1} = \Phi_{l1}.$$

Consequently, from (2.6), we have:

(2.13) $$(X-\lambda)^{w_{l1}+1} f_{l1} = 0.$$

Now, corresponding to the equations (2.1), we consider the equations as follows:

$$(X-\lambda) f_{ip} = f_{ip-1}+\Phi_{ip}. \quad (p = 2, 3, \dots, P_i) \tag{2.14}$$

Then, by the same way as on (2.7), we see that there exist regular solutions of (2.14) which are expanded as follows:

$$f_{ip} = x_{ip}+\dots\dots, \tag{2.15}$$

where the unwritten terms are of the second and higher orders. Then, from (2.14), we see that

$$(X-\lambda)^{w_{ip}+1} f_{ip} = 0. \tag{2.16}$$

Summarizing the above results, we see that the following theorem is valid for $x=S+1$.

Theorem 1. *The equations*

$$(X-\lambda_i) f^i_{ip} = f^i_{ip-1}+\Phi^i_{ip} \quad (f^i_{i0} \equiv 0, \quad \Phi^a_{ip} \equiv 0), \tag{C}$$

have regular solutions f^i_{ip} which are expanded in the vicinity of $x_k=0$ as follows:

$f^i_{ip}=x^i_{ip}$ + sum of the terms of the second and higher orders,

where

$$\Phi^x_{ip} \equiv L\left\{ \prod_{i=1}^{x-1} \prod_{m=1}^{L_i} \prod_{q=1}^{P^i_m} f^{i\ \overset{x_i}{p_{mq}}}_{mq} \right\}.$$

Here $L\{\dots\}$ denotes the linear combination of the arguments with suitable constant coefficients and the arguments are constructed for all $\overset{x}{p}{}^i_{mq}$ satisfying (1.5). For such f^i_{ip} and Φ^i_{ip}, it is valid that

$$(X-\lambda_i)^{w^i_{ip}+1} f^i_{ip} = 0 \quad \textit{and} \quad (X-\lambda_i)^{w^i_{i1}} \Phi^i_{ip} = 0. \tag{C'}$$

For $i=S+1$, we have just now proved the theorem. For $i=a$, from (2.1) and (2.3), the theorem is valid. For any x, by induction, we shall prove the theorem. We assume that the theorem is valid for $i=1, 2, \dots, x-1$.

Take any one of the terms in Φ^x_{ip} and let it be

$$\varphi = \prod_{i=1}^{x-1} \prod_{m=1}^{L_i} \prod_{q=1}^{P^i_m} f^{i\ \overset{x_i}{p_{mq}}}_{mq}. \tag{2.17}$$

By our assumption, the theorem is valid for factors f^i_{mq} of φ. If we operate X on φ, then we have:

$$X\varphi=\sum_{i=1}^{x-1}\sum_{m=1}^{L_i}\sum_{q=1}^{P_m^i}\overset{x}{p}{}^i_{mq}\frac{\varphi}{f^i_{mq}}(\lambda_i f^i_{mq}+f^i_{mq-1}+\Phi^i_{mq})$$

(2.18)

$$=\lambda_x\varphi+\sum_{i=1}^{x-1}\sum_{m=1}^{L_i}\sum_{q=1}^{P_m^i}\overset{x}{p}{}^i_{mq}\frac{\varphi}{f^i_{mq}}f^i_{mq-1}+\sum_{y=S+1}^{x-1}\sum_{m=1}^{L_y}\sum_{q=1}^{P_m^y}\overset{x}{p}{}^y_{mq}\frac{\varphi}{f^y_{mq}}\Phi^y_{mq}\,.$$

The second and third terms in the right-hand side are linear combinations of the functions of the same forms as φ. Let the weight of $\overset{x}{p}{}^i_{mq}$ be w. Then, as in the case where $i=S+1$, it can be easily seen that the second term is a linear combination of the functions with the weight $w-1$ and the order λ_x. Let the exponents of φ/f^y_{mq} and any term of Φ^y_{mq} be $'p^i_{lp}$ and $''p^i_{lp}$ respectively. Then the order of the third term in the right-hand side of (2.18) is calculated as follows:

$$\sum_{i=1}^{x-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}{}'p^i_{lp}\lambda^i_{lp}+\sum_{i=1}^{y-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}{}''p^i_{lp}\lambda^i_{lp}=\Big(\sum_{i=1}^{x-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}\overset{x}{p}{}^i_{lp}\lambda^i_{lp}-\lambda_y\Big)+\lambda_y=\lambda_x\,.$$

