On Solutions of the Linear Homogeneous Partial Differential Equations in the Vicinity of the Sigularnity. II.

By

Minoru URABE.

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§ 1. Introduction.

In the previous paper (1), we have considered the following equation

$$\sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i}=0, \tag{1.1}$$

where $X_i(x)$ are regular in the vicinity of $x_i=0$ and vanish there. Let the expansions of $X_i(x)$ in the vicinity of $x_i=0$ be

$$X_i(x)=\sum_{j=1}^{n} a_{ij}x_j+\ \ldots\ldots\ , \tag{1.2}$$

and let the eigen values of the matrix $A=\| a_{ij} \|$ be λ_i. In the previous paper we have assumed that λ_i satisfy Poincaré's condition. In this paper as Picard has done (2), we weaken Poincaré's condition and assume that λ_i satisfy the condition as follows:

Among λ_i, there exist λ_α ($\alpha=1, 2, \ldots, m$) such that

(I) *on a complex plane there exists a convex domain Ω which contains all λ_α's but not the origin,*

(II) $\lambda_1 p_1+\lambda_2 p_2+\ \ldots\ldots\ +\lambda_m p_m-\lambda_i\neq 0$ $(i=1, 2, \ldots\ldots, n)$ *for all non-negative integers $p_1, p_2, \ldots\ldots, p_m$ satisfying $p_1+p_2+\ \ldots\ldots\ +p_m\geqq 2$.*

We call this condition Picard's condition. When $m=n$, this condition coincides with Poincaré's condition. In this paper, we consider the case where $m<n$.

As in the previous paper, by means of the suitable linear transformation of the variables x_i, we transform the matrix A into that of Jordan's form. Then X_i are of the forms as follows:

$$X_i=\lambda_i x_i+\delta_i x_{i+1}+\ \ldots\ldots\ ,$$

where δ_i are unity or zero. At first, we search for the integrals $x_\nu=x_\nu(x_\alpha)$ $(\nu=m+1, \ldots\ldots, n)$ satisfying the equation

(1) M. Urabe, Jour. Sci. Hiroshima Univ. Vol. 14, No. 2, p. 115.

(2) Picard, Traité d'Analyse, t. III, p. 17.

(1.3) $$\sum_{\alpha} X_{\alpha} \frac{\partial x_{\nu}}{\partial x_{\alpha}} = X_{\nu}.$$

Next, putting $x_\nu = x_\nu(x_\alpha)$, we shall solve the equation (1.1).

§ 2. The case where all the eigen values are distinct.

In this case, the matrix A has the form $\begin{pmatrix} \lambda_1 & 0 \dots 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \lambda_{n-1} & 0 \\ 0 & 0 \dots 0 & \lambda_n \end{pmatrix}$, therefore X_i are of the forms as follows:

(2.1) $$X_i = \lambda_i x_i + v_i,$$

where v_i denote the sums of the terms of the second and higher orders.

We consider the equation as follows:

(2.2) $$\sum_{\alpha} (\lambda_\alpha x_\alpha + v_\alpha) \frac{\partial x_\nu}{\partial x_\alpha} = \lambda_\nu x_\nu + v_\nu$$

where $\nu = m+1, m+2, \dots\dots, n$. We are going to search for integrals $x_\nu = x_\nu(x_\alpha)$ of (2.2), which vanish for $x_\alpha = 0$. After having differentiated both sides of (2.2) p_α-times with respect to x_α respectively, put $x_\alpha = 0$, then we have the following relations among the values of the derivatives of x_ν for $x_\alpha = 0$:

$$(\lambda_1 p_1 + \lambda_2 p_2 + \dots\dots + \lambda_m p_m - \lambda_\nu) \frac{\partial^p x_\nu}{\partial x_1^{p_1} \partial x_2^{p_2} \dots\dots \partial x_m^{p_m}}$$

(2.3) $=$polynomial of the derivatives of $x_{m+1}, \dots\dots, x_n$ of the orders $p-1$ at most,

where $p = p_1 + p_2 + \dots\dots + p_m$. For the derivatives of the first order, we have

$$(\lambda_1 - \lambda_\nu) \frac{\partial x_\nu}{\partial x_1} = 0, \ (\lambda_2 - \lambda_\nu) \frac{\partial x_\nu}{\partial x_2} = 0, \dots\dots, (\lambda_m - \lambda_\nu) \frac{\partial x_\nu}{\partial x_m} = 0.$$

Because $\lambda_\alpha \neq \lambda_\nu$, $\frac{\partial x_\nu}{\partial x_\alpha} = 0$. For the derivatives of the second and higher orders, by means of (2.3) we can determine successively their values, for $\lambda_1 p_1 + \lambda_2 p_2 + \dots\dots + \lambda_m p_m - \lambda_\nu \neq 0$ because of the condition (II). Thus we obtain the power series which express x_ν formally.

