SOME GENERAL THEOREMS AND CONVERGENCE THEOREMS IN VECTOR LATTICES

By

Tôzirô OGASAWARA

(Received Nov. 30, 1948)

Several attempts have been made to investigate the fundamental properties of vector lattice. Among them most important are the spectral theory developed by F. Riesz[1] and H. Freudental[2], and the representation theory by S. kakutani[3], M.H. Stone[4] and others. In my previous works I investigated the general properties of vector lattice, representation theories of vector lattice and linear operators. The purpose of the present paper is to make some additional remarks on my previous works[5] (Part 1) and to study the convergence character of linear operators with range in vector lattice (Part 2). Especially in § 5 (Part 1) I shall introduce a complex Banach lattice $Z=\{X, X\}$ and show that the properties of Z are reduced to those of its component real Banach lattice X, e.g., Z is reflexive if and only if X is reflexive.

Part 1 Some General Theorems on Vector Lattice

§ 1. Remarks on (o)-bounded linear operators. Let X be a vector lattice and Y a complete vector lattice. Let T be an (o)-bounded linear operator from X to Y. If $x_n \rightarrow 0$ (o) implies $T.x_n \rightarrow 0$ (o), then we say that T is (o)-continuous. If in this statement directed sets play a roll instead of simple sequences, that is, $x_\delta \rightarrow 0$ (o) implies $T.x_\delta \rightarrow 0$ (o), then we say that T is (o)- continuous in the sense of Moore and Smith, or MS-continuous.

THEOREM I. 1. If T is (o)-continuous, then so are T_+, T_-, $|T|$.

PROOF. Let $x_n \downarrow 0$, and put $y=\bigwedge_n T_+.x_n$, then $T_+. x_n - T.(a\frown x_n) \leqq T_+.x_1 - T.a$ for $0\leqq a \leqq x_1$. By making use of this inequality we obtain $y \leqq T_+.x_1 - T.a$, since $a\frown x_n \downarrow 0$ and T is (o)-continuous. But by definition $T_+.x_1 = \bigvee_{0\leqq a \leqq x_1} T.a$, and so $y=0$.

THEOREM I. 2. If T is MS-continuous, then so are T_+, T_-, $|T|$.

The proof is very similar to that of Theorm I. I.

By an ideal J of X we mean a linear subset of X such that $|u| \leqq |v|$, $v \varepsilon J$ implies $u \varepsilon J$, and by a normal ideal the totality of elements of X orthogonal to each element of some subset of X. A normal ideal N of a complete vector lattice is characterized as a direct component of this vector lattice, or as such an ideal that l. u. b. of a subset of N,

(1) F. Riesz, Ann. of Math., **41** (1940), 174-209.
(2) H. Freudenthal, Proc. Acad. Amsterdam, **39** (1936) 641-651.
(3) S. Kakutani, Proc. Imp. Acad. Tokyo, **16** (1940) 63-67.
(4) M.H. Stone, Proc. Nat. Acad. Sci., **26** (1940), 280-283. Do **27** (1941), 83-87.
(5) T. Ogasawara, J. Sci Hiroshima Univ., **12** (1942), 37-100. Do **13** (1944), 41-162. (in Jap.). Lattice Theory II (1948) (in Jap.) which we refer to L.T.

if it exists, belongs to N. The totality of (o)-bounded linear operators from X to Y forms a complete vector lattice L. Let L' be the totality of (o)-continuous linear operators from X to Y.

THEOREM I. 3. L' is a normal ideal of L.

PROOF. It is clear that L' is an ideal of L. Let S be l.u.b. of a directed set of positive elements $T\delta \varepsilon L'$, then by definition we have $S.a = \bigvee T\delta.a$ for any $a \geqq 0$. Let $x_n \downarrow 0$, then $S.x_n - T\delta \cdot x_n \leqq S.x_1 - T\delta \cdot x_1$, and so $\bigwedge S \cdot x_n \leqq S.x_1 - T\delta \cdot x_1$, hence $\bigwedge S \cdot x_n = 0$. Thus we obtain the theorem.

Let L'' be the totality of MS-continuous operators from X to Y.

THEOREM I. 4. L'' is a normal ideal of L.

With slight modification the proof follows word by word the proof of Theorem 1. 3.

Let E be a Banach space or a Fréchet space, and let T be a linear operator from E to X. We say with L. Kantorovitch[(6)] that T is H_b^t-continuous when $\|u_n\| \to 0$ implies $T.u_n \to 0$ (*). Operators of this type will be considered in part 2.

§ **2.** Conjugate vector lattice. Let X be a vector lattice. By the conjugate vector lattice $\bar{X}$ of X is meant the totality of (o)-bounded linear functionals $f(x)$ on X. If X is σ-complete and satisfies the ascending $\aleph_1$-chain condition:

(2. 1) There exists no transfinite sequence of elements x_a such that $0 \leqq x_1 < x_2 < \dots < x_a < \dots$, $a < \Omega$, where Ω is the first ordinal of 3^{rd} class,

then it is clear that X is complete, and l.u.b. of a subset H of X is l.u.b. of a properly chosen enumerable subset of H. And $f(x)$ is MS-continuous if and only if it is (o)-continuous. This statement is also true if we assume the descending $\aleph_1$-chain condition:

(2. 2) There exists no transfinite sequence of elements x_a such that $x_1 > x_2 > \dots > x_a > \dots > 0$, $a \epsilon \Omega$, where Ω is the first ordinal of 3^{rd} class.

