Reduced Forms of Ordinary Differential Equations in the Vicinity of the Singularity of the Second Kind.

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§ 1. Introduction.

The differential equation $\frac{dy}{dx}=f(y, x)$, in the vicinity of the singularity of the second kind ($y=0$, $x=0$), by means of Weierstrass' preparation theorem, can be reduced to the equation of the following form[1]

(1. 1) $$\frac{dy}{dx}=x^S y^R \frac{q(y,x)}{p(y,x)} e^{G(y,x)}$$

where S and R are integers, $G(y, x)$ is regular in the vicinity of ($y=0$, $x=0$), $p(y, x)$ and $q(y, x)$ are relatively prime singular algebroid polynomial with the vertex ($y=0$, $x=0$), and $p(0, x)$, $q(0, x)$ do not vanish identically. It happens that the function $p(y, x)$ or $q(y, x)$ becomes a constant, but, even in that case, ($y=0$, $x=0$) is the singularity of the second kind of the differential equation.

Forsyth has shown that, if the equation (1. 1) has a determinate integral of regular class,[2] then, by means of Newton's polygon method, it can be reduced to the equation of the type either

I: $t^{\kappa}\frac{dv}{dt}=av^n+pt+\dots,$ or II: $t^{\kappa}\frac{dv}{dt}=\frac{av^n+pt+\dots}{bv^m+qt+\dots},$

where κ, m, n are positive-integers and $a, b\neq 0$.[3] Besides, he has proved that, the equation of the type II which has an integral of regular class, is reduced to the equation of the type I, when certain conditions upon the coefficients are satisfied.[4]

In this paper, by means of Newton's polygon method, we shall establish the general theory of reduction of the differential equation in the vicinity of the singularity of the second kind. Without any assumptions which Forsyth has laid, we shall obtain the final reduced forms as follows:

(A_1) $t^{\kappa}\frac{dv}{dt}=av+pt+\dots;$ $\kappa\geqq 2$, $a\neq 0$,

(A_2) $t\frac{dv}{dt}=v^{\lambda}(av^n+pt^m+\dots);$ $a,p\neq 0$, $n\geqq 0$, $m>0$, and $\lambda\geqq 1$ when $n=0$,

(B_0) $t\frac{dv}{dt}=\frac{av^n+pt+\dots}{bv^m+qt+\dots};$ $a,b\neq 0$, $m\geqq 1$, $n\geqq 1$,

(C_0) $\frac{dv}{dt}=t^{\nu}v^{\lambda}(av^n+pt^m+\dots);$ $a, p\neq 0$, $n\geqq 0$, $m>0$, $\nu\geqq 0$, $\lambda\geqq 1$.

1) A.R. Forsyth, Theory of Differential Equations, Part II, p. 88.

2) When an integral has an appropriate order in powers of the independent variable in the vicinity of the initial values, we call such an integral "an integral of regular class."

3) Forsyth, ibid. p. 123.

4) Forsyth, ibid. p. 204.

The equation of the form (C_0) is not contained in the equation of the types of Forsyth. If, as Forsyth has done, we assume that each equation has a determinate integral of regular class, then we obtain, as the final reduced forms, the form either (A_1) or (A_2) with $\lambda \geqq 0$, namely two special cases of the type I with $\chi=1$ or $n=1$. This means that the reduced forms of Forsyth are not satisfactory forms as the final reduced forms even under his assumptions.

§ 2. First reduction.

The equations of the form (1. 1) can be classified into the equations of the forms:

(A) $x^{\kappa}\dfrac{dy}{dx}=y^{\lambda}q(y,x)$, (B) $x^{\kappa}\dfrac{dy}{dx}=y^{\lambda}\dfrac{q(y,x)}{p(y,x)}$,

(C) $\dfrac{dy}{dx}=x^{\nu}y^{\lambda}q(y,x)$, (D) $\dfrac{dy}{dx}=x^{\nu}y^{\lambda}\dfrac{q(y,x)}{p(y,x)}$,

where $p(y, x)$ and $q(y, x)$ are regular in the vicinity of $(y=0,\ x=0)$ and $p(y, x)$ vanishes there, $p(0, x)$, $p(y, 0)$; $q(0, x)$, $q(y, 0)$ do not vanish identically, and χ, λ, ν are integers such that $\chi \geqq 1$ and $\nu \geqq 0$.

In this paragraph, by means of Newton's polygon method for differential equations,(1) we shall reduce the equation (1. 1) to one of the equations of the forms (A), (B), (C) and (D).

The functions $q(y, x)$ and $p(y, x)$ of (1. 1) are of the following forms:

$q(y, x)=a_0y^n+a_1(x)y^{n-1}+\ldots\ldots+a_n(x)$ and $p(y, x)=b_0y^m+b_1(x)y^{m-1}+\ldots\ldots+b_m(x)$,

where $a_i(x)$ and $b_k(x)$ are regular in the vicinity of $x=0$ and vanish there. We write the terms of lowest order in the expansions of $a_i(x)$ and $b_k(x)$ as $a_ix^{r_i}$ and $b_kx^{s_k}$ respectively. On the plane with two perpendicular axes 0ξ and 0η, we mark the points $A_l(n-l,\ r_l+1+S)$ and $B_k(m-R+1-k;\ s_k)$, and construct a Newton's polygon.