Next we shall prove the convergence of the series obtained just now. For sufficiently small positive number ρ, we may assume that all v_i's are regular for $|x_i| \leqq \rho$. Let the greatest value of all $|v_i|$ for $|x_i| \leqq \rho$ be M. We take the following function

(2.4) $$V=\frac{M}{1-\dfrac{x_1+x_2+\ldots\ldots+x_n}{\rho}}-M-M\frac{x_1+x_2+\ldots\ldots+x_n}{\rho}.$$

From the conditions (I) and (II), as in the previous paper, there exists a positive number ε such that

(2.5) $$\left|\frac{\lambda_1 p_1+\lambda_2 p_2+\ldots\ldots+\lambda_m p_m-\lambda_i}{p_1+p_2+\ldots\ldots+p_m-1}\right|>\varepsilon.$$

We consider the equation as follows:

(2.6) $$\varepsilon\left(\sum_\alpha x_\alpha\frac{\partial x_\nu}{\partial x_\alpha}-x_\nu\right)=V\left(\sum_\alpha\frac{\partial x_\nu}{\partial x_\alpha}+1\right).$$

This equation is symmetric with respect to $x_1, x_2, \ldots\ldots, x_m$ and $x_{m+1}, \ldots\ldots, x_n$ respectively. Therefore, putting $x_1+x_2+\ldots\ldots+x_m=x$ and $x_{m+1}=\ldots\ldots=x_n=y$, from (2.6) we have:

(2.7) $$\varepsilon\left(x\frac{dy}{dx}-y\right)=V\left(m\frac{dy}{dx}+1\right).$$

Put $V/\varepsilon=W$, then W can be written as follows:

(2.8) $$W=\frac{M}{\varepsilon}\cdot\frac{\left[\dfrac{x+(n-m)y}{\rho}\right]^2}{1-\dfrac{x+(n-m)y}{\rho}}.$$

Then, the equation (2.7) can be written as follows:

(2.9) $$(x-mW)\frac{dy}{dx}=y+W.$$

Put $y=xv$ and $W=x^2U$, then U is regular in the vicinity of $x=v=0$. From (2.9), we have:

$$(x-mx^2U)\left(v+x\frac{dv}{dx}\right)=xv+x^2U,$$

i.e.

(2.10) $$\frac{dv}{dx}=\frac{U(1+mv)}{1-mxU}.$$

Now the equation (2.10) has a regular integral v which vanishes for $x=0$, namely there exist regular integrals of (2.6) such that

(2.11) $$x_{m+1}=\ldots\ldots=x_n=(x_1+x_2+\ldots\ldots+x_m)^2\,\mathfrak{p}(x_1+x_2+\ldots\ldots+x_m),$$

where $\mathfrak{p}(t)$ denotes a function which is regular in the vicinity of $t=0$. Now the values of the derivatives of x_ν of (2.11) with respect to x_α for $x_\alpha=0$

are successively determined by performing the same process upon (2.6) as upon (2.2).

From $v_i \ll V$ and (2.5), we know that the power series which formally express x_ν satisfying (2.2), are convergent for sufficiently small absolute values of x_α.

Thus, there exist regular integrals x_ν of (2.2), which vanish for $x_\alpha=0$, and are sums of the terms of the second and higher orders of x_α.

§ 3. The case where all the eigen values are equal. (1)

When all the eigen values of A are equal, after the suitable linear transformation of variables x_i, the equation (1.1) can be written as follows:

$$(3.1)\quad (\lambda x_{11}+x_{12}+v_{11})\frac{\partial f}{\partial x_{11}}+(\lambda x_{12}+x_{13}+v_{12})\frac{\partial f}{\partial x_{12}}+\ \cdots\cdots\ +(\lambda x_{1k}+v_{1k})\frac{\partial f}{\partial x_{1k}}$$
$$+(\lambda x_{21}+x_{22}+v_{21})\frac{\partial f}{\partial x_{21}}+\ \cdots\cdots\ +(\lambda x_{2l}+v_{2l})\frac{\partial f}{\partial x_{2l}}$$
$$+\ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(\lambda x_{r1}+x_{r2}+v_{r1})\frac{\partial f}{\partial x_{r1}}+\ \cdots\cdots\ +(\lambda x_{rs}+v_{rs})\frac{\partial f}{\partial x_{rs}}$$
$$+\ \cdots\cdots\cdots\cdots\cdots\cdots=0,$$

where v_{ij} denote sums of the terms of the second and higher orders. From the condition (I), $\lambda \neq 0$.

We consider the equations as follows:

$$(3.2)\quad (\lambda x_{11}+x_{12}+v_{11})\frac{\partial x_{tu}}{\partial x_{11}}+(\lambda x_{12}+x_{13}+v_{12})\frac{\partial x_{tu}}{\partial x_{12}}+\ \cdots\cdots\ +(\lambda x_{1k}+v_{1k})\frac{\partial x_{tu}}{\partial x_{1k}}$$
$$+(\lambda x_{21}+x_{22}+v_{21})\frac{\partial x_{tu}}{\partial x_{21}}+\ \cdots\cdots\ +(\lambda x_{2l}+v_{2l})\frac{\partial x_{tu}}{\partial x_{2l}}$$
$$+\ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$=\lambda x_{tu}+\delta_{tu}x_{tu+1}+v_{tu},$$

where $t=r, r+1, \ldots\ldots$, and δ_{tu} is equal to 0 or 1 according as x_{tu} is the last one or not of the variables which have t as a first suffix. Dividing both sides of (3.2) by λ, we have:

$$(3.3)\qquad x_{11}\frac{\partial x_{tu}}{\partial x_{11}}+x_{12}\frac{\partial x_{tu}}{\partial x_{12}}+\ \cdots\cdots\ +x_{1k}\frac{\partial x_{tu}}{\partial x_{1k}}$$