LEMMA 2. 1. If X is such that there exist positive $f_n \varepsilon \bar{X}$, $n = 1, 2, \dots$ satisfying

(2. 3) $f_n(|x|) = 0$, $n = 1, 2, \dots$, imply $x = 0$,

then X satisfies (2. 1).

PROOF. The contrary implies a sequence $\{x_n\}$ of (2.1). But for some ordinal a we have $f_n(x_a) = f_{n+1}(x_a)$, $n = 1, 2, \dots$. Then (2. 3) shows that $x_a = x_{a+1}$, which is a contradiction.

Let J be an ideal of X, and let us designate by x^* the set of elements congruent to x mod J. Then X/J is a vector lattice with elements x^*. If $f \varepsilon \bar{X}$ annuls J, and if we put $F(x^*) = f(x)$, then $F \varepsilon \overline{(X/J)}$. Conversely if we write $f(x) = F(x^*)$ for any $F \varepsilon \overline{(X/J)}$, then $f \varepsilon \bar{X}$ and annuls J.

THEOREM 2. 1. Let X be σ-complete and let $0 < f_n \varepsilon \bar{X}$, $n = 1, 2, \dots$, be (o)-continuous. If we put $J = \{x; f_n(|x|) = 0, n = 1, 2, \dots\}$, then X/J is a complete vector lattice satisfying the ascending $\aleph_1$-chain condition and $F \varepsilon \overline{(X/J)}$ is MS-contiuuous if and only if f is (o)-continuous and anuuls J, otherwise stated, f belongs to the normal ideal N generated by f_n, $n = 1, 2, \dots$.

(6) L. Kantorovitch, Recueil Math., **49** (1940), 209-284.

PROOF. It is clear that J is a σ-ideal and F_n is (o)-continuous, and that F_n, $n=1, 2,\dots$, satisfy (2. 3) for X/J. Then Lemma 2. 1 shows that the first part of the theorem is true. Therefore F is MS-continuous if and only if F is (o)-continuous, which leads to the statement that f is (o)-continuous and annuls J. Consider any positive $f\varepsilon N$, then $f=\bigvee_n \{f\frown n\sum_1^n f_\nu\}$. It is readily seen that $f\frown n\sum_1^n f_\nu$ is (o)-continuous and annuls J. As a consequence of Theorem 1. 3 f is (o)-continuous and annuls J, and so F is MS-continuous. Conversely let F be MS-continuous. For our purpose it will be sufficient to assume that $f\frown f_n=0$, $f_n\leqq f_{n+1}$, $n=1, 2,\dots$, and to show that $f=0$. Suppose that $f(e)>0$ for some $e>0$. Since F, F_n are MS-continuous and $F\frown F_n=0$, $F_n\leqq F_{n+1}$, we can conclude (L.T. 36, Lemma 3) that there exist sequences $\{'e_n{}^*\}$, $\{''e_n{}^*\}$ such that $e^*='e_n{}^*+''e_n{}^*$, $'e_n{}^*\frown''e_n{}^*=0$, $'e^*_{n+1}\leqq 'e_n{}^*$ and $F_n('e_n{}^*)=F(''e_n{}^*)=0$. If we put $'e^*=\bigwedge 'e_n{}^*$, then $F('e^*)=lim\ F('e_n{}^*)=F(e^*)>0$. But $F_n('e^*)\leqq F_n('e_n^*)=0$ implies $'e^*=0$, and so $F('e^*)=0$. This is a contradiction.

COROLLARY. Let X be a σ-complete vector lattice such that any $f\varepsilon\bar{X}$ is (o)-continuous and there exist f_n satisfying (2. 3), then X is a complete vector lattice satisfying (2. 1) and $\bar{X}$ coincides with the normal ideal generated by f_n, $n=1, 2,\dots$. And any $f\varepsilon\bar{X}$ is MS-continuous. If $f=f_n$, $n=1, 2,\dots$, then f is a unit of $\bar{X}$.

As a partial converse of this corollary we have

LEMMA 2. 2. Let X be a vector lattice satisfying

(2. 4) There exist positive $f_n\varepsilon\bar{X}$, $n=1, 2,\dots$, such that $f\frown f_n=0$, $n=1, 2,\dots$, imply $f=0$,

(2. 5) For $x>0$, there exists $f>0$ such that $f(x)>0$,

then X satisfies (2. 3).

PROOF. Suppose that $f_n(x)=0$, $n=1, 2,\dots$, for an $x>0$. Using (2. 4) we obtain $f=\bigvee_n\{f\frown n\sum_1^n f_\nu\}$ for any $f\geqq 0$, whence $f(x)=0$. (2. 5) shows that $x=0$, which is a contradiction.

Further we shall consider the conditions which X may possess.

(2. 6) Let D be any directed set of positive elements of X. If $\underset{x\in D}{Sup}\, f(x)<+\infty$ for any $f>0$, $f\varepsilon X$, then $\bigvee_{x\in D} x$ exists.

(2. 7) Any $f\in\bar{X}$ is MS-continuous.

(2. 8) Any $\xi\in\bar{\bar{X}}$ is MS-continuous.

Clearly (2. 6) implies (2. 5). If X satisfies (2. 5), then X is Archimedean and is considered as a subset of $\bar{\bar{X}}$. When $X=\bar{\bar{X}}$ holds, we say that X is reflexive. In my previous works (L.T. 75. Theorem 2) I established

THEOREM 2. 2. (2.6)——(2. 8) is a necessary and sufficient condition for X to be reflexive. More precisely (2.6) (2.7) imply that X is the normal ideal N of $\bar{\bar{X}}$ consisting of all MS-continuous $\xi\in\bar{\bar{X}}$. (2.5) (2.7) imply that X is an ideal of $\bar{\bar{X}}$ and generates the normal ideal N.

