Representations of Orthocomplemented Modular Lattices

By

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Let L be a complemented modular lattice of order $n\geqq 4$, and $\mathfrak{S}$ the auxiliary ring of L. J.v.Neumann [1][1] has proved that L is isomorphic to the lattice $\bar{R}_{\mathfrak{S}_n}$ of all principal right ideals in $\mathfrak{S}_n$, which is the set of all n-rowed square matrices with elements in $\mathfrak{S}$. Let $a\to a^\perp$ be an involutory dual automorphism of L, such that $a\leqq a^\perp$ implies $a=o$. Then $a^\perp$ is the orthogonal complement of a. This dual automorphism $a\to a^\perp$ of L induces an involutory dual automorphism $\mathfrak{a}\to\mathfrak{a}^\perp$ of $\bar{R}_{\mathfrak{S}_n}$.

When L is of finite dimensions (in this case $\mathfrak{S}$ is a skew-field), Birkhoff and v.Neumann [1] has proved that there exists an involutory anti-automorphism $\alpha\to\alpha^*$ of $\mathfrak{S}$ and a definite diagonal Hermitian form $\sum_{i=1}^{n}\alpha_i^*\varphi_i\beta_i$ in $\mathfrak{S}$, such that $\varphi_i^*=\varphi_i$, aud $\sum_{i=1}^{n}\alpha_i^*\varphi_i\alpha_i=0$ implies $\alpha_i=0$ $(i=1,\cdots,n)$, and the anti-automophism $\alpha\to\alpha^*$ of $\mathfrak{S}$ generates the given automorphism $\mathfrak{a}\to\mathfrak{a}^\perp$ of $\bar{R}_{\mathfrak{S}_n}$. In this case we can define an inner product

$$(\mathbf{a},\ \mathbf{b})=\sum_{i=1}^{n}\alpha_i^*\varphi_i\beta_i$$

of two n-dimensional vectors $\mathbf{a}=(\alpha_1,\cdots,\alpha_n)$, $\mathbf{b}=(\beta_1,\cdots,\beta_n)$. And the orthogonality of $\mathbf{a}$ and $\mathbf{b}$ defined by $(\mathbf{a},\mathbf{b})=0$, is closely related to the orthogonal complements in L.[2]

In this paper, I shall prove that these results hold also in the general case, under the assumption that L has an *orthogonal* homogeneous basis of order $n\geqq 4$.

1. Let L be an orthocomplemented modular lattice. If we denote the orthogonal complement of a by $a^\perp$, then $a\to a^\perp$ is an involutory dual automorphism of L, such that $a\leqq a^\perp$ implies $a=0$.

In what follows, we shall assume that L has an orthogonal homogeneous basis of order $n\geqq 4$:

$$(a_k; k=1,\cdots,n)\perp, \qquad a_1\cup\cdots\cup a_n=1,$$
$$a_k\sim a_h\ (k,h=1,\cdots,n), \qquad a_h\leqq a_k^\perp\ (k\neq h).^{(3)}$$

In this case, $a_k^\perp=\bigvee_{h=1,h\neq k}^{n} a_h$. For, $\bigvee_{h=1,h\neq k}^{n} a_h\leqq a_k^\perp$, and $a_k^\perp$ and $\bigvee_{h=1,h\neq k}^{n} a_h$ are both complements of a_k.

2. Let $(a_k, c_{kh}; k,h=1,\cdots,n)$ be a normalized frame in L, which is obtained from the orthogonal homogeneous basis $(a_k; k=1,\cdots,n)$, and let $\mathfrak{S}$ be the auxiliary ring of L relative to this frame. Then L is isomorphic to $\bar{R}_{\mathfrak{S}_n}$.[4] And the involutory dual automorphism $a\to a^\perp$ of L induces a unique involutory anti-automorphism $A\to A^*$ of $\mathfrak{S}_n$,[5] and

(1) The numbers in square brackets refer to the list given at the end of this paper.

(2) Cf. also Birkhoff [1] 71-72.