The weight of the third term is as follows:

$$\sum_{i=1}^{x-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}{}'p^i_{lp}w^i_{lp}+\sum_{i=1}^{y-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}{}''p^i_{lp}w^i_{lp}$$

$$\leqq\Big(\sum_{i=1}^{x-1}\sum_{l=1}^{L_i}\sum_{p=1}^{P_l^i}\overset{x}{p}{}^i_{lp}w^i_{lp}-w^y_{mq}\Big)+\Big(w^y_{m1}-1\Big)$$

$$\leqq\ w-1\,.$$

Thus $(X-\lambda_x)\varphi$ becomes a linear combination of the functions of the same forms as φ with the order λ_x and the weight $w-1$ at most. Therefore, after having operated $X-\lambda_x$ on φ w-times, we have a linear combination of the functions of the same forms as φ with the weight zero. However, when the weight of φ is zero, $\overset{x}{p}{}^i_{mq}$ is zero except for $\overset{x}{p}{}^a_{m1}$. Then, for such φ, from (2.18), $(X-\lambda_x)\varphi=0$, because, in this case, there does not come out the second and the third terms in the right-hand side. Thus, for φ of (2.17), we have: $(X-\lambda_x)^{w+1}\varphi=0$. Now, from the definition, $w+1 \leqq w^x_{l1}$. Therefore $(X-\lambda_x)^{w^x_{l1}}\varphi=0$. Thus, for any linear combination Φ^x_{mp} of the functions of the same forms as φ with constant coefficients, it is valid that $(X-\lambda_x)^{w^x_{l1}}\Phi^x_{mp}=0$. Then, by repeating the same process as in the case where $i=S+1$, we can see that the theorem is valid also for the case where $i=x$. Thus, the theorem is completely proved for any i.

From the theorem, it is seen that, when the second condition of Poincaré's conditions is not assumed, the equations corresponding to (2.1) are the equations (C). In this sense, *we call the equations (C) the characteristic equations of the given equation* (1.1).

§ 3. Solution of the given linear homogeneous equation.

We put $f^i_{iP^i_i}=g^i_{iP^i_i}$ and define g^i_{ip} successively as follows:

$$(3.1)\qquad (X-\lambda_i)\,g^i_{ip} = g^i_{ip-1}\,. \qquad (p=P^i_i,\ P^i_i-1, \dots, 2, 1)$$

Then, from (C), we have:

$$(3.2)\quad \begin{cases} g^i_{iP^i_i} = f^i_{iP^i_i}\,, \\ g^i_{iP^i_i-1} = f^i_{iP^i_i-1}+\Phi^i_{iP^i_i}\,, \\ g^i_{iP^i_i-2} = f^i_{iP^i_i-2}+\Phi^i_{iP^i_i-1}+(X-\lambda_i)\,\Phi^i_{iP^i_i} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g^i_{i1} = f^i_{i1}+\Phi^i_{i2}+(X-\lambda_i)\,\Phi^i_{i3}+\cdots\cdots+(X-\lambda_i)^{P^i_i-2}\,\Phi^i_{iP^i_i}\,, \\ g^i_{i0} = \Phi^i_{i1}+(X-\lambda_i)\,\Phi^i_{i2}+\cdots\cdots\cdots\cdots+(X-\lambda_i)^{P^i_i-1}\,\Phi^i_{iP^i_i}\,. \end{cases}$$

When $g^i_{i0}\not\equiv 0$, we put $g^i_{i0}\equiv h^i_{i0}$ and define h^i_{ir} successively as follows:

$$(3.3)\qquad (X-\lambda_i)\,h^i_{ir} = h^i_{ir+1}\,. \qquad (r=0, 1, 2, \dots)$$

Then

$$h^i_{ir} = (X-\lambda_i)^r\,h^i_{i0} = (X-\lambda_i)^r\,g^i_{i0}\,.$$

However, from (C′), $h^i_{ir}\equiv 0$ for $r\geqq w^i_{i1}$. Let the minimum of r be H^i_i+1, for which $h^i_{ir}\equiv 0$. Then,

$$(3.4)\qquad (X-\lambda_i)\,h^i_{iH^i_i} = 0\,, \qquad h^i_{iH^i_i}\not\equiv 0\,.$$