We shall designate by N_X the normal ideal N indicated in Theorem 2. 2. It seems to be of some interest to investigate the possibility of replacing (2. 6), (2. 7), and (2. 8), partly or wholly, by

(2. 6′) Let $0 \leqq x_1 \leqq x_2 \leqq \ldots\ldots$ If $\underset{n}{Sup}\, f(x_n) < +\infty$ for each positive $f \varepsilon \bar{X}$, then $\vee x_n$ exists.

(2. 7′) Any $f \varepsilon \bar{X}$ is (o)-continuous.

(2. 8′) Any $\xi \varepsilon \bar{\bar{X}}$ is (o)-continuous.

LEMMA 2. 3. If there exist positive elements $e_n \varepsilon X$ such that

(2. 9) For any $f \geqq 0$, $f(e_n)=0$, $n=1, 2, 3, \ldots$, imply $f=0$,

then $\bar{X}$ satisies the ascending $\aleph_1$-chain condition, whence (2. 8′) is equivalent to (2.8).

LEMMA 2. 4. If $\bar{\bar{X}}$ is a σ-complete vector lattice and satisfies (2. 7′) and also

(2. 10) There exist positive elements $e_n \varepsilon X$ such that $x \frown e_n = 0$, $n = 1,2,3,\ldots$, imply $x=0$,

then $\{e_n\}$ satisfies (2. 9).

The proof of these lemmas is very similar to those of Lemmas 2. 1 and 2. 2.

By making use of the lemmas established in this § we obtain immediately

THEOREM 2. 3. If X satisfies (2. 4), (2. 10), then (2. 6′)——(2. 8′) is necessary and sufficient for X to be reflexive.

§ **3.** Regular vector lattice. For the later purpose we give

DEFINITION 3. 1. A σ-complete vector lattice will be said to be regular if it satisfies (2.2) and also[7]

(3. 1) If $x_{m,n} \downarrow 0$ for each m as $n \to \infty$, then there exists an increasing sequence of positive integers n_m such that $x_{m,n_m} \to 0(o)$.

DEFINITION 3. 2. A regular vector lattice X will be said to be of type (K') if it satisfies

(3. 2) A subset $E \leqq X$ is (o)-bounded if $\lambda_n \downarrow 0$, $x_n \varepsilon E$ implies $\lambda_n x_n \to 0(o)$.

DEFINITION 3. 3. A regular vector lattice will be said to be of type (K) if it satisfies

(3. 3) If $0 \leqq x_{m,n} \uparrow +\infty$ for each m as $n \to +\infty$, then there exists an increasing sequence of positive integers n_m such that $\{x_{m,n_m}\}$ is not (o)-bounded.

A regular vector lattice of type (K') $\{(K)\}$ is equivalent to that of type $\bar{K_6}$ $\{K_6\}$ introduced by L. Kantorovitch.[8] (3. 1) implies the equivalence of (o)-convergence and relative uniform (o)-convergence, and therefore implies (2. 7′). Owing to the descending $\aleph_1$-chain condition any $f \varepsilon \bar{X}$ is MS-continuous. By making use of Lemma 2. 3 a regular vector lattice with unit is reflexive if and only if it satisfies (2. 6′) and (2. 8′).

DEFINITION 3. 4. A vector lattice will be said to be a Bochner lattice if there exist positive (o)-continuous $f_p(x)$ such that

(3. 4) If $0 \leqq x_1 \leqq x_2 \leqq \ldots$, $\underset{n}{Sup}\, f_p(x_n) < +\infty$, $p=1, 2, \ldots$, then $\vee x_n$ exists.

This condition is a slight modification of the condition (L) given by S. Bochner.[9]

DEFINITION 3. 5. A vector lattice X will be said to be a Fréchet lattice if to each element $x \varepsilon X$ there corresponds a real number $\|x\|$, called quasi-norm, such that

(3. 5) $\|x\| \geqq 0$ and $\|x\| = 0$ if and only if $x=0$,

(7) S. Orihara, Proc. Imp. Acad. Tokyo, **18** (1942), 525-529.
(8) L. Kantorovitch, Recueil Math., **44** (1937), 121-165.
(9) S. Bochner, Proc. Nat. Acad. Sci., **26** (1940), 29-31.

(3. 6) $\|x+y\| \leqq \|x\|+\|y\|$, $\lim_{\lambda\to 0}\|\lambda x\|=0$, $\lim_{\|x\|\to 0}\|\lambda x\|=0$.

(3. 7) $|x| \geqq |y|$ implies $\|x\| \geqq \|y\|$.

(3. 8) X is complete in the norm sense.

If X satisfies the first three conditions, we shall say that X is a quasi-normed vector lattice.

In my previous works (L.T. 78-80) I established the results: A quasi-normed vector lattice is a regular Fréchet lattice if and only if it satisfies

(3. 9) $x_n \downarrow 0$ implies $\|x_n\| \to 0$,

(3. 10) If $0 \leqq x_1 \leqq x_2 \leqq \ldots$, $\lim_{n\to\infty} \lim_{p\to\infty} \|x_{n+p}-x_n\|=0$, then $\bigvee x_n$ exists.

And a regular Fréchet lattice is of type (K') if and only if it satisfies

(3. 11) If $0 \leqq x_1 \leqq x_2 \leqq \ldots$, $\lim_{\lambda\downarrow 0} \lim_{n\to\infty} \|\lambda x_n\|=0$, then $\bigvee x_n$ exists.