(1) In this case, Poincaré's condition is satisfied, consequently the discussions of this paragraph are of no use for solving the equation (1.1). The discussions of this paragraph are meant to make the lemma of § 4.

$$
\begin{aligned}
&+x_{21}\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots\cdots\cdots\cdots\cdots +x_{2l}\frac{\partial x_{tu}}{\partial x_{2l}} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&-x_{tu} \\
=&\delta'_{tu}x_{tu+1}+ax_{12}\frac{\partial x_{tu}}{\partial x_{11}}+ \cdots\cdots +ax_{1k}\frac{\partial x_{tu}}{\partial x_{1k-1}} \\
&\quad +ax_{22}\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots +ax_{2l}\frac{\partial x_{tu}}{\partial x_{2l-1}} \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad -v_{11}\frac{\partial x_{tu}}{\partial x_{11}}- \cdots\cdots\cdots\cdots\cdots\cdots -v_{1k}\frac{\partial x_{tu}}{\partial x_{1k}} \\
&\quad -v_{21}\frac{\partial x_{tu}}{\partial x_{21}}- \cdots\cdots\cdots\cdots\cdots\cdots -v_{2l}\frac{\partial x_{tu}}{\partial x_{2l}} \\
&\quad - \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad +v_{tu},
\end{aligned}
$$

where $a=-1/\lambda\neq 0$ and $\delta'_{tu}=\frac{1}{\lambda}\delta_{tu}$.

Put $|a|=A$. We take V which is a sum of the terms of the second and higher orders, so that $V\gg v_{ij}$. We take also a positive number ε_0 which is less than 1/3 and choose σ_{ij} so that $\sigma_{ij}=1-\varepsilon_{ij}$ where all ε_{ij} are distinct and $2\varepsilon_0>\varepsilon_{ij}>\varepsilon_0$.

Corresponding to the equations (3. 3), we consider the equations as follows:

$$
\begin{aligned}
(3.4) \qquad &\sigma_{11}x_{11}\frac{\partial x_{tu}}{\partial x_{11}}+ \cdots\cdots\cdots\cdots\cdots\cdots +\sigma_{1k}x_{1k}\frac{\partial x_{tu}}{\partial x_{1k}} \\
&+\sigma_{21}x_{21}\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots\cdots\cdots\cdots\cdots +\sigma_{2l}x_{2l}\frac{\partial x_{tu}}{\partial x_{2l}} \\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&-\sigma_{tu}x_{tu} \\
=&A\delta_{tu}x_{tu+1}+Ax_{12}\frac{\partial x_{tu}}{\partial x_{11}}+ \cdots\cdots +Ax_{1k}\frac{\partial x_{tu}}{\partial x_{1k-1}} \\
&\quad +Ax_{22}\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots +Ax_{2l}\frac{\partial x_{tu}}{\partial x_{2l-1}} \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad +V\Big(\frac{\partial x_{tu}}{\partial x_{11}}+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots +\frac{\partial x_{tu}}{\partial x_{1k}} \\
&\qquad +\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots +\frac{\partial x_{tu}}{\partial x_{2l}} \\
&\qquad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad +1\Big).
\end{aligned}
$$

At first, we shall prove that the equations (3.4) have regular integrals. The equations (3.4) are of the forms as follows:

(3.5)
$$
\begin{aligned}
&(\sigma_{11}x_{11}-Ax_{12}-V)\frac{\partial x_{tu}}{\partial x_{11}}+ \cdots\cdots +(\sigma_{1k}x_{1k}-V)\frac{\partial x_{tu}}{\partial x_{1k}}\\
&+(\sigma_{21}x_{21}-Ax_{22}-V)\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots +(\sigma_{2l}x_{2l}-V)\frac{\partial x_{tu}}{\partial x_{2l}}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\\
&=\sigma_{tu}x_{tu}+A\delta_{tu}x_{tu+1}+V.
\end{aligned}
$$

Corresponding to (3.5), we consider the equation as follows:

(3.6)
$$
\begin{aligned}
&(\sigma_{11}x_{11}-Ax_{12}-V)\frac{\partial f}{\partial x_{11}}+ \cdots\cdots +(\sigma_{1k}x_{1k}-V)\frac{\partial f}{\partial x_{1k}}\\
&+(\sigma_{21}x_{21}-Ax_{22}-V)\frac{\partial f}{\partial x_{21}}+ \cdots\cdots +(\sigma_{2l}x_{2l}-V)\frac{\partial f}{\partial x_{2l}}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\\
&+(\sigma_{t1}x_{t1}+Ax_{t2}+V)\frac{\partial f}{\partial x_{t1}}+ \cdots\cdots +(\sigma_{tv}x_{tv}+V)\frac{\partial f}{\partial x_{tv}}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\\
&=0.
\end{aligned}
$$