When m and n are positive, there exist at least four points $A_n(0, r_n+1+S)$, $A_0(n, 1+S)$, $B_m(1-R,\ s_m)$ and $B_0(1-R+m, 0)$, consequenently there exists a Newton's polygon. When $m=0$ and $n>0$, draw parallels A_0CP and A_nCQ through A_0 and A_n to $O\xi$ and $O\eta$ respectively, and let the point of their intersection be C. Then, there does not exist Newton's polygon only when the point B_0 $(1-R, 0)$ lies either on one of the sides of the angle PCQ or in that angle. In this case, $R\geqq 1$ and $S\geqq -1$, however, when $S\geqq 0$, $(y=0,\ x=0)$ is not the singularity of the second kind of (1. 1), therefore $S=-1$. Then the equation (1. 1) becomes as follows:

(a) $x\dfrac{dy}{dx}=y^{R}q(y, x)e^{G(y,x)}$, where $R\geqq 1$.

Likewise, when $n=0$ and $m>0$, there does not exist Newton's polygon, only when $S\leqq -1$ and $R=1$. In this case, the equation (1. 1) becomes as follows:

(b) $y\dfrac{dx}{dy}=x^{S'}p(y, x)e^{-G(y,x)}$, where $S'=-S\geqq 1$.

When m and n are zero, we can easily prove that there does not exist Newton's polygon

1) Forsyth, ibid. p. 123.

only when either $R>1$ and $S=-1$ or $R=1$ and $S<-1$.[1] In either case, we have the equation of the form either (a) or (b). Thus, when there does not exist Newton's polygon, the equation (1. 1) can be transformed to the equation of the form (A) with $\chi=1$ and $\lambda\gtreqless 1$, and conversely.

When there exists a Newton's polygon, the order μ of y in powers of x, if it exists in the vicinity of $(y=0,\ x=0)$, is determined as the tangent of the angle made by the side of the Newton's polygon and the negative direction of 0ξ. Then μ becomes a positive rational number,[2] consequently we may put $\mu=p/q$, where p and q are relatively prime positive integers. Put $x=t^q$, $y=t^p u$, where $u(0)=\rho\neq 0$. Substituting these into (1. 1), we have

$$\text{(2. 1)}\qquad t^{(1-R)p}\left(pu+t\frac{du}{dt}\right)\left\{\Sigma' b_k u^{m-k} t^{qs_k+p(m-k)}+\dots\right\}$$
$$=Cqt^{q(1+S)}u^R\left\{\Sigma' \bar{a}_l u^{n-l} t^{qr_l+p(n-l)}+\dots\right\},$$

where $C=e^{G(0,0)}$ and Σ' denotes the sum of the terms of lowest order of t. With regard to the sides of the Newton's polygon, there arise three cases.

Case I. The side contains only the points B_k.

In this case $q(r_l+1+S)+p(n-l)>qs_k+p(m-k+1-R)=N$ say. After having divided both sides of (2. 1) by t^N, we put $t=0$, then $\Sigma' b_k\rho^{m-k}=0$. Now, the terms of $\Sigma' b_k u^{m-k}$ are those corresponding to the points B_k on the side under consideration. Therefore $\Sigma' b_k u^{m-k}$ contains at least two terms, consequently the finite non-zero value ρ' of ρ is certainly determined. Namely, there exists an integral of regular class.

Case II. The side contains only the points A_l.

In this case $qs_k+p(m-k+1-R)>q(r_l+1+S)+p(n-l)=M$ say. After having divided both sides of (2. 1) by t^M, we put $t=0$. Then, as in the case I, we can determine the finite non-zero value ρ' of ρ by means of the equation $\Sigma' a_l\rho^{n-l}=0$. Namely, there exists an integral of regular class.

Case III. The side contains some points A_l and some points B_k.

In this case $q(r_l+1+S)+p(n-l)=qs_k+p(m-k+1-R)=N$ say. After having divided both sides of (2. 1) by t^N, we put $t=0$. If we put $Cq\Sigma' a_l u^{n-l}-pu^{1-R}\Sigma' b_k u^{m-k}=\Phi(u)$, then $\Phi(\rho)=0$. The expression $\Phi(u)$ is the sum of the terms corresponding to the points A_l and B_k on the side under consideration. For the expression $\Phi(u)$, there arise three cases.

(i) The expression $\Phi(u)$ contains at least two terms.

In this case, the finite non-zero value ρ' of ρ can be determined, consequently there exists an integral of regular class.

(ii) The expression $\Phi(u)$ contains only one term.

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1) When $S=-1$ and $R=1$, A_0 coincides with B_0 In such a case, we say that there exists a Newton's polygon which has the side consisting of two coincident points. For details, refer the explanations in the case III (ii).

2) When the side of Newton's polygon contains only two coincident points, one of the coincident points is A_l and the other is B_k. In this case μ is indeterminate, therefore is not necessarily rational. When μ is not rational, by the substitution $y=x^\mu u$, we shall obtain the equation involving an irrational power of x. In order to avoid such a case, we consider hereafter only positive rational values of μ.

