ON THE LOGARITHMIC FUNCTIONS OF MATRICES. I.

By

Kakutarō MORINAGA and Takayuki NŌNO.

(Received Sept. 30, 1949)

Let $\mathfrak{A}$ be the set of all matrices of order n over the field of complex numbers, $\mathfrak{M}$ the set of all regular matrices of order n, $\mathring{\mathfrak{A}}$ the set of all matrices of order n whose all different characteristic values μ_i have the imaginary parts $I(\mu_i)$ such that $-\pi\leqq I(\mu_i)<\pi$;[1] $\widetilde{\mathfrak{A}}$ the set of all matrices of order n whose all characteristic values μ_i have the imaginary parts $I(\mu_i)$ such that $-\pi<I(\mu_i)<\pi$, and $\widetilde{\mathfrak{M}}$ the set of all regular matrices of order n whose characteristic values are not negative.

The exponential function of a matrix C is defined by the series

$$expC=E+\sum_{r=1}^{\infty}\frac{C^r}{r!}.$$

As for the exponential functions of matrices we have already known the following:

$(1)_1$ There exists a neighbourhood of zero matrix O in $\mathfrak{A}$ which is mapped topologically onto a neibourhood of unit matrix E in $\mathfrak{M}$ by the exponential mapping $M=expA$.[2]

$(1)_2$ The set of all hermitian matrices of order n is mapped topologically onto the set of all positive definite hermitian matrices of order n by the exponential mapping $M=expA$[2]

(2) There exists a matrix A in $\mathring{\mathfrak{A}}$ such as $expA=M$ for $M\epsilon\mathfrak{M}$.[3]

(3) The general solutions of the matric equation $expA=M$ for a given regular matrix M have been discussed by many writers, called the logarithmic function of M, and denoted by $logM$.[3] (We shall also use this denotation in the following).

In this paper we shall prove the following:

(a) *There exists one and only one matrix A in $\mathring{\mathfrak{A}}$ such as $expA=M$ for $M\epsilon\mathfrak{M}$, (we shall denote this matrix by $L(M)$ in the following); namely the mapping $M=expA$ from the set $\mathring{\mathfrak{A}}$ onto $\mathfrak{M}$ is one to one.*[4]

(b) *The set $\widetilde{\mathfrak{A}}$ is mapped topologically onto $\widetilde{\mathfrak{M}}$ by the mapping $M=expA$.*[5]

(c) *We obtain the general solutions of the matric equation $expA=M$ for $M\epsilon\mathfrak{M}$; Using*

1) Even if we use the interval $a\leqq I(\mu_i)<a+2\pi$ (a is arbitrary real number) in the place of $-\pi\leqq I(\mu_i)<\pi$, we can similarly prove the following theorems.

2) C. Chevalley: Theory of Lie groups. I. (1946). p. 7, p. 14.

3) J.H.M. Wedderburn: Lectures on matrices (1934). p. 122-123.
K. Yosida: A matrix A such as det $A\neq 0$ is expressed by $A=expB$. Shijō-Sūgaku-Danwakai No. 72; (309) (1935). (In Japanese)
M. Nagumo: On the equation $A=e^X$ in a normed ring. ibid. No. 72, (310). (1935). (In Japanese).
K. Asano: On the solutions of matric equation $e^X=A$. ibid. No. 74; (326). (1936), (In Japanese.)

4) K. Morinaga and T. Nōno: On the logarithmic functions of matrices. Shijō-Sūgaku-Danwakai, (in the press).

5) Considering the periodicity of the logarithmic function $LogM$, it is impossible to extend this topological mapping to the larger domains.

the branch $L(M)$ of $log\ M$, we can make clear the periodicity of the logarithmic functions.

(*d*) *Using this expression of $logM$, we can determine the branches of $logM$ which are expressed by the polynomial of M.*[1]

In the following we shall denote by $\mathfrak{R}(C)$ the set of all the matrices commutative with C, and by $B\sim C$ the fact that B is transformable to C.