Then, of course,

$$(3.5)\qquad H^i_i \leqq w^i_{i1}-1\,.$$

When $i=a$, always $g^i_{i0}\equiv 0$. However, when $i=x$, it may happen also that $g^i_{i0}\equiv 0$. In these cases, we put as follows:

$$(3.6)\quad \begin{cases} \text{(i)}\ (p-1)!\ g^a_{ip} = (p-1)!\ f^a_{ip} = F^a_{ip-1}\,, \qquad (p=1, 2, \dots, P^a_i) \\ \text{(ii)}\ (p-1)!\ g^x_{ip} = (p-1)!\ [f^x_{ip}+\Phi^x_{ip+1}+(X-\lambda_x)\Phi^x_{ip+2}+\cdots\cdots \\ \qquad +(X-\lambda_x)^{P^x_i-p-1}\,\Phi^x_{iP^x_i}] = F^x_{ip-1}\,. \qquad (p=1, 2, \dots, P^x_i) \end{cases}$$

When $g^x_{i0} \neq 0$, we put as follows:

(3.6) (iii) $\quad (H^x_i - r)!\, h^x_{ir} = F^x_{lH^x_i - r}, \quad (H^x_i + p)!\, g^x_{ip} = F^x_{lH^x_i + p}.$

Then, from (3.1) and (3.3), we have:

$$(X-\lambda_i) F^i_{ip} = p F^i_{ip-1}. \tag{3.7}$$

Now, when $i=a$, $\Phi^a_{ip}=0$. Therefore F^a_{ip} is equal to f^a_{ip+1} except for a constant factor. When $i=S+1$, Φ^{S+1}_{ip} is a polynomial of f^a_{mq}, therefore Φ^{S+1}_{ip} is a polynomial of F^a_{mq}. From the discussions on (2.5) $(X-\lambda_{S+1})\Phi^{S+1}_{ip}$ is also a function of the same forms as Φ^{S+1}_{ip}, consequently it is also a polynomial of F^a_{mq}. Then, from (3.3), h^{S+1}_{ir} is also a polynomial of F^a_{mq}. Therefore, for $p=0, 1, \dots, H^{S+1}_i$, F^{S+1}_{ip} is a polynomial of F^a_{mq}. Then, from (3.2) and (3.6) (ii), (iii), we see also that f^{S+1}_{ip} is a sum of a constant multiple of F^{S+1}_{ip-1} or $F^{S+1}_{lH^{S+1}_i + p}$ and a polynomial of F^a_{mq}. Thus we see that the following lemma is valid for $i=S+1$.

Lemma. *f^x_{ip} is a sum of a constant multiple of F^x_{ip-1} or $F^x_{lH^x_i+p}$ and a polynomial of $F^1_{mq}, F^2_{mq}, \dots, F^{x-1}_{mq}$, and for $p=0, 1, \dots, H^x_i$, F^x_{ip} is also a polynomial of $F^1_{mq}, \dots, F^{x-1}_{mq}$.*

By induction, we shall prove this lemma. As stated above, the lemma is valid for $x=S+1$. We assume that the lemma is valid for $S+1, \dots, x-1$. Now, by the definition, Φ^x_{ip} is a polynomial of $f^1_{mq}, \dots, f^{x-1}_{mq}$, therefore, by by our assumption, Φ^x_{ip} is a polynomial of $F^1_{mq}, \dots, F^{x-1}_{mq}$. From the discussions on (2.18), $(X-\lambda_x)\Phi^x_{ip}$ is also a function of the same form as Φ^x_{ip}, consequently it is a polynomial of $F^1_{mq}, \dots, F^{x-1}_{mq}$. Then, from (3.2) and (3.6) (ii), (iii), we see that the lemma is valid also for x. Thus, we see that the lemma is valid for any x.

As in (I), we define U^i_{i0} as follows:

$$U^i_{i0} = F^i_{i1} / F^i_{i0}. \tag{3.8}$$

From this, $F^i_{i1} = U^i_{i0} F^i_{i0}$. According to the formula of Leibnitz upon differentiation, we define U^i_{ip} as follows:

$$F^i_{ip+1} = U^i_{ip} F^i_{i0} + \binom{p}{1} U^i_{ip-1} F^i_{i1} + \dots + \binom{p}{q} U^i_{ip-q} F^i_{iq} + \dots + U^i_{i0} F^i_{ip}. \tag{3.9}$$