In this case, the set of points B_k on the side under consideration must be coincident with the set of points A_l on the side except for at most one point. Among the points on the side, there exists one point P, the abscissa of which gives the exponent of u of the remaining term in $\Phi(u)$. When the abscissa of the point P is non-positive, it is readily seen that the ordinate of P is positive except for the case when P coincides with the origin O of the plane $\xi O \eta$ and belongs to both sets of points A_l and B_k. In this exceptional case, the point B_0 and the point A_n coincide with the origin, consequently $R=m+1$ and $S=-r_n-1$. The side contains only one point O with which B_0 and A_n coincide. Now, when the abscissa of the point P which does not coincide with the origin O is non-positive, we interchange the variable x with the variable y, and construct the points $B'_l(r_l+1+S,\ n-l)$ and $A'_k(s_k,\ m-R+1-k)$. Then we have a Newton's polygon symmetric to the initial one with respect to the bisector of the angle $\xi O \eta$. For the corresponding side, we have $\Psi(u)=\frac{1}{C}p\Sigma b_k u^{s_k}-qu^{l+S}\Sigma a_l u^{r_l}$ corresponding to $\Phi(u)$. Now, when a point A_l coincides with a point B_k, $n-l=m-R+1-k$ and $r_l+1+S=s_k$, consequently the remaining term in $\Phi(u)$ corresponds to the remaining term in $\Psi(u)$, and conversely. Then, on the corresponding side of the new polygon, the abscissa of the point P' corresponding to P gives an exponent of only one remaining term in $\Psi(u)$. Now, the abscissa of the point P' is equal to the ordinate of the point P, therefore, except for the above stated exceptional case, the abscissa of the point P' is positive. For a time, we exclude the exceptional case. Then, from the above reasonings, it is sufficient if we consider only the case in which the abscissa of the point P is positive. Then $\Phi(u)$ has the form $\Phi(u)=Au$, where K is a positive integer, and the value ρ' of ρ becomes zero. In this case, μ is not the order of y in powers of x, consequently for this μ, there exists no integral of regular class. However, because $u(0)=0$, if we put $u=v$, then the equation (2.1) is transformed to the following equation

$$t\frac{dv}{dt}=\frac{Av^{K+R}+t\{Cqv^{R}Q(v,t)-pvP(v,t)\}}{\Sigma b_k v^{m-k}+tP(v,t)}, \tag{2.2}$$

where $P(v,t)$ and $Q(v,t)$ are regular in the vicinity of $(v=0,\ t=0)$. This is transformed to the equation of the form either (A) or (B) with $\chi=1$.

The equation (2.2) has not any integral of regular class. For, if there exists an integral of regular class, we can find μ_1 and ρ_1 such as $v=t^{\mu_1}(\rho_1+v_1)$ where $\rho_1 \neq 0$ and $v_1(0)=0$. Then we have either $y=t^{p}v=x^{\frac{p+\mu_1}{q}}(\rho_1+v_1)$ or $x=y^{\frac{p+\mu_1}{q}}(\rho_1+v_1)$, i.e. $y=x^{\frac{q}{p+\mu_1}}(\rho_1+v_1)^{-\frac{q}{p+\mu_1}}$. Namely the integral y has an order in powers of x. This is a contradiction, because the integral for the present side has not any order.

Next, we consider the exceptional case. In this case $\Phi(u)=Cqa_n-pb_0$. The side under consideration contains only one point O, therefore it can be rotated about O in the certain angle $\hat{O}$. If, by any rotation in the angle $\hat{O}$, we cannot choose p and q so that $\Phi=0$, then there does not exist ρ such as $\Phi=0$, namely there is no integral of regular class. In such a case, we exclude such side A_nB_0 from our Newton's polygon. If, by suitable rotation in the angle $\hat{O}$, we can choose p aud q so that $\Phi=0$, then, for such p

and q_{k} $\Phi(u)$ vanishes identically. In such a case we suppose that our Newton's polygon involves such side A_nB_0. Now, when A_n and B_0 coincide with the origin, there exists a Newton's polygon which has only the sides of finite length, except for the case when $m=n=0$. To that polygon, we add the side A_nB_0 only when there exists a suitable direction so that Φ vanishes identically, and the new polygon constructed in this way we call hereafter a Newton's polygon,[1] generalizing the sense of a polygon. When $m=n=0$, $R=1$ and $S=-1$, consequently the equation is of the form (A) with $\kappa=\lambda=1$. Otherwise, there exists a Newton's polygon and for the sides except A_nB_0, there arise only the cases already discussed. For A_nB_0, $\Phi(u)$ vanishes identically. Next, we shall consider such a case.

(iii) The expression $\Phi(u)$ vanishes identically.

In this case, the set of points A_l on the side under consideration coincides with the set of points B_k on the side, and the value ρ' of ρ is indeterminate, accordingly there exist an infinite number of integrals of regular class. We can assign to ρ an arbitrary non-zero constant ρ'. From (2. 1), we have

$$\frac{du}{dt}=\frac{Cqu^{R}Q(u,t)-pnuP(u,t)}{\Sigma b_k u^{m-k}+tP(u,t)}. \tag{2. 3}$$

Let $u=\rho'+v$, where $v(0)=0$, then (2. 3) is transformed to the equation of the form (C) or (D). This is transformed to the equation of the form (D) only when ρ' satisfies $\Sigma b_k \rho'^{m-k}=0$.

When there exists a determinate integral of regular class, namely in the case I, II, or III (i), we put $u=\rho'+v$ where ρ' is the finite non-zero value of ρ. Then it is easily seen that the equation (2. 1) is reduced to the equation of the form either (A) or (B). If we denote $\Sigma b_k u^{m-k}$ by $F(u)$, $F(\rho)$ is the coefficient of the term of lowest order of t in the expansion of $p\{t^p(\rho+v), t^q\}$. Then, we see that the equation (2. 1) is reduced to that of the form (B) only when $F(\rho')=0$, and that the new denominator $p_1(v,t)$ is related to the old one as follows:

$$p(y,x)=p\{t^p(\rho+v), t^q\}=t^N\{F(\rho)+v^M p_1(v,t)\}, \tag{2. 4}$$

where N aud M are integers, and $N>0$, $M\geqq 0$.