Now, from the assumption, σ_{ij} are all distinct, therefore, in (3.6), we can reduce the matrix into that of diagonal form, the elements of which are the coefficients of the terms of the first order in the coefficients of $\frac{\partial f}{\partial x_{ij}}$. From the form of the equation (3.6), the form of the linear transformation on demand is as follows:

(3.7)
$$
\begin{cases}
y_i=\sum_j s_{ij}x_j, & (i, j=11, 12, \ldots\ldots, 1k),\\
y_i=\sum_j s_{ij}x_j, & (i, j=21, 22, \ldots\ldots, 2l),\\
\vdots &
\end{cases}
$$

And, after the transformation, the equation (3.6) is of the form as follows:

(3.8)
$$
\begin{aligned}
&(\sigma_{11}y_{11}+W_{11})\frac{\partial f}{\partial y_{11}}+ \cdots\cdots +(\sigma_{1k}y_{1k}+W_{1k})\frac{\partial f}{\partial y_{1k}}\\
&+(\sigma_{21}y_{21}+W_{21})\frac{\partial f}{\partial y_{21}}+ \cdots\cdots +(\sigma_{2l}y_{2l}+W_{2l})\frac{\partial f}{\partial y_{2l}}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\\
&+(\sigma_{t1}y_{t1}+W_{t1})\frac{\partial f}{\partial y_{t1}}+ \cdots\cdots +(\sigma_{tv}y_{tv}+W_{tv})\frac{\partial f}{\partial y_{tv}}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\\
&=0,
\end{aligned}
$$

where W_{ij} denote the sums of the second and higher orders of y. Writing briefly the suffices as follows:

$$\alpha, \beta, \ldots\ldots = 11, 12, \ldots\ldots, 1k;\ 21, 22, \ldots\ldots, 2l;\ \ldots\ldots;$$
$$\lambda, \mu, \ldots\ldots = r1, r2, \ldots\ldots, rs;\ \ t1, t2, \ldots\ldots, tv;\ \ldots\ldots,$$

we can write (3. 6) and (3. 8) as follows:

$$(3.\,6') \qquad \sum_\alpha X_\alpha \frac{\partial f}{\partial x_\alpha} + \sum_\lambda X_\lambda \frac{\partial f}{\partial x_\lambda} = 0,$$

$$(3.\,8') \qquad \sum_\alpha Y_\alpha \frac{\partial f}{\partial y_\alpha} + \sum_\lambda Y_\lambda \frac{\partial f}{\partial y_\lambda} = 0,$$

and we can write (3. 7) as follows:

$$(3.\,7') \qquad y_\alpha = \sum_\beta s_{\alpha\beta} x_\beta, \qquad y_\lambda = \sum_\mu s_{\lambda\mu} x_\mu.$$

When we substitute (3. 7′) into (3. 6′) and compare the resulting equation with (3. 8′), we have

$$(3.\,9) \qquad Y_\alpha = \sum_\beta s_{\alpha\beta} X_\beta, \qquad Y_\lambda = \sum_\mu s_{\lambda\mu} X_\mu.$$

Now, for any non-negative integers $p_{11}, \ldots\ldots, p_{1k}, p_{21}, \ldots\ldots, p_{2l}, \ldots\ldots$ such that $\sum p_{ij} \geqq 2$, we have

$$\begin{aligned} &\sigma_{11} p_{11} + \ldots\ldots + \sigma_{1k} p_{1k} + \sigma_{21} p_{21} + \ldots\ldots + \sigma_{2l} p_{2l} + \ldots\ldots - \sigma_{pq} \\ &= \sum (1-\varepsilon_{ij}) p_{ij} - (1-\varepsilon_{pq}) \\ &> (1-2\varepsilon_0)\,2 - (1-\varepsilon_0) = 1 - 3\varepsilon_0 > 0. \end{aligned}$$

Moreover $1-\varepsilon_0 > \sigma_{ij} > 1-2\varepsilon_0 > 0$. Therefore σ_{ij} satisfy the conditions (I) and (II). Consequently, by the result of § 1, corresponding to the equation (3. 8′), there exist regular integrals $y_\lambda = y_\lambda(y_\alpha)$ satisfying the equations as follows:

$$(3.\,10) \qquad \sum_\alpha Y_\alpha \frac{\partial y_\lambda}{\partial y_\alpha} = Y_\lambda.$$

Here $y_\lambda(y_\alpha)$ are sums of the terms of the second and higher orders of y_α.

Now, from $y_\lambda(y_\alpha) = y_\lambda(\sum_\beta s_{\alpha\beta} x_\beta) = y_\lambda(x_\alpha)$ (we put), we have $\frac{\partial y_\lambda}{\partial x_\alpha} = \sum_\beta \frac{\partial y_\lambda}{\partial y_\beta} s_{\beta\alpha}$, then $\sum_\alpha X_\alpha \frac{\partial y_\lambda}{\partial x_\alpha} = \sum_\alpha Y_\alpha \frac{\partial y_\lambda}{\partial y_\alpha}$, therefore

$$\sum_\alpha X_\alpha \frac{\partial y_\lambda}{\partial x_\alpha} = \sum_\mu s_{\lambda\mu} X_\mu.$$

From this, we can easily deduce the relations as follows:

$$\sum_\alpha X_\alpha \frac{\partial x_\lambda}{\partial x_\alpha} = X_\lambda. \tag{3.11}$$

Here $x_\lambda = \sum_\mu S_{\lambda\mu} y_\mu(x_\alpha) = x_\lambda(x_\alpha)$, where $\| S_{\lambda\mu} \|$ denotes the inverse matrix of $\| s_{\lambda\mu} \|$. Therefore $x_\lambda = x_\lambda(x_\alpha)$ are sums of the terms of the second and higher orders of x_α. The equation (3.11) is none other than (3.5), i.e. (3.4). Thus we have obtained the regular integrals $x_\lambda = x_\lambda(x_\alpha)$ of (3.4).