Let X be a Bochner lattice and if we put $\|x\| = \sum_1^\infty \frac{1}{2^n} \frac{f_n(|x|)}{1+f_n(|x|)}$, then it is easily verified that (3. 5)—(3. 7) and (3. 9)—(3. 11) hold, so that X is regular of type (K').

§ 4. Banach lattice.

DEFINITION 4. 1. A vector lattice will be said to be a normed vector lattice if to each element x there corresponds a real number $\|x\|$ which satisfies (3. 5), (3. 7) and also the conditions: $\|x+y\| \leqq \|x\|+\|y\|$, $\|\lambda x\| = |\lambda| \|x\|$. Moreover if it is complete in the norm sense, then we say that it is a Banach lattice.

DEFINITION 4. 2. A normed vector lattice X will be said to be a K space if it satisfies

(4. 1) If $0 \leqq x_1 \leqq x_2 \leqq \ldots$, $\operatorname{Sup}_n \|x_n\| < +\infty$, then there exists an $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Let X be a normed vector lattice. A linear funcional on X is (o)-bounded if it is bounded, and the converse is true if X is a Banach lattice. We shall mean the conjugate space $\bar{X}$ of X by the totality of the bounded linear functionals on X. Then $\bar{X}$ will be a Banach lattice with the norm $\|f\| = \operatorname{Sup}_{\|x\|\leqq 1} |f(x)|$. For any Banach lattice this definition of the conjugate space coincides with that given in § 2. K space is essentially equivalent to $\mathfrak{K}_2$ space introduced by L. Kantorovitch and B. Vulich.(10) In my previous works (L.T. 83-94) I established the following theorems:

THEOREM 4. 1. A Banach lattice is a K space if and only if one of the following conditions holds:

(4. 2) It is weakly complete.

(4. 3) It is regular of type (K'), or of type (K).

THEOREM 4. 2. If $\bar{X}$ is separable, then $\bar{X}$ is a K space.

As to the reflexivity of a Banach lattice holds

THEOREM 4.3. A Banach lattice is reflexive if and only if one of the following conditions holds:

(4. 4) It is locally weakly compact,

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(10) L. Kantrovitch and B. Vulich, Compositio Math., 5 (1939), 119—165.

(4. 5) It is weakly complete together with its conjugate space.

(4. 6) It is a K space together with its conjugate space.

It is remarkable that (4. 4) is sufficient for X to be reflexive, for it is not decided whether or not (4. 4) is sufficient for the reflexivity of a Banach space.

THEOREM 4. 4. If $\bar{\bar{X}}$ is separable, then the Banach lattice X is reflexive.

DEFINITION 4. 3. A Banach lattice X will be said to be an abstract (L^p), or simply (AL^p), if it satisfies

(4. 7) $x \wedge y = 0$ implies $\|x+y\| = \{\|x\|^p + \|y\|^p\}^{\frac{1}{p}}$, $1 \leqq p \leqq +\infty$,

where for $p=+\infty$ the right side of the equation means $max\{\|x\|, \|y\|\}$.

A normed vector lattice satisfying (4. 7) becomes an (AL^p) by the completion in the norm sense.

THEOREM 4. 5. The conjugate space of any nomed vector lattice satisfying (4. 7) is an (AL^q), where $\frac{1}{p}+\frac{1}{q}=1$. (AL^p), $1 \leqq p < +\infty$ is a K space, and (AL^p), $1 < p < +\infty$ is reflexive. And (AL^p), $1 \leqq p < +\infty$ has a concrete representation as (L^p) on a properly chosen set.

§ **5.** Complex vector latice. Let X be an Archimedean vector lattice such that $(x^2+y^2)^{\frac{1}{2}}$ exists for any $x, y \in X$, where $(x^2+y^2)^{\frac{1}{2}}$ means $\bigvee_{\theta}\{cos\theta \cdot x + sin\theta \cdot y\}$. Let $\{X, X\}$ be the totality of couples $\{x, y\}$, where $x, y \in X$. And we define

(1) $\{x_1, y_1\} = \{x_2, y_2\}$ if and only if $x_1 = x_2$, $y_1 = y_2$,

(2) $\{x_1, y_1\} + \{x_2, y_2\} = \{x_1+x_2, y_1+y_2\}$,

(3) $(\lambda + i\mu)\{x, y\} = \{\lambda x - \mu y, \mu x + \lambda y\}$, where λ, μ are real numbers,

(4) $|\{x, y\}| = (x^2+y^2)^{\frac{1}{2}}$.

Put $z = x+iy = \{x, y\}$, and we say that x, y are the real and the imaginary part of z respectively. Then we define that $\{X, X\}$ is a complex vector lattice constructed from X, and we shall denote $\{X, X\}$ by Z in the subsequent part of this §.

LEMMA 5. 1. $|x|, |y| \leqq |z| \leqq |x| + |y|$, $|az| = |a|\,|z|$, $|z| = |\bar{z}|$, where $a = \lambda + i\mu$, $z = x+iy$.

We say that a compiex linear fuctional $f(z)$ on Z is (o)-bounded if $\underset{|z| \leqq x}{Sup} |f(z)| < +\infty$ for any $x \geqq 0$, that f is real if $f(x)$ is real for any $x \in X$ and that $f \geqq 0$ if f is real and $f(x) \geqq 0$ for any $x \geqq 0$.

LEMMA 5. 2. Any functional $f(z)$ on Z is uniquely represented as $g(z)+ih(z)$ where g, h are real and called the real and the imaginary part of f. And $f_1(z) \equiv f_2(z)$ if and only if $f_1(x) \equiv f_2(x)$ on X.