(3) When L is a continuous geometry of order $n\geqq 4$ with orthogonal complements, there exists such an orthogonal homogeneous basis.

(4) v.Neumann [1] Theorem 14·1.

(5) v.Neumann [1] Theorem 4·3, Theorem 4·4.

(1°) $A^*A=0$ implies $A=0$,

(2°) for every $A\in\mathfrak{S}_n$ there exists a unique Hermitian idempotent E such that $(A)_r=(E)_r$, and $(1-E)_r$ is the orthogonal complement of $(E)_r$.[1]

3. To the normalized frame $(a_k, c_{kh}; k, h=1,\cdots,n)$ in L, there corresponds a normalized frame in $\bar{R}_{\mathfrak{S}_n}$:

$$\mathfrak{a}_k=(E_{kk})_r, \qquad \mathfrak{c}_{kh}=(E_{kk}-E_{kh})_r=(E_{hh}-E_{hk})_r\,;\ k,h=1,\cdots,n),$$

E_{kh} being matrices $(\eta_{ij}{}^{kh})$, where

$$\eta_{ij}{}^{kh}=\begin{cases}1, & \text{if } (i,j)=(k,h),\\ 0, & \text{if } (i,j)\neq(k,h).\text{[2]}\end{cases}$$

Since $\sum_{h=1,h\neq k}^{n} E_{hh}=1-E_{kk}$, $E_{ll}E_{hh}=0$ $(l\neq h)$, we have $\bigvee_{h=1,k\neq h}^{n} \mathfrak{a}_h=(1-E_{kk})_r$. Then by **1**, $(1-E_{kk})_r$ is the orthogonal complement of $(E_{kk})_r$. Hence by **2** (2°) E_{kk} $(k=1,\cdots,n)$ are Hermitian idempotents in $\mathfrak{S}_n$.

4. Since $E_{kh}=E_{kk}E_{kh}$, we have $E_{kh}^*=E_{kh}^*E_{kk}$. Hence if we put $E_{kh}^*=(\zeta_{ij}{}^{kh})$, then $j\neq k$ implies $\zeta_{ij}{}^{kh}=0$. Similarly from $E_{kh}=E_{kh}E_{hh}$, we have $E_{kh}^*=E_{hh}E_{hk}^*$. Hence $i\neq h$ implies $\zeta_{ij}{}^{kh}=0$. Consequently there exists φ_{kh}, such that

$$\zeta_{ij}{}^{kh}=\begin{cases}\varphi_{kh}, & \text{if } (i,j)=(h,k),\\ 0, & \text{if } (i,j)\neq(h,k).\end{cases}$$

Since $E_{hk}^*E_{kh}^*=E_{kk}^*=E_{kk}$, we have

$$\varphi_{hk}\varphi_{kh}=\varphi_{kk}=1 \qquad (k,h=1,\cdots,n).$$

5. Denote by $\mathfrak{R}(E_{11})$, the set of all A such that $AE_{11}=E_{11}A=A$. $\mathfrak{R}(E_{11})$ is a subring of $\mathfrak{S}_n$. If $A\in\mathfrak{R}(E_{11})$, then $E_{11}A^*=A^*E_{11}=A^*$, and $A^*\in\mathfrak{R}(E_{11})$. Therefore, $A\to A^*$ is an involutory anti-automorphism of $\mathfrak{R}(E_{11})$. For $A=(a_{ij})\in\mathfrak{R}(E_{11})$, $(i,j)\neq(1,1)$ implies $a_{ij}=0$, and $\mathfrak{R}(E_{11})$ is isomorphic to $\mathfrak{S}$ by the correspondence $A\to a_{11}$. Hence $A\to A^*$ induces an involutory anti-automorphism $a\to a^*$ of $\mathfrak{S}$, aud for $A=(a_{ij})\in\mathfrak{R}(E_{11})$, we have $A^*=(a_{ij}{}^*)$. ($(i,j)\neq(1,1)$ implies $a_{ij}=a_{ij}{}^*=0$.)

