On an Axiom of Continuous Geometry

By

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By a continuous geometry,[1] not necessarily irreducible, we mean a complemented, modular, and complete lattice such that the following (α) and its dual (α^*) are satisfied:

(α) *Let* (a_α; $\alpha<\Omega$) *be a system of elements, where* Ω *is any limit ordinal. If* $a_\alpha \uparrow a$, *then* $a_\alpha \cap b \uparrow a\cap b$ *for any* b.

In this paper I shall exhibit another condition which is equivalent to (α) in any complete lattice,[2] and show that a complemented modular lattice is a continuous geometry if and only if it is a generalized topological lattice.[3]

THEOREM I. *In a complete lattice* L, (α) *is equivalent to the following condition:*

(β) *Let* $(a_\delta;\delta\in D)\leqq L$, *where* D *is a directed set. If* $a_\delta\uparrow a$, *then* $a_\delta\cap b\uparrow a\cap b$ *for any* b.

PROOF. It is sufficient to prove that (α) implies (β).
If possible, suppose $\bigvee(a_\delta\cap b;\delta\in D)\neq\bigvee(a_\delta;\delta\in D)\cap b$ for some b, and M be a directed set with the least cardinal power such that $\bigvee(a_\delta\cap b;\delta\in M)\neq\bigvee(a_\delta;\delta\in M)\cap b$. Clearly M cannot be finite. Hence, by T. Iwamura's lemma,[4] there exists a transfinite sequence $\{M_\alpha;\alpha<\Omega\}$ of directed subsets of M with the following properties: (1) $\overline{\overline{M}}_\alpha<\overline{\overline{M}}$[5], (2) $\alpha<\beta<\Omega$ implies $M_\alpha\leqq M_\beta$, and (3) $M=\sum_{\alpha<\Omega}M_\alpha$. Then $\alpha<\beta<\Omega$ implies $\bigvee(a_\delta;\delta\in M_\alpha)\leqq \bigvee(a_\delta;\delta\in M_\beta)$ by (2), whence using (α),

$$\bigvee\{\bigvee(a_\delta;\delta\in M_\alpha)\cap b;\alpha<\Omega\}=\bigvee\{\bigvee(a_\delta;\delta\in M_\alpha);\alpha<\Omega\}\cap b.$$

While by making use of (1), $\bigvee(a_\delta;\delta\in M_\alpha)\cap b=\bigvee(a_\delta\cap b;\delta\in M_\alpha)$.
Hence, using (3) $\bigvee(a_\delta\cap b;\delta\in M)=\bigvee(a_\delta:\delta\in M)\cap b$, which contradicts the assumption of M. This completes the proof.

THEOREM II. *A complemented modular lattice* L *is a continuous geometry if and only if* L *is a generalized topological lattice.*

PROOF. Let $(a_\delta;\delta\in D)\leqq L$, D being a directed set. If $a_\delta\downarrow a$, then clearly $a_\delta\cap b\downarrow a\cap b$ for any $b\in L$, whence (β) is equivalent to the following:

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(1) Cf. J. v. Neumann, Lectures on continuous geometry, I, II (1936—1937).

(2) F. Maeda showed that, in any complete lattice, (α) is equivalent to the following condition: Let S be any system of elements. If $\bigvee(a;a\in\nu)\cap b=\bigvee(a\cap b;a\in\nu)$ for every finite subset ν of S, and any element b, then $\bigvee(a;a\in S)\cap b=\bigvee(a\cap b;a\in S)$. Cf. Zenkoku-Shijô-Sûgaku-Danwakai. **236** (1942), 1056.

(3) By (0)-lim $a_\delta=a$, it is meant the existence of $(v_\delta;\delta\in D)$, $(v_\delta;\delta\in D)$ and a such that $u_\delta\uparrow a$, $v_\delta\downarrow a$ and $u_\delta\leqq a_\delta\leqq v_\delta$. By generalized topological lattice, we mean a complete lattice such that (0)-lim $a_\delta=a$, and (0)-lim $b_\delta=b$ imply (0)-lim $a_\delta\cap b_\delta=a\cap b$, and (0)-lim $a_\delta\cup b_\delta=a\cup b$. Cf. G. Birkhoff, Lattice Theory, New York (1940), 30

(4) Cf. T. Iwamura, Zenkoku-Shijô-Sûgaku-Danwakai, **202** (1944), 107.

(5) By $\overline{\overline{M}}$, it is meant the cardinal power of M.

If (0)-lim $a_\delta = a$, then (0)-lim $a_\delta \cap b = a \cap b$ for any b.
And this is equivalent to the following:[1]

$$\text{(0)-lim } a_\delta,\ \text{(0)-lim } b_\delta = b \text{ imply (0)-lim } a_\delta \cap b_\delta = a \cap b.$$

Hence we infer that (β) is equivalent to the continuity of the lattice operation $\cap$ in generalized (0)-topology. The proof is completed by duality and theorem I.

(1) (0)-continuity of the meet in each variable implies simultaneous (0)-continuity in the two variables. Cf. G. Birkhoff, loc. cit., 30.