§ 1. First we shall obtain a matrix A in $\mathring{\mathfrak{A}}$ such as $expA=M$ for $M\epsilon\mathfrak{M}$. We transform M into the canonical form[2] by P, and denote by M_i the block for a characteristic value λ_i, i.e.,

$$\left.\begin{aligned} &P^{-1}MP=\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}, \quad M_i=\begin{pmatrix} M_{i1} & & 0 \\ & \ddots & \\ 0 & & M_{ip_i} \end{pmatrix} = \lambda_i E_i + N_i \quad (i=1,\dots,p) \\ &M_{ia}=\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_i \end{pmatrix}, \quad (a=1,\dots,p_i), \quad \lambda_i \neq \lambda_k \quad \text{for} \quad i\neq k, \end{aligned}\right\} \tag{1}$$

where M_i and M_{ia} are the matrices of order n_i and n_{ia} respectively, E_i is the unit matrix of order n_i, N_i satisfies $N_i^{\,n_i}=O$, and since $M\epsilon\mathfrak{M}$, $\lambda_i\neq 0$.

Let us now consider the matrices

$$\mathring{L}(M_i)=Log\lambda_i.E_i+\sum_{r=1}^{n_i-1}(-1)^{r-1}\frac{N_i^r}{r\lambda_i^r}, \qquad (i=1,\dots,p.) \tag{2}$$

where $Log\lambda_i$ denotes the principal value of $log\lambda_i$, i. e., $-\pi\leqq I(Log\lambda_i)<\pi$. Since $N_i^{\,n_i}=0$ and $EN_i=N_iE$, these matrices are obtained by substituting E_i, N_i and λ_i for 1, x and λ respectively in the next identitiy which is valid for complex numbers

$$Log(\lambda+x)=Log\lambda+\sum_{r=1}^{\infty}(-1)^{r-1}\frac{x^r}{r\lambda^r}, \qquad (\lambda\neq 0\ |\frac{x}{\lambda}|<1)^{(3)}$$

Hence from an identity

$$exp\cdot Log(\lambda+x)=\lambda+x,$$

we have

$$exp\cdot\mathring{L}(M_i)=\lambda_iE_i+N_i=M_i \tag{3}$$

Next if we put

$$L(M)=P\begin{pmatrix} \mathring{L}(M_1) & & 0 \\ & \mathring{L}(M_2) & \\ & & \ddots \\ 0 & & \mathring{L}(M_p) \end{pmatrix}P^{-1}, \tag{4}$$

then we obtain

$$exp\cdot L(M)=M,\ L(M)\epsilon\ \mathring{\mathfrak{A}}. \tag{5}$$

Moreover, since $\mathring{L}(M_i)$ is the polynomial of N_i, i. e., M_i, $L(M)$ is a polynomial of M.

1) When we call the polynomial of a matrix C, it means the polynomial of a matrix C and the unit matrix with the coefficients of complex numbers which may depend on a matrix C.

2) The canonical form means the Jordan's canonical form, and so in the following.

3) Since $N_i^{\,n_i}=O$, here the condition of convergence, i.e., $|\frac{x}{\lambda}|<1$ is unneccesary.

Thus we have the result:[1]

RESULT 1. *$L(M)$ is a solution of $\exp A = M$ and a polynomial of M.*

§ **2.** We shall consider the general solutions of the matric equation

$$\exp A = M \qquad \text{for } M \in \mathfrak{M}. \tag{6}$$

From (6) we have

$$\exp(P^{-1}AP) = P^{-1}MP = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}, \tag{7}$$

hence from (7) we get

$$(P^{-1}AP)\,(P^{-1}MP) = (P^{-1}MP)\,(P^{-1}AP), \tag{8}$$

and by means of (7) and the above it follows that

$$P^{-1}AP = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}, \qquad \exp A_i = M_i \ (i = 1, \ldots, p), \tag{9}$$

where A_i is the matrix of the same order as M_i.

Here we transform A_i into the canonical form, i. e.,

$$Q_i^{-1}A_iQ_i = \begin{pmatrix} A_{i1} & & 0 \\ & \ddots & \\ 0 & & A_{iq_i} \end{pmatrix}, \qquad A_{i\beta} = \begin{pmatrix} \mu_{i\beta} & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu_{i\beta} \end{pmatrix}, \qquad (\beta = 1, \ldots, q_i) \tag{10}$$

where $A_{i\beta}$ $(\beta = 1, \ldots, q_i)$ are the matrices of order $m_{i\beta}$. By means of (9) and (10) we have

$$\begin{pmatrix} \exp A_{i1} & & 0 \\ & \ddots & \\ 0 & & \exp A_{iq_i} \end{pmatrix} = Q_i^{-1}M_iQ_i, \tag{11}$$

and

$$\exp A_{i\beta} = e^{\mu_{i\beta}} \exp \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \sim \begin{pmatrix} e^{\mu_{i\beta}} & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & e^{\mu_{i\beta}} \end{pmatrix}.[2] \tag{12}$$

