On the Extension of Semi-simple Sub-nuclei in Lie Groups

By

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Let H' be a sufficiently small sub-nucleus in a Lie group G. Generally there exists no Lie subgroup of G with H' as its nucleus. When such a subgroup exists, we say that H' is extensible to a Lie subgroup in G. Here we shall investigate under what conditions a semi-simple sub-nucleus H' is extensible to a Lie subgroup in G.

§ 1. Some preliminaries.

Let H' be a sub-nucleus in a Lie group G and let H be the abstract subgroup of G generated by H' i.e. $H=\sum H'^n$. H^a, the closure of H, is the Lie subgroup generated by H'. We denote by H^* the simply connected Lie group locally isomorphic with H'.

LEMMA 1.[1] The local isomorphism from H^* to H defines uniqely a representation f[2] from H^* to H.

LEMMA 2.[3] The following conditions are equivalent to each other:

(1) $H=H^a$, namely H is a Lie group,

(2) H' is a nucleus of H^a,

(3) H' is a nucleus of H,

(4) f (in Lemma 1) is a homomorphism[2] from H^* in H.

§ 2. Local structure of H^a.

In this and the following §§ we assume that H' is a semi-simple sub-nucleus of G.

LEMMA 3. Let Z_G and Z_H be the center of G and H respectively. If a nucleus U of G is sufficiently small, then $Z_H \cap U \subset Z_G$.

Proof. We consider the adjoint representation $x \to D(x)$ of G. Then it is clear that $x \in Z_G$ is equivalent to $D(x)=E$ (unit matrix). Since H' is semi-simple, $D(x), x \in H'$ is similar to

$$\begin{pmatrix} D_1(x) & & & \\ & D_2(x) & & \\ & & \ddots & \\ & & & D_n(x) \end{pmatrix},$$

where $x \to D_i(x)$, $x \in H$ is an irreducible representation of H', whence det. $D_i(x)=1$. Now for any given $z \in Z_H$ we have $D_i(x)D_i(z)=D_i(z)D_i(x)$, $x \in H$. So by I. Schur's lemma we obtain

$$D_i(z)=a_i E_i, \quad a_i^{d_i}=1 \quad (i=1, 2, \ldots, n),$$

where d_i is the degree of $D_i(x)$ and E_i is a unit matrix of degree d_i. If we take U sufficiently small, then $z \in U \cap Z_H$ implies $a_1=a_2=\ldots=a_n=1$. Hence $D(z)=E$, which is equivalent to $z \in Z_G$.

THEOREM 1. *If H' is sufficiently small and U is a properly chosen nucleus of G, then we have*

(1) $U \cap H^a = H' \times (Z_H{}^a \cap U)$ (*direct product*),

(2) $U \cap H = H' \times (Z_H \cap U)$ (*direct product*),

(3) $Z_H{}^a \cap U = Z_H^a \cap U$.

Proof. (1) It is clear that H' is an invariant sub-nucleus of H^a. Cartan's theorem shows that H' is a direct factor in a nucleus of H^a. So we can write $U \cap H^a = H' \times Z$. Hence Z is commutable with H' and so with H^a. Namely $Z \subset Z_{H^a}$ and we have $U \cap H^a = H' \times (Z_{H^a} \cap U)$.

(2) If $h \in U \cap H \subset U \cap H^a = H' \times (Z_{H^a} \cap U)$, then $h = h'z$, where $h' \in H'$ and $z \in Z_{H^a} \cap U$. Hence $z = h'^{-1} h \in H$ and so $z \in Z_H$. From this it is easily seen that $U \cap H = H' \times (Z_H \cap U)$.

(3) If we take U sufficiently small, then (3) follows from (1) and (2).

§ **3**. Sufficient conditions for $H^a = H$.

Using Theorem I we have

THEOREM 2. *A sub-nucleus H' of a Lie group G is extensible to a Lie subgroup in the following cases:*

(1) *The center Z_G of G is discrete.*

(2) *The center $Z_H(Z_{H*})$ of $H(H^*)$ is finite.*

(3) *The fundamental group of G is finite.*

Proof. (1) Since Z_G is discrete, there exists a nucleus U such that $Z_H \cap U$ contains no elements of G except the unit element e. Then $H \cap U = H'$ and hence the condition (3) of Lemma 2 is satisfied.