The values of the derivatives of $x_\lambda(x_\alpha)$ satisfying (3.4) for $x_\alpha = 0$ are determined successively by means of the formulae obtained by putting $x_\alpha = 0$ after having differentiated p_α-times both sides of (3.4) with respect to x_α as follows:

$$\begin{aligned}
&(\sigma_{11}p_{11} + \dots + \sigma_{1k}p_{1k} + \sigma_{21}p_{21} + \dots + \sigma_{2l}p_{2l} + \dots - \sigma_{tu}) \frac{\partial^p x_{tu}}{\partial x_{11}^{p_{11}} \dots \partial x_{1k}^{p_{1k}} \partial x_{21}^{p_{21}} \dots} \\
&= A\delta_{tu} \frac{\partial^p x_{tu+1}}{\partial x_{11}^{p_{11}} \dots \partial x_{1k}^{p_{1k}} \partial x_{21}^{p_{21}} \dots} + A_{p_{12}} \frac{\partial^p x_{tu}}{\partial x_{11}^{p_{11}+1} \partial x_{12}^{p_{12}-1} \partial x_{13}^{p_{13}} \dots \partial x_{1k}^{p_{1k}} \dots} + \dots\dots \\
&\quad + A_{p_{1k}} \frac{\partial^p x_{tu}}{\partial x_{11}^{p_{11}} \dots \partial x_{1k-1}^{p_{1k-1}+1} \partial x_{1k}^{p_{1k}-1} \partial x_{21}^{p_{21}} \dots} + A_{p_{22}} \frac{\partial^p x_{tu}}{\partial x_{11}^{p_{11}} \dots \partial x_{21}^{p_{21}+1} \partial x_{22}^{p_{22}-1}} + \dots\dots \\
&\quad + A_{p_{2l}} \frac{\partial^p x_{tu}}{\partial x_{11}^{p_{11}} \dots \partial x_{21}^{p_{21}} \dots \partial x_{2l-1}^{p_{2l-1}+1} \partial x_{2l}^{p_{2l}-1} \dots} \\
&\quad + \dots\dots\dots\dots\dots\dots\dots\dots \\
&\quad + (\text{polynomial of the derivatives of the orders } p-1 \text{ at most}),
\end{aligned} \tag{3.12}$$

where $p = p_{11} + \dots\dots + p_{1k} + p_{21} + \dots\dots + p_{2l} + \dots\dots$. For (3.3), we can make an analogous formulae as (3.12), but in this case, the coefficient of $\dfrac{\partial^p x_{tu}}{\partial x_{11}^{p_{11}} \dots \partial x_{21}^{p_{21}} \dots}$ is $p-1$. Now

$$\begin{aligned}
\sum \sigma_{ij} p_{ij} - \sigma_{tu} &= \sum (1-\varepsilon_{ij}) p_{ij} - (1-\varepsilon_{tu}) \\
&= (p-1) - (\sum \varepsilon_{ij} p_{ij} - \varepsilon_{tu}),
\end{aligned}$$

therefore $\quad (p-1) - (\sum \sigma_{ij} p_{ij} - \sigma_{tu}) = \sum \varepsilon_{ij} p_{ij} - \varepsilon_{tu} > 2\varepsilon_0 - 2\varepsilon_0 = 0,$

i.e. $\quad p-1 > (\sum \sigma_{ij} p_{ij} - \sigma_{tu}) > 0.$

Then we see easily that the absolute values of the derivatives of the functions x_{tu} which satisfy (3.3) formally and consist of the terms of the second and higher orders of x_α (1), are not greater than the absolute values of the derivatives of the functions x_{tu} satisfying (3.4), namely we see that

(1) Among the derivatives of the first order of general solution x_{tu} satisfying formally (3.3), $\dfrac{\partial x_{tu}}{\partial x_{1k}}$, $\dfrac{\partial x_{tu}}{\partial x_{2l}}$, ... are indeterminate. However, if we put these indeterminate derivatives zero, then we see that all the derivatives of the first order vanish. Here we adopt such solutions x_{tu}.

the equations (3. 3), i. e. (3. 2) have regular integrals $x_{tu}=x_{tu}(x_a)$ which are sums of the terms of the second and higher orders of x_a.

§ 4. General case.