By considering (1), (11) and (12) we can obtain the following relations

$$q_i = p_i, \qquad m_{i\alpha} = n_{i\alpha} \qquad (\alpha = 1, \ldots, p_i), \tag{13}$$

and

$$e^{\mu_{i\alpha}} = \lambda_i, \qquad \text{i. e.,}\ \mu_{i\alpha} = Log\lambda_i + 2\pi\sqrt{-1}\, f_{i\alpha}, \tag{14}$$

---

1) Since each of $\begin{pmatrix} E_1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}, \ldots\ldots, \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & E_p \end{pmatrix}$ is expressed by polynomial of $\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}$ and $\begin{pmatrix} f(M_1) & & 0 \\ & \ddots & \\ 0 & & f_p(M_p) \end{pmatrix} = f_1\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}\begin{pmatrix} E_1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} + \ldots\ldots + f_p\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}\begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & E_p \end{pmatrix}$ for any polynomials $f_i\ (M_i)$ of M_i, $\begin{pmatrix} f_1(M_1) & & 0 \\ & \ddots & \\ 0 & & f_p(M_p) \end{pmatrix}$ is a polynomial of $\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_p \end{pmatrix}$.

2) If $\gamma_1 \ldots\ldots \gamma_{m-1} \neq 0$, then $\begin{pmatrix} \alpha & \gamma_1 & & \# \\ & \ddots & \ddots & \\ & & & \gamma_{m-1} \\ 0 & & & \alpha \end{pmatrix} \sim \begin{pmatrix} \alpha & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \alpha \end{pmatrix}$; for these two matrices have the same elementary divisers.

where f_{ia} are the arbitrary integers. Consequently, from (10), (13) and (14) we have

$$A_{ia}=\begin{pmatrix} Log\lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & Log\lambda_i \end{pmatrix}+2\pi\sqrt{-1}\, f_{ia}\, E_{ia},$$

where E_{ia} is the unit matrix of order n_{ia}. And remembering the definition of $\mathring{L}(M_{ia})$ (see (2)) we see

$$A_{ia}\sim\mathring{L}(M_{ia})+2\pi\sqrt{-1}\, f_{ia}\, E_{ia}.^{(1)} \tag{15}$$

Accordingly, from (10) and (15) we deduce that by some matrix S_i,

$$A_i=S_i^{-1}(\mathring{L}(M_i)+F_i)S_i, \tag{16}$$

where

$$F_i=2\pi\sqrt{-1}\begin{pmatrix} f_{i1}E_{i1} & & 0 \\ & \ddots & \\ 0 & & f_{ip_i}E_{ip_i} \end{pmatrix}. \tag{16$'$}$$

Furthermore, from (4), (9) and (16) we get

$$A=S^{-1}(L(M)+F)S, \tag{17}$$

where

$$S=P\begin{pmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_p \end{pmatrix}P^{-1}, \tag{17$'$}$$

and

$$F=P\begin{pmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_p \end{pmatrix}P^{-1} \tag{17$''$}$$

and since from (1) and (16)′ $M_iF_i=F_iM_i$, by means of (14) and (17)″ it follows

$$MF=FM \tag{18}$$

From Result 1 and (18) we have

$$FL(M)=L(M)F$$

hence

$$exp(L(M)+F)=exp\, L(M)\cdot exp\, F,$$

moreover by (16)′ we have

$$exp(L(M)+F)=exp\, L(M)=M., \tag{19}$$

From (6) and (17) we get

$$M=exp\, A=S^{-1}\{exp(L(M)+F)\}S,$$

substituting (19) into the above, we have

$$M=S^{-1}MS,\ \text{that is,}\ S\epsilon\mathfrak{K}(M). \tag{20}$$

Ths, since $S^{-1}L(M)S=L(M)$ by (20) and result 1, the equation (17) is reduced to

$$A=L(M)+S^{-1}FS,\quad S\epsilon\mathfrak{K}(M),^{(2)} \tag{21}$$

where

$$F=P\begin{pmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_p \end{pmatrix}P^{-1}\quad and\ F_i=2\sqrt{-1}\begin{pmatrix} f_{i1}E_{i1} & & 0 \\ & \ddots & \\ 0 & & f_{ip_i}E_{ip_i} \end{pmatrix} \tag{21$'$}$$

Here P is a fixed matrix which transforms M into its canonical form, and S is the

---

1) See the foot note 2) p. 109.