PROOF. Put $g(z) = \frac{1}{2}(f(z) + \overline{f(\bar{z})})$ and $h(z) = \frac{1}{2i}(f(z) - \overline{f(\bar{z})})$, then g, h are real and we obtain $f(z) = g(z) + ih(z)$. That such a representation is unique follows from the fact that g is real if and only if $\overline{g(z)} = g(\bar{z})$. The last part of the lemma is clear from the equation $f(z) = f(x) + if(y)$.

LEMMA 5. 3. Any linear functional $f(x)$ on X is uniquely extended to a real linear functional $f(z)$ on Z.

PROOF. It is enough to define $f(z)=f(x)+if(y)$ to prove the lemma.

LEMMA 5. 4. $f(z)$ is (o)-bounded if and only if its real and imaginary parts are (o)-bounded on X.

PROOF. It suffices to show the lemma when $f(z)$ is real. But it is clear from Lemma 5. 1.

Now we shall define the conjugate complex vector lattice $\bar{Z}$ of Z. $\bar{Z}$ is the totality of (o)-bounded functional $f(z)$ on Z. By making use of Lemmas 5. 2, 5. 4, we see that the real parts of such functionals form a complete vector lattice isomorphic with $\bar{X}$, and so we identify them. Clearly $\bar{X}$ satisfies the requirements stated in the beginning of this §.

LEMMA 5. 5. $g(x)+h(y)\leqq(g^2+h^2)^{\frac{1}{2}}\{(x^2+y^2)^{\frac{1}{2}}\}$, where $g, h\epsilon\bar{X}$, $x, y\epsilon X$.

PROOF. Let ξ, η, ζ be the functionals determined by $x, y, (x^2+y^2)^{\frac{1}{2}}$ such that $\xi(g)=g(x)$ and so on. Then they are MS-continuons. To prove the lemma it suffices to assume that $e=(g^2+h^2)^{\frac{1}{2}}$ is a unit of $\bar{X}$, and to show that $\xi(g)+\eta(h)\leqq\zeta(e)$. Let A be the Boolean lattice with e as unit in $\bar{X}$, and let us consider the Boolean space Ω corresponding to A. Then $\bar{X}$ is linear-lattice isomorphic with a vector lattice of continuous functions on Ω in which e corresponds to the constant function 1 (L.T. Chap. 4). We denote by $g(\mathfrak{p})$ the function corresponding to g. Then $g(\mathfrak{p})^2+h(\mathfrak{p})^2=1$. We can define a completely additive measure $m(E)$ on the family of Borel sets of Ω such that $m(G)=\zeta(a)$, where G is the bicompact open set representing $a\epsilon A$. Because of MS-continuity of ξ, η, ζ and Radon-Nikodym's theorem we can find continuous functions $\xi(\mathfrak{p}), \eta(\mathfrak{p}), \zeta(\mathfrak{p})$ such that $\xi(g)=\int_\Omega\xi(\mathfrak{p})g(\mathfrak{p})dm$, $\eta(g)=\int_\Omega\eta(\mathfrak{p})g(\mathfrak{p})dm$, $\zeta(g)=\int_\Omega\zeta(\mathfrak{p})g(\mathfrak{p})dm$. Then $(\xi^2+\eta^2)^{\frac{1}{2}}\leqq\zeta$ implies $\xi(g)+\eta(h)\leqq\int_\Omega\zeta(\mathfrak{p})\{g(\mathfrak{p})^2+h(\mathfrak{p})^2\}^{\frac{1}{2}}dm=\int_\Omega\zeta(\mathfrak{p})dm$. Therefore we obtain $\xi(g)+\eta(h)\leqq\zeta(e)$.

LEMMA 5. 6. $(g^2+h^2)^{\frac{1}{2}}(a)=Sup\{g(x)+h(y);\ (x^2+y^2)^{\frac{1}{2}}\leqq a\}$, where $g, h\epsilon\bar{X}$, $a\geqq0$, $a\epsilon X$.

PROOF. By Lemma 5. 5 it suffices to show that $(g^2+h^2)^{\frac{1}{2}}(a)\leqq Sup\{g(x)+h(y));\ (x^2+y^2)^{\frac{1}{2}}\leqq a\}$.

$$\begin{aligned}
&Sup\ \{g(x)+h(y);\ (x^2+y^2)^{\frac{1}{2}}\leqq a\}\\
&\quad= Sup\left\{\sum_{\nu=1}^{n}{}'(g(x_\nu)+h(y_\nu));\ \sum_{\nu=1}^{n}{}'|z_\nu|\leqq a\right\},\quad where\ z_\nu=x_\nu+iy_\nu,\\
&\quad= Sup\left\{\sum_{\nu=1}^{n}{}'(|g|(|x_\nu|)+|h|(|y_\nu|));\ \sum_{\nu=1}^{n}{}'|z_\nu|\leqq a\right\}\\
&\quad\geqq Sup\left\{\sum_{\nu=1}^{n}{}'(cos\theta_\nu.\ |g|(|z_\nu|)+sin\theta_\nu.\ |h|(|z_\nu|));\ \sum_{\nu=1}^{n}{}'|z_\nu|\leqq a\right\}\\
&\quad=(g^2+h^2)^{\frac{1}{2}}(a).
\end{aligned}$$

LEMMA 5. 7. For any $f\epsilon\bar{Z}$, we have $|f|(a)=\underset{|z|\leqq a}{Sup}|f(z)|=\underset{|z|\leqq a}{Sup}\Re f(z)=Sup\left\{\sum_{1}^{n}{}'|f(z_\nu)|;\ \sum_{1}^{n}{}'|z_\nu|\leqq a\right\}$ for any $a\geqq0$.