6. If $A\in\mathfrak{S}_n$, put $A_{kh}=E_{1k}AE_{h1}$, then $A_{kh}\in\mathfrak{R}(E_{11})$ and $A=\sum_{k,h}E_{k1}A_{kh}E_{1h}$. Hence we have

$$A^*=\sum_{k,h}E_{1h}^*A_{kh}^*E_{k1}^*.$$

Let $A=(a_{ij})$, $A^*=(\gamma_{ij})$, and put the values of E_{1h}^*, E_{k1}^*, A_{kh}^* obtained in **4** and **5**. Then we have

$$\gamma_{ij}=\varphi_{1i}a_{ji}^*\varphi_{j1} \qquad (i,j=1,\cdots,n).$$

Put $\varphi_i=\varphi_{i1}$ $(i=1,\cdots,n)$. (Of course $\varphi_1=\varphi_{11}=1$.) Then from **4** $\varphi_{i1}\varphi_{1i}=\varphi_{1i}\varphi_{i1}=1$. Hence $\varphi_{1i}=\varphi_{i1}^{-1}=\varphi_i^{-1}$, and φ_i $(i=1,\cdots,n)$ are regular elements. And we have

$$\gamma_{ij}=\varphi_i^{-1}a_{ji}^*\varphi_j \qquad (i,j=1,\cdots,n).$$

Since $E_{i1}^{**}=E_{i1}$, we have $\varphi_i^{-1}\varphi_i^*\varphi_1=1$, that is $\varphi_i=\varphi_i^*$. Therefore $\varphi_i (i=1,\cdots,n)$ are Hermitian elements.

7. Let $a_i (i=1,\cdots,n)$ be elements of $\mathfrak{S}$, snch that $\sum_{i=1}^{n}a_i^*\varphi_i a_i=0$. For $A=(a_{ij})$, where

$$a_{ij}=\begin{cases}a_i, & \text{if } j=1,\\ 0, & \text{if } j\neq1,\end{cases}$$

(1) v.Neumann [1] Theorem 4·5.

(2) Cf. v.Neumann [1] 113. $(E_{kh}; k,h=1,\ldots,n)$ is a system of matrix units of $\mathfrak{S}_n$.

we have by **6** $A^*=(\gamma_{ij})$, where

$$\gamma_{ij}=\varphi_i^{-1}\alpha_{ji}^*\varphi_j=\begin{cases}\alpha_j^*\varphi_j, & \text{if } i=1,\\ 0, & \text{if } i\neq 1.\end{cases}$$

Hence $A^*A=0$, and by **2** (1°) we have $A=0$. That is $\alpha_i=0$ $(i=1,\cdots,n)$.

8. Summing up the results of **5**, **6**, **7**, we obtain:

Corresponding to the involutory anti-automorphism $A\to A^*$ of $\mathfrak{S}_n$ obtained in **2**, *there exists an involutory anti-automorphism $\alpha\to\alpha^*$ of $\mathfrak{S}$ and a definite diagonal Hermitian form $\sum_{i=1}^n \alpha_i^*\varphi_i\beta_i$, where $\varphi_i (i=1,\cdots,n)$ are fixed Hermitian regular elements, and $\sum_{i=1}^n \alpha_i^*\varphi_i\alpha_i=0$ implies $\alpha_i=0$ $(i=1,\cdots,n)$, and $A^*=(\varphi_i^{-1}\alpha_{ji}^*\varphi_j)$ corresponds to $A=(\alpha_{ij})$.*

9. Between two n-dimensional vectors $\mathbf{a}=(\alpha_1,\cdots,\alpha_n)$. $\mathbf{b}=(\beta_1,\cdots,\beta_n)$, $(\alpha_i,\beta_i\in\mathfrak{S})$, we define the *inner product* by

$$(\mathbf{a},\mathbf{b})=\sum_{i=1}^n\alpha_i^*\varphi_i\beta_i.$$

When $(\mathbf{a},\mathbf{b})=0$, we say that $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal*, and write $\mathbf{a}\perp\mathbf{b}$. And two right linear sets $\mathfrak{M}$, $\mathfrak{N}$ of n-dimensional vectors are said to be *orthogonal*, when $\mathbf{a}\perp\mathbf{b}$ for every $\mathbf{a}\in\mathfrak{M}$, $\mathbf{b}\in\mathfrak{N}$.