2) $S^{-1}FS$ is the general solution of $XM=MX$, $exp X=E$.

arbitrary matrix such that $SM=MS$

Conversely it is evident that these matrices satisfy the relation $exp\, A=M$. Thus we have the theorem:[1]

THEOREM I. *The general solutions of the matric equation $expA=M$ for $M\epsilon\mathfrak{M}$ are given by*

$$A=\log M=L(M)+S^{-1}FS$$

where $F=P\begin{pmatrix}F_1 & & 0\\ & \ddots & \\ 0 & & F_p\end{pmatrix}P^{-1}$, $F_i=2\pi\sqrt{-1}\begin{pmatrix}f_{i1}E_{i1} & & 0\\ & \ddots & \\ 0 & & f_{ip_i}E_{ip_i}\end{pmatrix}$, *and f_{ia} is an arbitrary integer; P is a fixed matrix which transforms M into its canonical form, and S is the arbitrary matrix such that $SM=MS$.*

By this theorem we obtain the following results:

If $A\equiv L(M)+S^{-1}FS$ belongs to $\mathring{\mathfrak{A}}$, then F must be zero matrix O. So we have the theorem:

THEOREM II. *There exists one and only one matrix A in $\mathring{\mathfrak{A}}$ such as $expA=M$ for $M\epsilon\mathfrak{M}$, and this matrix is $L(M)$.*[2]

By thorem II we can immediately deduce the theorem;

THEOREM III. *The set $\mathring{\mathfrak{A}}$ is mapped topologically onto $\widetilde{\mathfrak{M}}$ by the mapping $M=expA$.*

Next if $A\equiv L(M)+S^{-1}FS$ is the polynomial of M, then this matrix must be permutable with all matrices of $\mathfrak{K}(M)$. We shall denote these branches of $logM$ by $LogM$. And $S\epsilon\mathfrak{K}(M)$, hence we have

$$S(L(M)+S^{-1}FS)S^{-1}=(L(M)+S^{-1}FS) \tag{22}$$

and using $SL(M)S^{-1}=L(M)$ we get

$$S^{-1}FS=F \tag{23}$$

that is,

$$Log\, M=L(M)+F \tag{24}$$

Furthermore, since

$$FK=KF \quad \text{for all } K\epsilon\mathfrak{K}(M), \tag{25}$$

we have

$$F_i=2\pi\sqrt{-1}\; f_iE_i, \quad f_i \text{ is an arbitrary integer;} \tag{26}$$

this means that $f_{ia}=f_i$. $(a=1,\dots,p_i)$ in (21)'. Therefore it must be

$$Log\, M=L(M)+2\pi\sqrt{-1}\,P\begin{pmatrix}f_1E_1 & & 0\\ & \ddots & \\ 0 & & f_pE_p\end{pmatrix}P^{-1}, \tag{27}$$

and it is evident that these matrices in the right hand member are the polynomials of M.

---

1) Let $\bar{P}$ be the other matrix which transforms M into its same canonical form, then $\bar{P}=RP$, $R\epsilon\mathfrak{K}(M)$, hence we have $\bar{L}(M)\equiv\bar{P}\begin{pmatrix}\mathring{L}(M_1) & & 0\\ & \ddots & \\ 0 & & \mathring{L}(M_p)\end{pmatrix}\bar{P}^{-1}=RL(M)R^{-1}=L(M)$, $\bar{F}\equiv\bar{P}\begin{pmatrix}F_1 & & 0\\ & \ddots & \\ 0 & & F_p\end{pmatrix}\bar{P}^{-1}$ $=RFR^{-1}$,
that is,
$\bar{L}(M)+\bar{S}^{-1}\bar{F}\bar{S}=L(M)+\bar{S}^{-1}RFR^{-1}\bar{S}=L(M)+S^{-1}FS$,
where $\bar{S}\epsilon\mathfrak{K}(M)$ and $\bar{S}=RS$.

2) K. Morinaga and T. Nōno. ibid. theorems 1, 2 and 3.