PROOF. For any $z=x+iy$ we can find a complex number $e^{i\theta}$ such that $\Re\{f(z)\}=f(e^{i\theta}z)$.

Therefore we obtain $\underset{|z|\leqq a}{Sup}\ |f(z)| = \underset{|z|\leqq a}{Sup}\ \Re\{f(z)\}$. And Lemma 5.6 and its proof show that $\underset{|z|\leqq a}{Sup}\ \Re\{f(z)\} = Sup\left\{\sum_1^n |f(z_\nu)|\ ;\ \sum_1^n |z_\nu| \leqq a\right\} = |f|(a)$.

As a consequence of Lemmas 5.2, 5.4 and 5.7 we have

THEOREM 5.1. $\bar{Z} = \{\bar{X}, \bar{X}\}$.

By virtue of this theorem most properties of a complex vector lattice $Z=\{\bar{X}, \bar{X}\}$ can be reduced to those of its real component X, e.g., Z is reflexive if and only if X is reflexive. Results obtained in the preceding § will be trivially true. So we shall confine ourselves to Banach lattices. Z is called a normed complex vector lattice if to each element $z \varepsilon Z$ there corresponds a real number $\|z\|$ satisfying the norm conditions completely similar to those given in Definition 4.1. Moreover if it is complete in the norm sense, then Z is called a complex Banach lattice. By an easily veified relation $\|x\|$, $\|y\| \leqq \|z\| \leqq \|x\| + \|y\|$ we see that a normed complex vector lattice is a Banach lattice if and only if X is a Banach lattice. The conjugate space of a normed complex vector lattice Z is, by definition, the totality of bounded linear functionals $f(z)$ on Z. We remark that $f(z)$ is bounded if and only if its real and imaginary parts are bounded on X.

THEOREM 5.2. The conjugate space of a normed complex vector lattice $Z=\{X, X\}$ is a complex Banach lattice denoted by $\bar{Z}=\{\bar{X}, \bar{X}\}$ where $\bar{X}$ means the conjugate space defined in § 4.

PROOF. Only to show that $|f| \geqq |g|$ implies $\|f\| \geqq \|g\|$. Using Lemma 5.7 we see that $\|f\| = \underset{\|z\|\leqq 1}{Sup}\ |f(z)| = \underset{\|x\|\leqq 1}{Sup}\ |f|(x) = \|\,|f|\,\| \geqq \|\,|g|\,\| = \|g\|$.

We shall say that Z is complete if X is a complete vector lattice. Then we have

THEOREM 5.3. Let Z be a separable complex vector lattice. If the norm $\|z\|$ is (o)-continuous, then Z is complete.

PROOF. Let $x_1 \geqq x_2 \geqq \ldots > 0$. Suppose that $\underset{n}{\wedge} x_n$ does not exist. Because of the separability of X we can find a sequence $\{y_n\}$ such that $0 < y_m \leqq y_{m+1} < x_n$, $(x_n - y_n) \downarrow 0$. Then by the assumption we have $\|x_n - y_n\| \to 0$. Using the inequality $0 \leqq x_n - x_{n+p} \leqq (x_n - y_n) - (x_{n+p} - y_{n+p})$ we see that there exists an x such that $\|x_n - x\| \to 0$. This implies $x = \wedge x_n$ which is a contradiction.

A normed complex vector lattice $Z=\{X, X\}$ is said to be a complex K space if the condition (4.I) is satisfied. Then Z is a K space if and only if X is a K space. Theorems 4.1, 4.2, 4.3 and 4.4 replaced X, $\bar{X}$ by Z, $\bar{X}$ hold also for a complex Banach lattice. Here that Z is regular means that X is regular, and so on.

LEMMA 5.8. Let $Z=\{X, X\}$ be a normed complex vector lattice such that

(I) $|z_1| \wedge |z_2| = 0$ implies $\|z_1 + z_2\| = \{\|z_1\|^p + \|z_2\|^p\}^{\frac{1}{p}}$, $1 \leqq p \leqq +\infty$ where for $p = +\infty$ the right side of the equation means $max\{\|z_1\|, \|z_2\|\}$, then the conjugate Banach lattice satisfies also (I) with q, where $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. Let $|f| \wedge |g| = 0$. Then $|f| + |g| = |f+g|$ and $\|f+g\| = \|\,|f| + |g|\,\|$. Hence Theorem 3.5 shows that $\|f+g\| = \{\|f\|^q + \|g\|^q\}^{\frac{1}{q}}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

A complex Banach lattice Z satisfying (I) is called a complex (AL^p). By making use of Lemma 5.8 it is easy to see that Theorem 4.5 holds also for the complex case.