Then, by (I), from (3.7), it is seen that the following functions are solutions of the equation $Xf=0$:

$$
(3.10)\qquad \begin{cases} F^{i}_{i0}{}^{\frac{1}{\lambda_i}}/F^{1}_{10}{}^{\frac{1}{\lambda_1}} \quad \text{(except for the case where } i=l=1\text{)},\\ U^{i}_{i0}-\dfrac{1}{\lambda_i}\log F^{i}_{i0},\ U^{i}_{ip}\,. \end{cases}
$$

Now the functions g^{i}_{ip} are independent of one another. For, in the vicinity of $x_k=0$, g^{i}_{ip} are expanded as follows:

$g^{i}_{ip}=x^{i}_{ip}+$sum of terms of the second and higher orders,

because Φ^{i}_{ip} are polynomials of f^{i}_{mq} of the second and higher orders, consequently the expansions of Φ^{i}_{ip} are sums of the terms of the second and higher orders. Therefore the value of the Jacobian of g^{i}_{ip} with respect to x_k is not zero for $x_k=0$. For the case (3.6) (ii), and (3.6) (iii), we denote the functions F^{z}_{ip} by F^{y}_{ip} and F^{z}_{ip} respectively. Then, from (3.6), the value of the Jacobian of F^{a}_{ip}, F^{y}_{ip} and $F^{z}_{lH^z_i+p}(p=1,2,\dots,P^z_i)$ is not zero for $x_k=0$, consequently these functions are independent of one another. Then we can prove that, among the functions of (3.10), the following functions are independent of one another:

$$
(3.11)\qquad \begin{cases} F^{a}_{i0}{}^{\frac{1}{\lambda_a}}/F^{1}_{10}{}^{\frac{1}{\lambda_1}}, \quad \text{(except for the case where } a=l=1\text{)},\\ U^{a}_{i0}-\dfrac{1}{\lambda_a}\log F^{a}_{i0},\ U^{a}_{ip}, \quad (p=1,2,\dots,P^{a}_{i}-2);\\ F^{y}_{i0}{}^{\frac{1}{\lambda_y}}/F^{1}_{10}{}^{\frac{1}{\lambda_1}},\ U^{y}_{i0}-\dfrac{1}{\lambda_y}\log F^{y}_{i0},\ U^{y}_{ip}, \quad (p=1,2,\dots,P^{y}_{i}-2);\\ \text{and for the case where } H^{z}_{i}\neq 0,\ U^{z}_{lH^z_i+p}, \quad (p=0,1,\dots,P^{z}_{i}-1);\\ \qquad \text{for the case where } H^{z}_{i}=0,\ U^{z}_{i0}-\dfrac{1}{\lambda_z}\log F^{z}_{i0},\ U^{z}_{ip};\\ \qquad\qquad (p=1,2,\dots,P^{z}_{i}-1). \end{cases}
$$

For, from (3.8) and (3.9), U^{a}_{ip} is a rational function of $F^{a}_{i0},\dots,F^{a}_{ip+1}$ and actually depends upon F^{a}_{ip+1}. Likewise, U^{y}_{ip} is a rational function of $F^{y}_{i0},\dots,F^{y}_{ip+1}$ and actually depends upon F^{y}_{ip+1}. For the case where $H^{z}_{i}\neq 0$, from (3.8) and (3.9), $U^{z}_{lH^z_i+p}$ is a rational function of $F^{z}_{i0},\dots,F^{z}_{lH^z_i+p+1}$ and actually depends upon $F^{z}_{lH^z_i+p+1}$. However, by the lemma, $F^{z}_{i0},\dots,F^{z}_{lH^z_i}$ are polynomials of F^{a}_{mq}, F^{y}_{mq} and F^{w}_{mq} where y and w are not greater than $z-1$. Therefore, $U^{z}_{lH^z_i+p}$ is a rational function of these F^{a}_{mq}, F^{y}_{mq} and F^{w}_{mq} and $F^{z}_{lH^z_i+1},\dots,F^{z}_{lH^z_i+p+1}$ and actually depends upon $F^{z}_{lH^z_i+p+1}$. For the case where $H^{z}_{i}=0$, it is readily seen that U^{z}_{ip} has an analogous form as $U^{z}_{lH^z_i+p}$ for the case where $H^{z}_{i}\neq 0$ and actually depends upon F^{z}_{ip+1}. Therefore the Jacobian of the functions of (3.11) with respect to F^{α}_{ip} (except for F^{1}_{10}), F^{y}_{ip} and $F^{s}_{lH^s_i+p}$ is not identically zero, namely the functions of (3.11) are independent of one another as the functions of F^{α}_{ip}, F^{y}_{ip} and $F^{s}_{lH^s_i+p}$. However, these functions are independent of one another. Thus, the functions of (3.11) are also independent of one another as the functions of x_k.