Summarizing the results, for the reduced equations of (1. 1), we have the following results:

Case	Newton's polygon	Integrals of regular class	Reduced equation: Form	Reduced equation: κ, λ
$m=0$ & $n>0$ or $n=0$ & $m>0$	none	none	(A)	$\kappa=1$, $\lambda\geqq 1$
$m=n=0$	none	none	(A)	$\kappa=1$, $\lambda=1$
	exists	exist infinitely		
I, II, III (i)	exists	exist	(A) or (B)	$\kappa=1$
III (ii)		none	(A) or (B)	
III (iii)		exist infinitely	(C) or (D)	

1) When a point A_l and a point B_k coincides with a vertex C of a Newton's polygon which has only sides of finite length, by executing the same reasonings on the vertex C as on the origin O, we can construct a generalized Newton's polygon. Hereafter we use the word "Newton's polygon" in this sense.

§ 3. Reduction of the equation of the form (B) or (D).

The equation of the form (D) is written as follows:

$$\frac{dv}{dt}=t^{\nu}v\lambda\frac{q_1(v,t)}{p_1(v,t)}=t^{\nu}v\lambda\frac{a_0v^n+a_1t^{r_1}v^{n-1}+\dots+a_nt^{r_n}+\dots}{b_0v^m+b_1t^{s_1}v^{m-1}+\dots+b_mt^{s_m}+\dots},\tag{3.1}$$

where $\nu\geqq 0$, a_0, b_0, a_n, $b_m\neq 0$, $m\geqq 1$ and r_i, s_k are positive integers. Taking only the terms of lowest order of v and t in the expansions of $q_1(v,t)$ and $p_1(v,t)$ as written in (3.1), we again apply the processes of § 2 to (3.1). Because of the condition that $m\geqq 1$ and $\nu\geqq 0$, there exists a Newton's polygon. Then we can determine p_1, q_1 and the value ρ'_1 of ρ_1[1] such that the substitution $t=t_1^{q_1}$, $v=t_1^{p_1}(\rho'_1+v_1)$ reduces the equation (3.1) to one of the equations of the forms (A), (B), (C) and (D). The equation (3.1) is reduced again to the equation of the form (D), only when the value ρ'_1 of ρ_1 becomes indeterminate and the term of lowest order of t_1 in the expansion of $p_1\{t^{p_1}(\rho_1+v_1),t^{q_1}\}$ vanishes for certain value ρ'_1 of ρ_1. If (3.1) is the reduced equation of (1.1), it is readily seen that the relation (2.4) holds good in this case also. Then, when the reduced equation of (3.1) is again of the form (D), by the repetition of the relation (2.4), it follows:

$$\begin{aligned}p(y,x)&=t^NF(\rho)+t^Nv^Mt_1^{N_1}\{F_1(\rho_1)+v_1^{M_1}p_2(v_1,t_1)\}\\&=t^NF(\rho)+t^Nt_1^{N_1}v^MF_1(\rho_1)+t^Nt_1^{N_1}v^Mv_1^{M_1}p_2(v_1,t_1),\end{aligned}$$

where $p_2(v_1,t_1)$ is the new denominator and $F(\rho')=F_1(\rho'_1)=0$.

Now, suppose that the equation of the form (D) recurs infinitely by the repetition of the above processes. Then, we obtain the infinite sequence of p, q, ρ'; p_1, q_1, ρ'_1; ...., such that the substitutions

$$\begin{cases}x=t^q,\\y=t^p(\rho'+v),\end{cases}\qquad\begin{cases}t=t_1^{q_1},\\v=t_1^{p_1}(\rho'_1+v_1),\end{cases}\qquad\begin{cases}t_1=t_2^{q_2},\\v_1=t_2^{p_2}(\rho'_2+v_2),\end{cases}\quad\dots,\tag{3.2}$$

into $p(y,x)$ make vanish the coefficients of each term of powers of t one after another. Namely, such p, q, ρ'; p_1, q_1, ρ'_1; .... give the expansion of y, satisfying $p(y,x)=0$, in powers of x. However, on the other hand, it is readily seen that the expansion of the function y, for which a singular algebroid polynomial vanishes, is determined in the same manner as we determine the expansion of the algebraic fnnction, namely, it is determined by means of Newton's polygon method, and that, as in the case of the algebraic function, there exists a finite number i such that $q_j=1$ for all $j\geqq i$. Then, if we put $t_{i-1}=\theta$ and $qq_1\dots q_{i-1}=r$, we have

$$\begin{cases}x=x(\theta)\equiv\theta^r,\\y=y(\theta)\equiv\rho'\theta^s+\rho'_1\theta^{s_1}+\dots+\rho'_i\theta^{s_i}+\theta^{s_i}v_i,\end{cases}\tag{3.3}$$

where $s=pq_1q_2\dots q_{i-1}$, $s_1=s+p_1q_2\dots q_{i-1}$, ..., $s_i=s_{i-1}+p_i$, and $v_i=v_i(\theta)\equiv\rho'_{i+1}\theta^{p_{i+1}}+\dots$ is regular in the vicinity of $\theta=0$ and vanishes there. When (3.3) is substituted into (1.1), the equation (1.1) is satisfied formally, and moreover $y(\theta)$ is regular in the vicinity of $\theta=0$. Therefore $y(\theta)$ is an integral of (1.1). Then, substituting (3.3) into (1.1),

1) When the equation determining ρ' is of the form $A\rho^L=0$ where $L\leqq 0$, we say that the value ρ' of ρ is an infinity, and we consider the value of ρ inclusive of an infinity.

we have

$$O \equiv p\{y(\theta), x(\theta)\}\frac{dy}{d\theta} = x^S y^R q\{y(\theta), x(\theta)\} e^{G(y,x)} \frac{dx}{d\theta}.$$

Therefore $q\{y(\theta), x(\theta)\} \equiv 0$. Thus $p(y, x)$ and $q(y, x)$ have a common factor $D(y, x)$, which is a singular algebroid polynomial and vanishes identically for $y=y(\theta)$, $x=x(\theta)$. This is a contradiction, for we have at first assumed that $p(y, x)$ and $q(y, x)$ are relatively prime.