Part 2 Convergence Theorems

§ 6. (o)-limit in an extended sense. Let X be a σ-complete vector lattice, and let $\mathbf{N}$ be the set of all normal ideals $\mathfrak{a}$ of X (L.T. Chap. 4). Then $\mathbf{N}$ is a complete Boolean lattice with respect to the inclusion relation. We denote by Ω_{N} the Boolean space corresponding to $\mathbf{N}$. And let $\mathfrak{L}$ be the totality of continuous functions $\varphi(\mathfrak{p})$ on Ω_{N} which are finite except for a set of 1st category. Then $\mathfrak{L}$ forms a complete vector lattice, in which the order relation $\varphi \geqq \psi$ is defined by $\varphi(\mathfrak{p}) \geqq \psi(\mathfrak{p})$ and $\varphi = \varphi_1 + \varphi_2$ is defined by a continuous function $\varphi(\mathfrak{p})$ which coincides with $\varphi_1(\mathfrak{p}) + \varphi_2(\mathfrak{p})$ at points $\mathfrak{p}$ where $\varphi_1(\mathfrak{p})$, $\varphi_2(\mathfrak{p})$ are finite. The statement $\varphi_n \to \varphi(o)$ is equivalent to the following: $\varphi_n(\mathfrak{p}) \to \varphi(\mathfrak{p})$ except for a set of 1st category. Thus the (o)-convergence in $\mathfrak{L}$ is the same as the pointwise convergence except for a set of 1st category. On the other hand, X is represented linear-lattice isomorphically by a subset of $\mathfrak{L}$ which we denote by $\mathfrak{L}_X$. Let $x(\mathfrak{p})$ be the representing function of $x \in X$. Then $X_n \to x(o)$ in X is equivalent to the statement that x_n are (o)-bounded in X and $x_n \to x(o)$ in $\mathfrak{L}$. We say that $x_n \to x(o)$ on e if $P_e x_n \to P_e x(o)$ where $P_e x$ denotes the component of x in the principal ideal generated by e, and similarly we say that x_n are (o)-bounded on e if $P_e x_n$ are (o)-bounded.

DEFINITION 6.1. $x_n \in X$ (o')-converge to $x \in X$ if for any given $0 < a \in X$ there exists $o < e \leqq a$ such that $x_n \to x(o)$ on e.

From the above discussion it is clear that $x_n \to x(o')$ in X is nothing more than $x_n \to x(o)$ in $\mathfrak{L}$.

DEFINITION 6.2. We say that $\bigvee |x_n| \equiv +\infty$ on a, where $x_n \in X$, $o < a \in X$, if there exists no $o < e \leqq a$ such that x_n are (o)-bounded on e.

It is clear that $\bigvee |x_n| \equiv +\infty$ on a means that $Sup_n |x_n(\mathfrak{p})| \doteqdot +\infty$ on the bicompact open set G except for a set of 1st category, where G is the set corresponding to the principal ideal generated by a.

Now we define an abstract (S), simply (AS), by generalizing the ordinary (S).

DEFINION 3.3. A σ-complete vector lattice X with unit e will be said to be (AS) if the following conditions are satisfied.

(6.1) There exists a complete additive non-negative functional $m(a)$ on the Boolean sub-lattice A of X, with e as unit, such that $m(a)$ implies $a=0$.

(6,2) If $x_n \wedge x_m = 0$ $(n \neq m)$, $n, m = 1, 2, \ldots$, then $\bigvee x_n$ exists.

Let X be an (AS). Then A will be lattice-isomorphic with $\mathbf{N}$, since A is complete by (6.1). By (6.2) we obtain $\mathfrak{L}_X = \mathfrak{L}$. Let e be represented by the constant function 1. We define the complete additive measure $m(E)$ such that if $G \leqq \Omega_N$ is the bicompact open set corresponding to $a \in A$, then $m(G) = m(a)$. Let $\frac{|x|}{e+|x|}$ be the element represented by $\frac{|x(\mathfrak{p})|}{1+|x(\mathfrak{p})|}$. If we define $\|x\|$ by $\int_{\Omega_{\mathrm{N}}} \frac{|x(\mathfrak{p})|}{1+|(\mathfrak{p})|} dm$, then it will be easily verified (L.T. 93-94) that the conditions (3.5)-(3.10) are satisfied and X will be a regular Fréchet lattice of type (K'). Thus X is an (S) considered on a properly chosen set. As well known any

(L) is a subspace of (S). The same situation holds also for a K space. To see this we consider a K space X with unit e, and let h be a unit of $\overline{X}$, then $h(a)$ is a complete additive non-negative functional on A, satisfying (6. I), and $\mathfrak{L}$ will be an (AS) with $\mathfrak{L}_X$ as its ideal. In this case $\mathfrak{L}$ may be obtained by another way (L.T. 93–94). For any $\varphi\varepsilon\mathfrak{L}$ we define $\frac{|\varphi|}{e+|\varphi|}$ by the element of X representing $\frac{|\varphi(\mathfrak{p})|}{1+|\varphi(\mathfrak{p})|}$, and we put $\rho(\varphi)=\|\frac{|\varphi|}{e+|\varphi|}\|$. Then $\mathfrak{L}$ will be a regular Fréchet lattice of type (K'), and will be obtained also by the completion of X with respect to the quasi-norm $\rho(x)$.

§ 7. Resonance theorems. In this and the following § § we assume that X is a σ-complete vector lattice, E a Fréchet or Banach space, and T_n a sequence of H_b^l-continuous operators from E to X. Now consider the following conditions:

(α) There exists a subset $\mathfrak{B}$ of (o)-continuous functionals $f(x)$ on X such that if $0\leqq x_1\leqq x_2\leqq\ldots$, $\underset{n}{Sup}\, f(x_n)<+\infty$ for each $f\varepsilon\mathfrak{B}$, then $\vee x_n$ exists.

(β) X is a regular Fréchet lattice, or a regular Banach lattice.

For example K space, conjugate vector lattice, conjugate Banach lattice, and Bochner lattice satisfy (α).

THEOREM 7. 1. Let (α) or (β) be satisfied. If there exists a set $H\leqq E$ of 2^{nd} category such that for any $u\epsilon H$

(7. I) $\{T_n,u\}$ is (o)-bounded,

then (7. I) holds for any $u\epsilon E$.