In the general case, after the suitable linear transformation of variables x_i, the equation (1. 1) can be written as follows:

$$(4.1)\quad (\lambda_1 x_{11}+x_{12}+v_{11})\frac{\partial f}{\partial x_{11}}+(\lambda_1 x_{12}+x_{13}+v_{12})\frac{\partial f}{\partial x_{12}}+ \cdots\cdots +(\lambda_1 x_{1k}+v_{1k})\frac{\partial f}{\partial x_{1k}}$$
$$+(\lambda_1 x_{21}+x_{22}+v_{21})\frac{\partial f}{\partial x_{21}}+ \cdots\cdots +(\lambda_1 x_{2l}+v_{2l})\frac{\partial f}{\partial x_{2l}}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(\lambda_2 x_{r1}+x_{r2}+v_{r1})\frac{\partial f}{\partial x_{r1}}+ \cdots\cdots +(\lambda_2 x_{rs}+v_{rs})\frac{\partial f}{\partial x_{rs}}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$=0,$$

where v_{ij} are sums of the terms of the second and higher orders.

We consider the equations as follows:

$$(4.2)\quad (\lambda_1 x_{11}+x_{12}+v_{11})\frac{\partial x_{tu}}{\partial x_{11}}+(\lambda_1 x_{12}+x_{13}+v_{12})\frac{\partial x_{tu}}{\partial x_{12}}+ \cdots\cdots +(\lambda_1 x_{1k}+v_{1k})\frac{\partial x_{tu}}{\partial x_{1k}}$$
$$+(\lambda_1 x_{21}+x_{22}+v_{21})\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots +(\lambda_1 x_{2l}+v_{2l})\frac{\partial x_{tu}}{\partial x_{2l}}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+(\lambda_2 x_{r1}+x_{r2}+v_{r1})\frac{\partial x_{tu}}{\partial x_{r1}}+ \cdots\cdots +(\lambda_2 x_{rs}+v_{rs})\frac{\partial x_{tu}}{\partial x_{rs}}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$=\lambda_p x_{tu}+\delta_{tu}x_{tu+1}+v_{tu},$$

where $p=m+1,\ m+2, \ldots\ldots$ and δ_{tu} is equal to 0 or 1 according as x_{tu} is the last one or not of the variables which have t as a first suffix. We rewrite (4. 2) as follows:

$$(4.3)\quad \lambda_1 x_{11}\frac{\partial x_{tu}}{\partial x_{11}}+ \cdots\cdots +\lambda_1 x_{1k}\frac{\partial x_{tu}}{\partial x_{1k}}$$
$$+\lambda_1 x_{21}\frac{\partial x_{tu}}{\partial x_{21}}+ \cdots\cdots +\lambda_1 x_{2l}\frac{\partial x_{tu}}{\partial x_{2l}}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\lambda_2 x_{r1}\frac{\partial x_{tu}}{\partial x_{r1}}+ \cdots\cdots +\lambda_2 x_{rs}\frac{\partial x_{tu}}{\partial x_{rs}}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\lambda_p x_{tu}$$

$$
\begin{aligned}
=\delta_{tu}x_{tu+1}-x_{12}\frac{\partial x_{tu}}{\partial x_{11}}-\ \ldots\ldots\ -x_{1k}\frac{\partial x_{tu}}{\partial x_{1k-1}} \\
-x_{22}\frac{\partial x_{tu}}{\partial x_{21}}-\ \ldots\ldots\ -x_{2l}\frac{\partial x_{tu}}{\partial x_{2l-1}} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
-v_{11}\frac{\partial x_{tu}}{\partial x_{11}}-\ \ldots\ldots\ -v_{1k}\frac{\partial x_{tu}}{\partial x_{1k}} \\
-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+v_{tu}.
\end{aligned}
$$

As stated in § 2, when λ_i satisfy the conditions (I) and (II), there exists a positive number ε so that

$$
(4.4)\qquad \left|\frac{\lambda_1 p_{11}+\ \ldots\ldots\ +\lambda_1 p_{1k}+\lambda_1 p_{21}+\ \ldots\ldots\ +\lambda_1 p_{2l}+\ \ldots\ldots\ -\lambda_p}{p_{11}+\ \ldots\ldots\ +p_{1k}+p_{21}+\ \ldots\ldots\ +p_{2l}+\ \ldots\ldots\ -1}\right|>\varepsilon
$$

for all non-negative integers $p_{11},\ldots\ldots,p_{1k},p_{21},\ldots\ldots,p_{2l},\ldots\ldots$ such that $p_{11}+\ \ldots\ldots\ +p_{1k}+p_{21}+\ \ldots\ldots\ +p_{2l}+\ \ldots\ldots\ \geqq 2$. Corresponding to (4. 3), we consider the equations as follows:

$$
(4.5)\qquad
\begin{aligned}
\varepsilon\Big(x_{11}\frac{\partial x_{tu}}{\partial x_{11}}+\ \ldots\ldots\ +x_{1k}\frac{\partial x_{tu}}{\partial x_{1k}} \\
+x_{21}\frac{\partial x_{tu}}{\partial x_{21}}+\ \ldots\ldots\ +x_{2l}\frac{\partial x_{tu}}{\partial x_{2l}} \\
+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+x_{r1}\frac{\partial x_{tu}}{\partial x_{r1}}+\ \ldots\ldots\ +x_{rs}\frac{\partial x_{tu}}{\partial x_{rs}} \\
+\ \ldots\ldots\ldots\ldots\ldots\ -x_{tu}\Big) \\
=\delta_{tu}x_{tu+1}+x_{12}\frac{\partial x_{tu}}{\partial x_{11}}+\ \ldots\ldots\ +x_{1k}\frac{\partial x_{tu}}{\partial x_{1k-1}} \\
+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+V\Big(\frac{\partial x_{tu}}{\partial x_{11}}+\ \ldots\ldots\ +\frac{\partial x_{tu}}{\partial x_{1k}}+\ \ldots\ldots\ +1\Big),
\end{aligned}
$$

where V is a function such that $v_{ij}\ll V$.