Thus we have the theorem:

THEOREM IV. *The branches Log M of log M which are expressed by the polynomials of M are given by*

$$Log\, M \equiv L(M) + 2\pi\sqrt{-1}\; P\begin{pmatrix} f_1 E_1 & & 0 \\ & \ddots & \\ 0 & & f_p E_p \end{pmatrix} P^{-1}$$

From this theorem we get the following corollaries:

COROLLARY 1. *If and only if $MN=NM$ for $M, N \in \mathfrak{M}$, then*

$$Log\, M \cdot Log\, N = Log\, N \cdot Log\, M.$$

COROLLARY 2. *When and only when $M(\in \mathfrak{M})$ has the form*

$$M=\begin{pmatrix} U & W \\ 0 & V \end{pmatrix}$$

where U and V are the matrices of order l and m respectively, then Log M has also the form

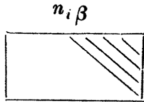
$$Log\, M=\begin{pmatrix} H & L \\ O & K \end{pmatrix}$$

where H and K are the matrices of order l and m respectively.

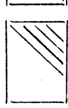
§ 3. Finally we shall investigate the terms which express the periodicity of the logarithmic functions of matrices, that is, the terms $S^{-1}FS$ where S is the arbitrary matrix belonging to $\mathfrak{K}(M)$.

Since $S \in \mathfrak{K}(M)$, we have

$$P^{-1}SP=\begin{pmatrix} S_1 & & 0 \\ & S_2 & \\ & & \ddots \\ 0 & & S_p \end{pmatrix}, \quad S_i=\begin{pmatrix} S_{i11} & \cdots & S_{i1p_i} \\ \vdots & & \vdots \\ S_{ip_i1} & \cdots & S_{ip_ip_i} \end{pmatrix}, \quad (i=1,\dots,p,) \tag{28}$$

$S_{i\alpha\beta}=$  $n_{i\beta}$ columns, $n_{i\alpha}$ rows, *for* $n_{i\beta}>n_{i\alpha}$,

$=$  *for* $n_{i\beta}=n_{i\alpha}$,

$=$  *for* $n_{i\beta}<n_{i\alpha}$.

where S_i is the matrix of order n_i, and $S_{i\alpha\beta}$ is the matrix having $n_{i\alpha}$ rows and $n_{i\beta}$ columns, all elements of which on the obliques are equal and arbitrary, and the other elements of which are zero.

Now we shall consider the mapping $S \rightarrow S^{-1}FS$ in $\mathfrak{K}(M)$, regarding F as fixed. The kernel $\mathring{\mathfrak{S}}$ of this mapping is the set $\mathfrak{K}(M) \cap \mathfrak{K}(F)$, and a normal subgroup of $\mathfrak{K}(M)$, regarding these set as the group with respect to the matrix multiplication. Furthermore, the form of the matrix $S_0 \in \mathring{\mathfrak{S}}$ is given by

$$\mathring{S}=P\begin{pmatrix} \mathring{S}_1 & & 0 \\ & \ddots & \\ 0 & & \mathring{S}_p \end{pmatrix} P^{-1}, \quad \mathring{S}_i=\begin{pmatrix} \mathring{S}_{i11} & \cdots & \mathring{S}_{i1p_i} \\ \vdots & & \vdots \\ \mathring{S}_{ip_i1} & \cdots & \mathring{S}_{ip_ip_i} \end{pmatrix}, \tag{29}$$

where if $f_{i\alpha} \neq f_{i\beta}$, then $\mathring{S}_{i\alpha\beta}=\mathring{S}_{i\beta\alpha}=0$; and $\mathring{S}_{i\alpha\beta}$ is the matrix of the same type as $S_{i\alpha\beta}$. Moreover let $\mathfrak{S}'$ be the set of all matrices S' of the following type:

$$S'=P\begin{pmatrix} S_1' & & 0 \\ & \ddots & \\ 0 & & S'_p \end{pmatrix}P^{-1}, \quad S_i'=\begin{pmatrix} E_{i11} & \cdots & S'_{i1p_i} \\ \vdots & \ddots & \vdots \\ S'_{ip_i1} & \cdots & E_{ip_ip_i} \end{pmatrix}, \tag{30}$$

where if $f_{i\alpha}=f_{i\beta}$, then $S'_{i\alpha\beta}=S'_{i\beta\alpha}=0$; and $S'_{i\alpha\beta}$ is the matrix of the same type as $S_{i\alpha\beta}$. Then $\mathfrak{R}(M)$ is decomposed into a direct product of $\mathring{\mathfrak{S}}$ and $\mathfrak{S}'$; that is,

$$\mathfrak{R}(M)=\mathring{\mathfrak{S}}\times\mathfrak{S}' \tag{31}$$

Accordingly, let $S\epsilon\mathfrak{R}(M)$, then $S=\mathring{S}\cdot S'$, $\mathring{S}\epsilon\mathring{\mathfrak{S}}$, $S'\epsilon\mathfrak{S}'$. (uniquely!). Hence we can conclude that

$$S^{-1}FS=S'^{-1}FS', \tag{32}$$

and if S' and T' ($S', T'\epsilon\mathfrak{S}'$) are distinct, then $S'^{-1}FS'$, and distinct.