Thus, we see that *the equation of the form (1.1) can be reduced, by the finite repetition of Newton's polygon processes, to one of the equations of the forms (A), (B) and (C).*

The equation of the form (B) can be written as follows:

$$(3.4) \qquad \frac{dv}{dt} = t^{-\kappa} v^{\lambda} \frac{q_1(v,t)}{p_1(v,t)} = t^{-\kappa} v^{\lambda} \frac{a_0 v^n + a_1 t^{r_1} v^{n-1} + \dots + a_n t^{r_n} + \dots}{b_0 v^m + b_1 t^{s_1} v^{m-1} + \dots + b_m t^{s_m} + \dots},$$

where $\varkappa \geqq 1$, a_0, b_0, a_n, $b_m \neq 0$, $m \geqq 1$ and r_i, s_k are positive integers. When $n=0$, interchanging v with t, we have the equation of the form either (A) or (C). Therefore we may assume that $n \geqq 1$. Taking only the terms of lowest order of v and t in the expansions of $q_1(v, t)$ and $p_1(v, t)$ as written in (3.4), we again apply the processes of § 2 to (3.4). Because $m \geqq 1$ and $n \geqq 1$, there exists a Newton's polygon. Then we can determine p_1, q_1, and the value ρ'_1 of ρ_1 such that the substitution $t=t_1^{q_1}$, $v=t_1^{p}(\rho'_1+v_1)$ reduces the equation (3.4) to one of the equations of the forms (A), (B), (C) and (D). But, when the value ρ'_1 becomes an infinity, we must adopt the substitution $v=t_1^{q_1}$, $t=t_1^{p_1}v_1$ instead of the former substitution.

If the equation (3.4) has not any integral of regular class for some side of the Newton's polygon, then, according to the results of § 2, for that side, (3.4) is reduced to the equation of the form (A) or (B) with $\varkappa=1$, and, in this case, the reduced equation has not any integral of regular class. If the equation (3.4) has at least one integral of regular class for some side,[1] then, for that side, (3.4) is reduced to one of the equations of the forms (A), (B), (C) and (D), by means of the substitution $t=t_1^{q_1}$, $v=t_1^{p_1}(\rho'_1+v_1)$ where $\rho'_1 \neq 0$, and moreover, (3.4) is reduced to the equation of the form either (B) or (D) only when the term of lowest order of t_1 in the expansion of $p_1\{t_1^{p_1}(\rho_1+v_1), t_1^{q_1}\}$ vanishes for $\rho_1=\rho'_1$. Suppose that, when we repeat the above processes, every reduced equation has at least one integral of regular class and is of the form either (B) or (D). Then, if (3.4) is the reduced equation of (1.1), then, by the analogous reasonings as in the reduction of the equation of the form (D), we see that $p(y, x)$ and $q(y, x)$ have a common factor $D(y, x)$. This is a contradiction. Thus, we see that, after the finite repetition of our reductions, the reduced equation has not any integral of regular class, or is of the form either (A) or (C). The equation of the form either (B) or (D) having not any integral of regular class is reduced to the equation of the form (A) or (B) with $\varkappa=1$.

Summarizing the results, we see that *the equation (1.1) is reduced to one of the equations of the forms (A), (C), and (B) with $\varkappa=1$ which has not any integral of regular class.*

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1) We include the case where there exist an infinite number of integrals of regular class.

§ 4. Reduction of the equation of the form either (A) or (C).

The equation of the form (C) can be written as follows:

$$(4.1)\quad \frac{dv}{dt}=t^{\nu}v^{\lambda}q_1(v,t)=t^{\nu}v^{\lambda}(a_0v^n+a_1t^{r_1}v^{n-1}+\dots+a_nt^{r_n}+\dots),$$

where $\nu\geqq 0$, a_0, $a_n\neq 0$, and r_i are positive integers. We again apply Newton's polygon method to this equation, namely, on the plane with two perpendicular axes, we mark the point $B_0(1-\lambda,\ 0)$ and the points $A_l(n-l,\ r_l+1+\nu)$. When $\lambda\geqq 1$, there exists no Newton's polygon. When $\lambda\leqq 0$, there exists a Newton's polygon, and, because $r_l+1+\nu\geqq 1$, the point B_0 does not coincide with any point A_l. Therefore, in this case, there exist a finite number of integrals of regular class, consequently the order $\mu=p/q$ of v in powers of t is determined, and if we put

$$(4.2)\quad t=\theta^q,\qquad v=\theta^p(\rho+V),$$

where $V(0)=0$, the finite non-zero value ρ' of ρ is determined. For the side of the Newton's polygon, there arise two cases.

Case I. The side contains B_0 and some A_l.