After having differentiated both sides of (4. 3) $p_{11},\ldots\ldots,p_{1k},p_{21},\ldots\ldots,$ $p_{2l},\ldots\ldots$ times with respect to $x_{11},\ldots\ldots,x_{1k},x_{21},\ldots\ldots,x_{2l},\ldots\ldots$ respectively, put these independent variables zero. We have

$$
\begin{aligned}
(4.6)\qquad (\lambda_1 p_{11}+\ \ldots\ldots\ +\lambda_1 p_{1k}+\lambda_1 p_{21}+\ \ldots\ldots\ +\lambda_1 p_{2l}+\ \ldots\ldots\ +\lambda_2 p_{r1}+\ \ldots\ldots \\
+\lambda_2 p_{rs}+\ \ldots\ldots\ -\lambda_p)\frac{\partial^q x_{tu}}{\partial x_{11}^{p_{11}}\ldots\partial x_{1k}^{p_{1k}}\partial x_{21}^{p_{21}}\ldots}
\end{aligned}
$$

$$=\delta_{tu}\frac{\partial^q x_{tu+1}}{\partial x_{11}^{p_{11}}\cdots\partial x_{1k}^{p_{1k}}\partial x_{21}^{p_{21}}\cdots}-p_{12}\frac{\partial^q x_{tu}}{\partial x_{11}^{p_{11}+1}\partial x_{12}^{p_{12}-1}\cdots\partial x_{1k}^{p_{1k}}\cdots}-\cdots\cdots$$

$$-p_{1k}\frac{\partial^q x_{tu}}{\partial x_{11}^{p_{11}}\cdots\partial x_{1k-1}^{p_{1k-1}+1}\partial x_{1k}^{p_{1k}-1}\cdots}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

+(polynomial of the derivatives of the orders $q-1$ at most),

where $q=p_{11}+\cdots\cdots+p_{1k}+p_{21}+\cdots\cdots+p_{2l}+\cdots\cdots+p_{r1}+\cdots\cdots+p_{rs}+\cdots\cdots$.

When $q=1$, we have:

$$\left\{\begin{aligned}&(\lambda_1-\lambda_p)\frac{\partial x_{tu}}{\partial x_{11}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{11}}\\&(\lambda_1-\lambda_p)\frac{\partial x_{tu}}{\partial x_{12}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{12}}-\frac{\partial x_{tu}}{\partial x_{11}}\\&\quad\vdots\\&(\lambda_1-\lambda_p)\frac{\partial x_{tu}}{\partial x_{1k}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{1k}}-\frac{\partial x_{tu}}{\partial x_{1k-1}}\end{aligned}\right.,\quad\left\{\begin{aligned}&(\lambda_1-\lambda_p)\frac{\partial x_{tu}}{\partial x_{21}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{21}}\\&(\lambda_1-\lambda_p)\frac{\partial x_{tu}}{\partial x_{22}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{22}}-\frac{\partial x_{tu}}{\partial x_{21}}\\&\quad\vdots\\&(\lambda_1-\lambda_p)\frac{\partial x_{tu}}{\partial x_{2l}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{2l}}-\frac{\partial x_{tu}}{\partial x_{2l-1}}\end{aligned}\right.,\cdots\cdots,$$

$$\left\{\begin{aligned}&(\lambda_2-\lambda_p)\frac{\partial x_{tu}}{\partial x_{r1}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{r1}}\\&(\lambda_2-\lambda_p)\frac{\partial x_{tu}}{\partial x_{r2}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{r2}}-\frac{\partial x_{tu}}{\partial x_{r1}}\\&\quad\vdots\\&(\lambda_2-\lambda_p)\frac{\partial x_{tu}}{\partial x_{rs}}=\delta_{tu}\frac{\partial x_{tu+1}}{\partial x_{rs}}-\frac{\partial x_{tu}}{\partial x_{rs-1}}\end{aligned}\right.,\cdots\cdots.$$

Here $\lambda_1-\lambda_p,\ \lambda_2-\lambda_p,\ \cdots\cdots\neq0$. Then we see easily that all the derivatives of the first order are zero. The values of the derivatives of the second and higher orders are determined successively by means of (4. 6).