The right ideals $\mathfrak{a}$ in $\mathfrak{S}_n$ and the right linear sets $\mathfrak{M}$ of n-dimensional vectors are in a one-to-one correspondence, corresponding elements being linked by the following relations:

$(\alpha_{ij})\in\mathfrak{a}$ if and only if $(\alpha_{1j},\cdots,\alpha_{nj})\in\mathfrak{M}$ for $j=1,\cdots,n$;

$(\alpha_1,\cdots,\alpha_n)\in\mathfrak{M}$ if and only if there exists $(\alpha_{ij})\in\mathfrak{a}$ with $\alpha_{i1}=\alpha_i$ $(i=1,\cdots,n)$.

And $R_{\mathfrak{S}_n}$ is lattice-isomorphic to the system of all right linear sets. In this case the right linear set $\mathfrak{M}$ corresponds to a principal right ideal $\mathfrak{a}=((\alpha_{ij}))_r$ if and only if $\mathfrak{M}$ is the set of all linear combinations of vectors $\mathbf{a}_1,\cdots,\mathbf{a}_n$ for which

$$\mathbf{a}_j=(\alpha_{1j},\cdots,\alpha_{nj}),\qquad (j=1,\cdots,n).^{(1)}$$

But by **2**, for a principal right ideal $\mathfrak{a}$ in $\mathfrak{S}_n$, there exists a Hermitian idempotent E such that $\mathfrak{a}=(E)_r$, and $\mathfrak{a}^\perp=(1-E)_r$. Let $E=(\eta_{ij})$; then by **8** $E^*=(\varphi_i^{-1}\eta_{ji}^*\varphi_j)$. Therefore $E=E^*$ and $E^2=E$ mean

$$\varphi_i^{-1}\eta_{ji}^*\varphi_j=\eta_{ij},\qquad \sum_k\eta_{ik}\eta_{kj}=\eta_{ij}\qquad (i,j=1,\cdots,n).$$

To $\mathfrak{a}=(E)_r$ corresponds $\mathfrak{M}(\mathfrak{a})=(\mathbf{a}_1,\cdots,\mathbf{a}_n)$ where

$$\mathbf{a}_j=(\eta_{1j},\cdots,\eta_{nj})\qquad (j=1,\cdots,n),$$

and to $\mathfrak{a}^\perp=(1-E)_r$ corresponds $\mathfrak{M}(\mathfrak{a}^\perp)=(\mathbf{b}_1,\cdots,\mathbf{b}_n)$ where

$$\mathbf{b}_j=(-\eta_{1j},\cdots,-\eta_{j-1,j},\ 1-\eta_{jj},-\eta_{j+1,j},\cdots,-\eta_{nj})\qquad (j=1,\cdots,n).$$

Hence

$$(\mathbf{a}_i,\mathbf{b}_j)=-\sum_k\eta_{ki}^*\varphi_k\eta_{kj}+\eta_{ji}^*\varphi_j=\varphi_i(-\sum_k\varphi_i^{-1}\eta_{ki}^*\varphi_k\eta_{kj}+\varphi_i^{-1}\eta_{ji}^*\varphi_j)$$
$$=\varphi_i(-\sum_k\eta_{ik}\eta_{kj}+\eta_{ij})=0.$$

Therefore, two right linear sets $\mathfrak{M}(\mathfrak{a})$ and $\mathfrak{M}(\mathfrak{a}^\perp)$ which correspond to $\mathfrak{a}\in\bar{R}_{\mathfrak{S}_n}$ and its orthogonal complement $\mathfrak{a}^\perp$, are orthogonal.

Consequently, *an orthocomplemented modular lattice, which has an orthogonal homogeneous basis of order $n\geqq 4$, is lattice-isomorphic to a system of right linear sets of n-dimensional vectors, and this correspondence preserves the orthogonal property.*

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(1) v.Neumann [1] 27.

References.

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