Substituting (4.2) into (4.1), we have

$$(4.3)\quad \theta^{-q\nu}\{p\theta^{p-1}(\rho+V)+\theta^p\frac{dV}{d\theta}\}/q\theta^{q-1}=\theta^{p\lambda}(\rho+V)^{\lambda}\{\Sigma a_l\theta^{qr_l+p(n-l)}(\rho+V)^{n-l}+\cdots\},$$

where Σ denotes the sum of the terms cerresponding to the points A_l on the side. Then $p-q(1+\nu)=p\lambda+q^{r_l}+p(n-l)=N$ say. After having divided both sides of (4.3) by θ^N, put $\theta=0$, then the finite non-zero value ρ' of ρ is certainly determined, and the equation (4.3) is reduced to the equation of the form (A) with $\varkappa=1$ and $\lambda\geqq 0$.

Case II. The side contains only the points A_l.

By the substitution (4.2) into (4.1), we have again (4.3). However, in this case, $p-q(1+\nu)>p\lambda+q^{r}l+p(n-l)=N$. Therefore, dividing both sides of (4.3) by θ^N, we have

$$(4.4)\quad \theta^{\sigma}\{p(\rho+V)+\theta\frac{dV}{d\theta}\}=q(\rho+V)^{\lambda}\{\Sigma a_l(\rho+V)^{n-l}+\theta P(V,\theta)\},$$

where $\sigma=p-q(1+\nu)-N\geqq 1$ and $P(V,\theta)$ is regular in the vicinity of $(V=0,\ \theta=0)$. Putting $\theta=0$ in (4.4), we can determine the finite non-zero value ρ' of ρ, and the equation (4.4) is reduced to the equation of the form (A) with $\varkappa\geqq 2$ and $\lambda\geqq 0$.

Thus, *the equation of the form (C) with $\lambda\leqq 0$, is reduced to the equation of the form (A) with $\varkappa\geqq 1$ and $\lambda\geqq 0$. The equation of the form (C) with $\lambda\geqq 1$ has not any integral of regular class.*

Next, we consider the equation of the form (A):

$$(4.5)\quad t^{\varkappa}\frac{dv}{dt}=v^{\lambda}(a_0v^n+a_1t^{r_1}v^{n-1}+\dots+a_nt^{r_n}+\dots),$$

where $\varkappa\geqq 1$, a_0, $a_n\neq 0$, and r_i are positive integers. We apply Newton's polygon method to this equation, namely we mark the point $B_0(1-\lambda,\ 0)$ and the points $A_l(n-l.\ r_l+1-\varkappa)$.

When $n=0$ and $\lambda\leqq 0$, (4.5) is transformed to the equation of the form (C) with $\lambda\geqq 1$, and therefore there exists no Newton's polygon. When $n=0$ and $\lambda\geqq 1$, there exists a Newton's polygon only when either $\varkappa\geqq 2$ and $\lambda\geqq 2$ or $\varkappa=\lambda=1$. When $\varkappa\geqq 2$ and $\lambda\geqq 2$, the point A_0 does not coincide with the point B_0, therefore there exists one integral of regular class. When $\varkappa=\lambda=1$, the point A_0 coincides with the point B_0. This case is discussed at the end of § 2, and the result is that, either there exists no integral of regular class or there exist an infinite number of integrals of regular class.[1] In either cases, the equation is of the form (A) with $\varkappa=\lambda=1$.

Next, we consider the case where $n\geqq 1$. There does not exist a Newton's polygon only when $\varkappa=1$ and $\lambda\geqq 1$. In the other case, there exists a Newton's polygon, consequently the order μ of y in powers of x is determined and can be put $\mu=p/q$, where p and q are relatively prime positive integers. For the sides of the polygon, there arise two cases.

Case I. The side contains only the points A_l.

Put $t=\theta^q$, $v=\theta^p(\rho+V)$, where $V(0)=0$. Substituting this into (4.5), we have:

$$\theta^{q\varkappa}\left\{p\theta^{p-1}(\rho+V)+\theta^p\frac{dV}{d\theta}\right\}/q\theta^{q-1}=\theta^{p\lambda}(\rho+V)^{\lambda}\left\{\Sigma a_l\theta^{qr_l+p(n-l)}(\rho+V)^{n-l}+\dots\right\}, \tag{4.6}$$

where Σ denotes the sum of the terms corresponding to the points A_l on the side under consideration. Then $p+q(\varkappa-1)>p\lambda+qr_l+p(n-l)=N$ say. Dividing both sides of (4.6) by θ^N, we have the same equation as (4.4). Put $\Sigma a_l\rho^{n-l}=G(\rho)$. Then, putting $\theta=0$ in (4.4), the value ρ' of ρ is determined by $G(\rho')=0$, and there exists a finite non-zero value ρ'. Then, for $\rho=\rho'$, (4.4) becomes as follows:

$$\theta^{\sigma+1}\frac{dV}{d\theta}=q(\rho'+V)^{\lambda}\left\{VG'(\rho')+\dots+\frac{V^n}{n!}G^{(n)}(\rho')\right\}+\theta Q(V,\theta), \tag{4.7}$$

where $Q(V,\theta)$ is regular in the vicinity of $(V=0,\ \theta=0)$. This is the equation of the form (A) with $\varkappa\geqq 2$ and $\lambda\geqq 0$.

Case II. The side contains B_0 and some points A_l.