For the equations (4. 5), putting all the derivatives of the first order zero, we have analogous results. And because of (4. 4), the absolute values of the derivatives of the functions satisfying (4. 3) are not greater than those of the functions satisfying (4. 5). However, the equation (4. 5) are of the similar form as (3. 2). Therefore, making use of the results in § 3, or doing analogous reasonings directly on (4. 5) as on (3. 2), we see that the equations (4. 5) have regular integrals consisting of the terms of the second and higher orders. Thus, ultimately, we see that the equations (4. 2) have regular integrals $x_{tu}=x_{tu}(x_{11},\cdots\cdots,x_{1k},x_{21},\cdots\cdots,x_{2l},\cdots\cdots)$ which are sums of the terms of the second and higher orders.

§ 5. Solutions of the equation (1. 1).

After the suitable linear transformation of variables x_i, the equation

(1.1) can be written as (4.1). We write briefly the equation (4.1) as follows:

$$\sum_{\alpha} X_{\alpha}\frac{\partial f}{\partial x_{\alpha}}+\sum_{\lambda} X_{\lambda}\frac{\partial f}{\partial x_{\lambda}}=0, \tag{5.1}$$

where X_{α} denote the coefficients corresponding to the eigen values λ_{α} which lie in the convex domain Ω, and X_{λ} denote the other coefficients. By the results of § 4, there exist regular integrals $x_{\lambda}=x_{\lambda}(x_{\alpha})$ which are sums of the terms of the second and higher orders of x_{α} and satisfy the equations as follows:

$$\sum_{\alpha} X_{\alpha}\frac{\partial x_{\lambda}}{\partial x_{\alpha}}=X_{\lambda}. \tag{5.2}$$

Let $f(x_i)$ be any integral of the equation (5.1). Substituting $x_{\lambda}=x_{\lambda}(x_{\alpha})$ into $f(x_i)=f(x_{\alpha}, x_{\lambda})$, we write the reduced function $f(x_{\alpha}, x_{\lambda}(x_{\alpha}))$ as $g(x_{\alpha})$. Then

$$\begin{aligned}0&=\sum_{i} X_{i}\frac{\partial f}{\partial x_{i}}\\&=\sum_{\alpha} X_{\alpha}\frac{\partial f}{\partial x_{\alpha}}+\sum_{\lambda} X_{\lambda}\frac{\partial f}{\partial x_{\lambda}}\\&=\sum_{\alpha} X_{\alpha}\frac{\partial f}{\partial x_{\alpha}}+\sum_{\alpha,\lambda} X_{\alpha}\frac{\partial x_{\lambda}}{\partial x_{\alpha}}\frac{\partial f}{\partial x_{\lambda}}\\&=\sum_{\alpha} X_{\alpha}\frac{\partial g}{\partial x_{\alpha}},\end{aligned}$$

i.e.

$$\sum_{\alpha} X_{\alpha}\frac{\partial g}{\partial x_{\alpha}}=0. \tag{5.3}$$

Here $X_{\alpha}=X_{\alpha}(x_{\beta})=X_{\alpha}(x_{\beta}, x_{\lambda}(x_{\beta}))$, and $x_{\lambda}(x_{\alpha})$ are sums of the second and higher orders of x_{α}. Therefore $X_{\alpha}(x_i)$ and $X_{\alpha}(x_{\beta})$ do not differ from each other in the terms of the first order. Now, because of the conditions (I) and (II), the eigen values λ_{α} satisfy the Poincaré's two conditions with respect to the variables x_{α}. Then, by the results of the previous paper [1], we can obtain independent integrals of (5.3) as follows:

$$\begin{cases} U-\dfrac{1}{\lambda_1}\log\varphi,\ U_1,\ U_2,\ \ldots\ldots,\ U_{k-2}\,; \\ \theta/\varphi,\ U-\dfrac{1}{\lambda_1}\log\theta,\ V_1,\ V_2,\ \ldots\ldots,\ V_{\iota-2}\,; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{5.4}$$

(1) M. Urabe, ibid.

$$\left| \begin{array}{l} \omega^{\frac{1}{\lambda_2}}/\varphi^{\frac{1}{\lambda_1}},\ W-\frac{1}{\lambda_2}\log\omega,\ W_1,\ \ldots\ldots,\ W_{s-2}\,; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right.$$

We denote these functions by g_σ.

Put $x_\lambda - x_\lambda(x_\alpha) \equiv g_\lambda(x_\alpha, x_\lambda)$. Then

$$\sum_i X_i \frac{\partial g_\lambda}{\partial x_i} = X_\lambda - \sum_\alpha X_\alpha \frac{\partial x_\lambda}{\partial x_\alpha} = 0,$$

and

$$\sum_i X_i \frac{\partial g_\sigma}{\partial x_i} = \sum_\alpha X_\alpha \frac{\partial g_\sigma}{\partial x_\alpha} = 0.$$

Evidently g_σ and g_λ are independent.

Thus we have:

Under the condition that $g_\lambda(x_\alpha, x_\lambda)=0$, any integral $f(x_i)$ of (5.1) are functions of $g_\sigma(x_\alpha)$, and under the same condition, $g_\sigma(x_\alpha)$ and $g_\lambda(x_\alpha, x_\lambda)$ constitute $n-1$ independent integrals of (5.1).

Our conclusion does not furnish integrals in ordinary sense of the equation (1.1). It is a problem remained unsolved to seek the integrals in ordinary sense under Picard's condition. In future, we want to attack this problem.

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Mathematical Institute, Hiroshima University.