(2) The proof is similar to that of (1).

(3) Let us suppose the contrary to be true. By Ado-Cartan's theorem,[6] there exists a linear representation locally isomorphic to a nucleus of G. Let $x \in U \mapsto D(x)$ be such a representation, where U is a nucleus of G taken sufficiently small. Let $z (\neq e) \in Z_H \cap U$. We construct a path l_1 in U leading from e to z and a path l_2 in H from z to e, and we consider the product $l = l_1 l_2$. Extending the representation $x \mapsto D(x)$ along the path l, we obtain $D(z)$ ($\neq$ unit matrix) at the end of l. Hence such pathes l defined for different elements z are not homotopic to each other. Since Z_H is dense in a nucleus of Z_{H^a}, the fundamental group of G has elements with orders as large as we want. Thus we arrive at the contradiction.

From these results we obtain

THEOREM 3. *A semi-simple sub-nucleus H' of a Lie group G is extensible to a Lie subgroup in the following cases:*

(a) *G is semi-simple,*

(b) *H' is complex semi-simple,*

(c) *G is a complex Lie group or its Lie subgroup,*

(d) *There exists a complex Lie group K with H^* as its Lie subgroup (e.g. H^* is compact),*

(e) *G is simply connected,*

(f) *G is compact.*

Proof. (a) G is semi-simple, so that the center of G is discrete. Then Theorem 2 (1) is satisfied.

(b) Let H_u^* be the compact form of H^*. If we denote by $Z_{H_u^*}$ the center of H_u^*, then using Cartan's theorem[6] we obtain $Z_{H^*}=Z_{H_u^*}$. Hence Z_{H^*} as well as $Z_{H_u^*}$ is finite. Then Theorem 2 (2) is satisfied.

(c) It is sufficient to show when G is a complex Lie group. If we denote by H'_c the complexing of H' in G, then H'_c is semi-simple. Hence it follows from (a) and (b) that H' is extensible.

(d) (c) shows that there exists a complexing H_c^* of H^* in K. Then H_c^* is complex semi-simple. Since the center of H_c^* is finite, so the center Z_{H^*} of H^* is also finite. Hence Theorem 2 (2) is satisfied.

(e) This case is the consequence of Theorem 2 (3).

(f) G has an isomorphic linear representation, so is contained in a complex Lie group. Therefore (f) is reduced to the case (c).

§ 4. An example for $H^a \neq H$.

We denote by g a semi-simple Lie group whose fundamental group contains a free cyclic group $\{mc\}$ (m:integer). For example, so is the homographic group of a real variable, because it is homeomorphic to the inside of a torus. Let g^* be the universal covering group of g. Then $\{mc\}$ is contained in its center. We form the direct product $g^*\times R$, where R is an adittive group of real numbers. If we denote by D the totality of $(mc, m\tau-n)$ for an irrational number τ, then D is the discrete normal Abelian subgroup contained in the center of $g^*\times R$. Now we consider a Lie group $G=g^*\times R/D$. Let $g^{*\prime}$ be the sufficiently small nucleus of g^* and let $H'=g^{*\prime}\times\{o\}/D$. Then H' is a semi-simple sub-nucleus of G. The abstract subgroup generated by H' is $H=g^*\times\{o\}/D$. Since $(mc, o)/D=(o, -m\tau+n)/D$, H will be dense in a nucleus of G by Kronecker's theorem.[7] Hence $H^a=(g^*\times\{o\}/D)^a=G$.

References

[1] L. Pontrjagin, Topological Groups, (1939), Princeton, Theorem 63.

[2] For these terminologies see A. Weil, L'intégration dans les groupes topologiques et ses applications, (1940), Paris.

[3] L. Pontrjagin, loc. cit., Theorem 13.

[4] E. Cartan, Thèse, 53.

[5] E. Cartan, Les représentations linéaires des groupes de Lie, Journal de Math., 17 (1938), 1—12.

[6] E. Cartan, La géometrie des groupes simples, Annali di Mat., 4 (1927), 209—256, specially 250—251. F. Gantmacher, On the classification of real simple Lie groups, Recueil Math., 47 (1939), 217—249, § 4.

[7] L. Pontrjagin, loc. cit., 150.

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