By the same substitution as in the case I, we have (4.6). However, in this case, $p+q(\varkappa-1)=p\lambda+qr_l+p(n-l)=N$. Dividing both sides of (4.6) by θ^N, we have

$$p(\rho+V)+\theta\frac{dV}{d\theta}=q(\rho+V)^{\lambda}\{\Sigma a_l(\rho+V)^{n-l}+\theta P(V,\theta)\}. \tag{4.8}$$

Put $q\Sigma a_l\rho^{n-l}-p\rho^{1-\lambda}=\Phi(\rho)$. Putting $\theta=0$ in (4.8), we have $\Phi(\rho)=0$, namely the value ρ' of ρ is determined by the equation $\Phi(\rho')=0$. For the expression $\Phi(\rho)$, the following cases may occur.

(i) $\Phi(\rho)$ contains at least two terms.

In this case, there exists at least one finite non-zero value ρ' of ρ satisfying $\Phi(\rho')=0$. For this value ρ', from (4.8) it follows that

$$\theta\frac{dV}{d\theta}=(\rho'+V)^{\lambda}\{V\Phi'(\rho')+\frac{V^2}{2!}\Phi''(\rho')+\dots\}+\theta Q(V,\theta).$$

This is the equation of the form (A) with $\varkappa=1$ and $\lambda\geqq 0$.

(ii) $\Phi(\rho)$ contains only one term.

This case occurs only when the side contains only one point, with which B_0 and

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1) For example, consider the equation $t\frac{dv}{dt}=av$. The general integral is $v=ct^a$ where c is an arbitrary constant. Then, if the real part of a is positive, then, in general, v and t tend to zero simultaneously, but otherwise, in general, v does not tend to zero when t tends to zero. Namely, in the former case, there exist an infinite number of integrals of regular class of the order a and, in the latter case, there exists no integral of regular class.

certain A_l coincide. Then it must be $\lambda \leqq 1$, and the exponent of the remaining term in $\Phi(\rho)$ is equal to the abscissa of the coincident point P. Let the abscissa of the point P be K, then $K \geqq 0$.

When $K=0$, the point P is the origin, consequently it must be that $\lambda=1$ and $\varkappa=r_n+1$. As stated at the end of § 2, we can reduce this special case to the case when Φ vanishes identically.

When $K>0$, $\lambda \leqq 0$ and $\Phi(\rho)$ has the form $\Phi=Au^K$, consequently the value ρ' of ρ satisfying $\Phi(\rho)=0$ becomes zero. In this case we write V for $\rho+V$. Then, (4. 8) is transformed to the following equation.

(4. 9) $$\theta \frac{dV}{d\theta}=V^{\lambda}\{AV^{K}+\theta P_1(V, \theta)\},$$

where $P_1(V, \theta)$ is regular in the vicinity of $(V=0, \theta=0)$. This is of the form (A) with $\varkappa=1$, and, by the same reason as in § 2, has not any integral of regular class.

Now the side under consideration contains only two coincident points, therefore p and q are indeterminate. Therefore, except for the case where the point is the origin, for the arbitrary side which passes through the point P and lies in the suitable angle, namely, for arbitrary p and q satisfying certain conditions, we have the equation of the form (4. 9). Namely, the equation (4. 9) contains an arbitrary constant.

(iii) $\Phi(\rho)$ vanishes identically.

This case occurs only when B_0 coincides with certain A_l and the side contains only the point with which B_0 and A_l coincide. In this case the value of ρ is indeterminate and we can assign to it an arbitrary non-zero constant ρ'. For $\rho=\rho'$, the equation (4. 8) is transformed to the equation of the form:

(4. 10) $$\frac{dV}{d\theta}=q(\rho'+V)^{\lambda}P(V, \theta),$$

namely the equation of the form (C) with $\lambda \geqq 0$. Now, it is easily seen that the equation of the form (C) with $\lambda=0$ is reduced to the equation of the form (A) with $\varkappa=1$ and $\lambda \geqq 0$. Thus, in this case, the equation (4. 10) is reduced to the equation of the form either (A) with $\varkappa=1$ and $\lambda \geqq 0$ or (C) with $\lambda \geqq 1$.

When $n=0$, if $\varkappa \geqq 2$ and $\lambda \geqq 2$, then only side of the Newton's polygon is A_0B_0, therefore $p=\varkappa-1$ and $q=\lambda-1$. From (4. 8), the reduced equation becomes as follows:

$$\theta\frac{dV}{d\theta}=a_0q(\rho'+V)^{\lambda}-p(\rho'+V)+\dots\dots$$

$$=(\lambda-1)(\varkappa-1)V+\frac{\lambda(\lambda-1)}{2}a_0q\rho'^{\lambda-2}V^2+\dots\dots+a_0qV^{\lambda}+\dots\dots,$$

where $\rho'=\left(\frac{1}{a_0}\cdot\frac{\varkappa-1}{\lambda-1}\right)^{\frac{1}{\lambda-1}}$. This is of the form (A) in which $\varkappa=1$, $\lambda \geqq 0$ and $n+\lambda=1$.