The total number of the functions of (3.11) is $\left(\sum_{\alpha=1}^{S}\sum_{i=1}^{L_\alpha}P^\alpha_i-1\right)+\sum_{x=S+1}^{R}\sum_{i=1}^{L_x}P^x_i=n-1$ where n denotes the number of the variables x_k. Thus we have the following theorem.

Theorem 2. *In the equation* $Xf\equiv\sum_{k=1}^{n}X_k\dfrac{\partial f}{\partial x_k}=0$, *we assume that* $X_k(x)$ *can be expanded as follows:*

$X_k(x)=\sum_{j=1}^{n}a_{kj}x_j+$*sum of the terms of the second and higher orders. If all the eigen values* λ_i*'s of the matrix* $\|a_{kj}\|$ *lie in a convex domain which does not contain the origin, then, in the vicinity of the origin, the equation* $Xf=0$ *has* $n-1$ *independent solutions given by* (3.11).

§ 4. Extension to the case of Picard's condition.

Up to the present, we have considered the case where all the eigen values λ_i of the matrix A lie in a convex domain which does not contain the origin. In this paragraph, as Picard has done, we consider the case where some of the eigen values λ_i lie in a convex domain which does not contain the origin. In this case, if we draw a suitable line L, then some of λ_i, for example $\lambda_a\,(a=1,\dots,m)$ lie in one side of the line L, and the others which we denote by λ_x, lie either on L or in another side. Then, as stated in § 1, without loss of generality, we can assume that $\Re(\lambda_a)>0$ and $\Re(\lambda_x)\leqq 0$.

As in § 1, we transform the matrix A into one of Jordan's form and assume that

$X_i=\lambda_i x_i+\delta_i x_{i-1}+$sum of the terms of the second and higher orders, where $\delta_i=1$ or 0. We write X_i corresponding to λ_a and λ_x as X_a and X_σ respectively.

From our assumption that $\Re(\lambda_a)>0$ and $\Re(\lambda_x)\leqq 0$, it does not hold that $\lambda_x=\sum_{a=1}^{m}\lambda_a p_a$ for non-negative integers p_a. Then, by our previous paper[1], there exist regular solutions $x_\sigma = x_\sigma(x_\alpha)$ of the equations as follows:

$$\sum_\alpha X_\alpha \frac{\partial x_\sigma}{\partial x^\alpha} = X_\sigma, \tag{4.1}$$

where $x_\sigma(x_\alpha)$ are regular in the vicinity of $x_\alpha=0$ and their expansions are sums of the terms of the second and higher orders. Put $X_\alpha\{x_\beta, x_\sigma(x_\alpha)\} = \bar{X}_\alpha(x_\beta)$, then, in the terms of the first order, $X_\alpha(x_\iota)$ and $\bar{X}_\alpha(x_\beta)$ coincide with each other, for $x_\sigma(x_\alpha)$ are sums of the terms of the second and higher orders. Then, by our assumption that $\Re(\lambda_\alpha)>0$, from the theorem 2, there exist $m'-1$ independent solutions $g_\beta(x_\alpha)$ of the forms of (3.11) satisfying the following equation:

$$\sum_\alpha \bar{X}_\alpha \frac{\partial g}{\partial x_\alpha} = 0. \tag{4.2}$$

Here m' denotes the number of the variables x_α. Put $x_\sigma - x_\sigma(x_\alpha) = g_\sigma(x_\alpha, x_\sigma)$. Then, by (II), $g_\sigma(x_\alpha, x_\sigma)$ and $g_\beta(x_\alpha)$ are solutions of the equation (1.1) under the condition that $g_\sigma(x_\alpha, x_\sigma)=0$. Moreover, it is evident that $g_\sigma(x_\alpha, x_\sigma)$ and $g_\beta(x_\alpha)$ are independent of one another. Thus, *under the condition that $g_\sigma(x_\alpha, x_\sigma)=0$, we have $n-1$ independent solutions $g_\sigma(x_\alpha, x_\sigma)$ and $g_\beta(x_\alpha)$ of the equation* (1.1).

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(1) M. Urabe, *On Solutions of the Linear Homogeneous Partial Differential Equation in the Vicinity of the Singularity, II.* This Journal, Vol. 14, No. 3, p. 195. In the following we denote this paper by (II).