Thns we have theorem:

THEOREM V. *The general solutions, log M of the matric equation exp $A=M$ for $M\epsilon\mathfrak{M}$ are reduced to* $\log M=L(M)+S'^{-1}FS'$,

where $S'=P\begin{pmatrix} S'_1 & & 0 \\ & \ddots & \\ 0 & & S'_p \end{pmatrix}P^{-1}$, $S'_i=\begin{pmatrix} E_{i11} & \cdots & S'_{i1p_i} \\ \vdots & & \vdots \\ S_{ip_i1} & \cdots & E_{ip_ip_i} \end{pmatrix}$

if $f_{i\alpha}=f_{i\beta}$, *then* $S'_{i\alpha\beta}=S'_{i\beta\alpha}=0$; and $S'_{i\alpha\beta}$ *is the matrix of type* (28). *And all arbitrary elements in* S' *are contained in log M as the essential parameters.*

Next, in order to calculate the number ν of the essential arbitrary clements contained in S', we shall first calculate such a number ν_i for S'_i. Since, from the form of $S_{i\alpha\beta}$ in (28), the number of the arbitrary elements contained in $S'_{i\alpha\beta}$ is equal to the minimum of $n_{i\alpha}$ and $n_{i\beta}$, i. e., $min(n_{i\alpha}, n_{i\beta})$, and if $f_{i\alpha}=f_{i\beta}$, then from theorem V, $S'_{i\alpha\beta}=S'_{i\alpha\beta}=0$, so we have

$$\nu_i=\sum_{f_{i\alpha}\neq f_{i\beta}} min(n_{i\alpha}, n_{i\beta})$$

accordingly

$$\nu=\sum_{i=1}^{p}\sum_{f_{i\alpha}\neq f_{i\beta}} min\,(n_{i\alpha}, n_{i\beta}) \tag{33}$$

or

$$\nu=2\sum_{i=1}^{p}\sum_{=1}^{p_i} n_{i\lambda}\rho_{i\lambda} \tag{34}$$

where $\rho_{i\lambda}$ denotes the times of the case when it happens that $n_{i\lambda}<n_{i\mu}$ or $n_{i\lambda}=n_{i\mu}$ ($\lambda<\mu$), being $f_{i\lambda}\neq f_{i\mu}$ for the fixed i and λ.[1]

Furthermore we shall calculate the nnmber χ of the essential arbitrary integers contained in F. The number χ is equal to the number of the distinct integers among $f_{i\alpha}$ ($i=1,\ldots,p;\alpha=1,\ldots,p_i$). And we have

$$\chi\leqq\sum_{i=1}^{p}p_i \tag{35}$$

Thus we have the theorem:

1) Moreover, from the first we may take the normal form such that $n_{i1}\leqq n_{i2}\leqq\ldots\leqq n_{ip_i}$, then we have $\rho_{i\lambda}\leqq p_i-\lambda$ accordingly
$$\nu\leqq 2\sum_{i=1}^{p}\sum_{\lambda=1}^{p_i} n_{i\lambda}(p_i-\lambda).$$

THEOREM VI. *There exist the periodicities of two kinds of* $\log M$ *which have the essentially distinct meaning. The one is the periodicity contained in F, whose cardinal number is equal to* $\aleph_0^x$, *where x is equal to the number of the distinct integers among* f_{ia} $(i=1,\ldots,p;\alpha=1,\ldots,p_i)$. *The other is the periodicity contained in S, whose cardinal number is equal to* $\aleph^\nu$:

$$\nu=\sum_{i=1}^{p}\sum_{f_{i\alpha}\neq f_{i\beta}} min(n_{i\alpha}, n_{i\beta})=2\sum_{i=1}^{p}\sum_{\lambda=1}^{p_i} n_{i\lambda}\rho_{i\lambda}$$

where $\rho_{i\lambda}$ denotes the times of the case when it happens that $n_{i\lambda}<n_{i\mu}$ or $n_{i\lambda}=n_{i\mu}$ $(\lambda<\mu)$, *being* $f_{i\lambda}\neq f_{i\mu}$ for the fixed i, λ.

Mathematical Institute, Hiroshima University.