Summarizing the results, we see that the equation of the form (A) is reduced in the manner indicated in the following schema:

Initial equation				Case	Reduced equation		
n	$\kappa,\ \lambda$	Newton's polygon	Integrals of regluar class		Form	$\kappa,\ \lambda$	$\kappa+\lambda$
0	$\lambda\leqq 0$	none	none		(C)	$\lambda\geqq 1$	
	$\kappa=1$ & $\lambda=1$	none or exists	none or infinite		(A)	$\kappa=\lambda=1$	1
	$\kappa=1$ & $\lambda\geqq 2$	none	none		(A)	$\kappa\leqq 1$ & $\lambda\geqq 2$	
	$\kappa\geqq 2$ & $\lambda=1$					$\kappa\geqq 2$ & $\lambda=1$	1
	$\kappa\geqq 2$ & $\lambda\geqq 2$	exists	exist		(A)	$\kappa=1$ & $\lambda\geqq 0$	1
$n\geqq 1$	$\kappa=1$ & $\lambda\geqq 1$	none	none		(A)	$\kappa=1$ & $\lambda\geqq 1$	
	$\kappa=1$ & $\lambda\leqq 0$ or $\kappa\geqq 2$	exists	exist	I	(A)	$\kappa\geqq 2$ & $\lambda\geqq 0$	
				II (i)	(A)	$\kappa=1$ & $\lambda\geqq 0$	
			none	II (ii)	(A)	$\kappa=1$	
			infinite	II(iii)	(A)	$\kappa=1$ & $\lambda\geqq 0$	
					(C)	$\lambda\geqq 1$	

From the above schema, we see that, for the reduced equation which is of the form (A), except the case when $n+\lambda=1$, χ does not become unity only when, for the original equation of the form (A), there exists an integral of regular class and moreover the side contains only the points A_l. In this case, from (4. 7), we see that the reduced equation has the form:

$$\text{(4. 11)}\qquad \theta^{\kappa'}\frac{dV}{d\theta}=a'V^{n'_1}+\ldots=V^{\lambda'}(a'V^{n'}+\ldots),$$

where $\chi'=\sigma+1$, $1\leqq n'+\lambda'=n'_1\leqq n$, and $\lambda'\geqq 0$. When $n=n'_1$, ρ' is a n-tiple root of the equation $G(\rho)=0$, consequently $G(\rho)=a_0(\rho-\rho')^n$. Then all the points A_l lies on the side under consideration, therefore it follows that $q=1$ and $N=p\lambda+qr_l+p(n-l)=p(\lambda+n)$. Then $\chi'=\sigma+1=\chi-p(n+\lambda-1)$. Namely, when $n+\lambda\geqq 1$, χ decreases except for the case $n+\lambda=1$. But, after the first reduction, from (4. 11), $n+\lambda\geqq 1$ aud $\lambda\geqq 0$. Then we see that, by the finite repetition of the above reductions, the equation of the form (A) is reduced to the equation of the form (A) with either $\chi=1$ or $n+\lambda=1(\lambda\geqq 0)$, whenever the reduced equation is of the form (A).

Thus we see that *the equation of the form (A) is reduced to the equation of the form either (A) or (C), and that, for the reduced equation of the form (A), either* $\chi=1$ *or* $n+\lambda=1(\lambda\geqq 0)$.

§ 5. Conclusion.

Summarizing the results, we have the following conclusion:

The differential equation $\dfrac{dy}{dx}=f(y,x)$, *in the vicinity of the singularity of the second kind, is reduced to one of the equations of the following forms:*

(A$_1$) $t^{\kappa}\dfrac{dv}{dt}=av+pt+\dots$; $\kappa\geqq 2,\ \ a\neq 0,$

(A$_2$) $t\dfrac{dv}{dt}=v^{\lambda}(av^n+pt^m+\dots)$; $a,\ p\neq 0,\ n\geqq 0,\ m>0,$ and $\lambda\geqq 1$ when $n=0,$

(C$_o$) $\dfrac{dv}{dt}=t^{\nu}v^{\lambda}(av^n+pt^m+\dots)$; $a,\ p\neq 0,\ n\geqq 0,\ m>0,\ \nu\geqq 0,\ \lambda\geqq 1;$

(B$_o$) $t\dfrac{dv}{dt}=\dfrac{av^n+pt+\dots}{bv^m+qt+\dots}$; $a,\ b\neq 0,\ m\geqq 1,\ n\geqq 1.$

Here the equations (A_2) with negative λ, (Co) and (Bo) represent the equations which have not any integral of regular class.

If we assume, as Forsyth does, that each equation has a determinate integral of regular class, then, after the first reduction, the reduced equation is of the form either (A) or (B). Next, from the discussions in § 3, it follows that the equation of the form (B) is reduced to the equation of the form (A), by the finite repetition of Newton's polygon methods. By the results of § 4, the equation of the form (A) is reduced to the equation of the form (A) with $\lambda\geqq 0$ and either $\kappa=1$ or $n+\lambda=1$. Thus, under the assumptions which Forsyth has laid, the original equation $\dfrac{dy}{dx}=f(y,x)$ is reduced to the equation of the form (A$_1$) or (A$_2$) with $\lambda\geqq 0$. The final reduced forms are special forms of the type I which Forsyth has derived. Consequently the equation of the type I is the one admitting more reduction to the simpler forms, namely the forms (A$_1$) and (A$_2$) just obtained, and, in this sense, the type I is not the satisfactory form as the final reduced form.

In conclusion, we wish to direct attention to the following fact. Our reduction is carried on by means of Newton's polygon method. Accordingly our reduced equation, in general, is the equation which the reduced part[1] of an integral of regular class of the original equation satisfies. Therefore, if the original equation, besides integrals of regular class, has other integrals, then such integrals, in general, are independent of the reduced equations. Thus, in general, the set of reduced equations is not equivalent to the original equation, but it rather expresses a restricted equation.

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1) When an integral of regular class is expressd as follows: $y=x^{\mu}(\rho+v)$ where $\rho\neq 0$ and $v(0)=0$, we have called the function v "a reduced part of the integral y".