PROOF. *Case I.* Let (α) be satisfied. Take any $f\varepsilon\mathfrak{B}$, and we put $H_n^f=\{u;\ \underset{m}{Sup}\, f(|T_1\cdot u|\smile\ldots\smile|T_m\cdot u|)\leqq n\}$. Then clearly H_n^f is a closed subset of E and $H\leqq\underset{n}{\Sigma}H_n^f$. It is easy to see that $\underset{n}{\Sigma}H_n^f$ is a linear subspace of E which is a set of 2^{nd} category with Baire's property. Therefore we obtain $E=\Sigma H_n^f$ for any $f\varepsilon\mathfrak{B}$,(11) whence (7. I) must hold for any $u\varepsilon E$. *Case 2.* Let (β) be satisfied. Let E' be the set of all $u\epsilon E$ such (7. I) holds. Then E' is linear. If we put $H_{n,\nu}=\{u;\ \underset{p}{Sup}\||T_1\cdot u|\smile\ldots\smile|T_{\nu+p}\cdot u|-|T_1\cdot u|\smile\ldots|T_\nu\cdot u|\|\leqq\frac{1}{n}\}$, then $H_{n,\nu}$ is a closed subset of E and $E'=\underset{n}{\Pi}\underset{\nu}{\Sigma}H_{n,\nu}$, so that E' has Baire's property and is of 2^{nd} category. Therefore we have $E=E'$.

THEOREM 7. 2. Let (6. I) be satisfied. If there exists a subset $H\leqq E$ such that for any $u\varepsilon H$

(7. 2) $\{T_n\cdot u\}$ is (o)-bounded in $\mathfrak{L}$, where $\mathfrak{L}$ is a vector lattce described in § 6,

then (7. 2) holds for any $u\varepsilon E$

PROOF. Suppose that there exists a $u\varepsilon E$ such that $\vee T_n\cdot u\equiv+\infty$ on $a>0$. To reach a contradiction we may assume that a is a unit element of X. Then by (6. I) $\mathfrak{L}$ becomes an (AS), whence $\mathfrak{L}$ must satisfy (β). Theorem 7.I shows that this is contradictory.

THEOREM 7. 3. Let (β) be satisfied. If there exists a $u_0\epsilon E$ such that

(7. 3) $V|T_n\cdot u_0|\equiv+\infty$ on $e>0$,

then (7. 3) holds for any $u\varepsilon E$ except for a set of 1^{st} category.

PROOF. Let A be a Boolean sub-lattice with e as unit in X. If we suppose the

(11) C. Kuratowski, Topologie, I. (1933), 74.

contrary, there exists a set $H \leqq E$ of 2^{nd} category such that if $u \varepsilon H$, then $\{T_n \cdot u\}$ is (o)-bounded on some positive element of A. Put $K_n = \{u;\ \bigvee_k |T_k \cdot u| \leqq ne$ on $e_{u,n} \varepsilon A$, $\|e_{u,n}\| \geqq \frac{1}{n}\}$, then K_n is a closed subset of E. Indeed if $u_m \to u$, $u_m \varepsilon K_n$, then $T_k \cdot u_m \to T_k \cdot u(*)$. But we may assume that $T_k \cdot u_m \to T_k \cdot u$ (o), for if necessary it is sufficient to select a subsequence of $\{T_k \cdot u_m\}$ which (o)-converges to $T_k \cdot u$. If we put $e_{u,n} = (o)\text{-}\overline{lim}\ e_{u_m,n}$, then $\|e_{u,n}\| \geqq \frac{1}{n}$ and $\bigvee |T_k \cdot u| \leqq ne$ on $e_{u,n}$. Therefore K_n is closed. Because of the relation $\Sigma K_n \leqq H$ we infer that K_n contains a sphere $\|u-u_1\| \leqq \delta$. Consider $v \varepsilon E$ such that $\|v\| \leqq \delta$. Then $\|\lambda v + u_1 - u_1\| \leqq \delta$ for sufficiently small number $0 < \lambda < 1$. For some $\lambda, \lambda', \lambda \neq \lambda'$ we must obtain $e_1 = e_{\lambda v + u_1, n} \frown e_{\lambda' v + u_1, n} \neq 0$, since the contrary implies that the ascending $\aleph_1$-chain condition does not hold in X. Owing to the relation $(\lambda - \lambda')v = \lambda v + u_1 - (\lambda' v + u_1)$ we have $|T_k \cdot (\lambda - \lambda')v| \leqq 2ne$ on e_1 and therefore $|T_k \cdot v| \leqq \frac{2n}{a|\lambda - \lambda'|} e$ on e_1. Let a be a positive number such that $\|au_0\| \leqq \delta$, then $|T_k \cdot u_0| \leqq \frac{2n}{a|\lambda - \lambda'|} e$ on e_1, which is a contradiction.

§ **8.** Convergence theorems.

THEORM 8.1. Let (α) or (β) be satisfied. Suppose that the hypothesis of Theorm 7.1 is satisfied and also that there exists a dense subset D such that $\{T_n \cdot u\}$ (o)-converges for any $u \varepsilon D$, then $\{T_n \cdot u\}$ (o)-converges for any $u \varepsilon E$.

PROOF. Theorem 7.1 shows that $\{T_n \cdot u\}$ is (o)-bounded for any $u \varepsilon E$. Let $\widetilde{T} \cdot u = (o)\text{-}\overline{lim}\ T_n u - (o)\text{-}\underline{lim}\ T_n u$ and $V.u = \bigvee |T_n u|$. Clearly we obtain (1°) $|\widetilde{T} \cdot u - \widetilde{T} \cdot v| \leqq \widetilde{T}(u-v)$ (2°) $o \leqq \widetilde{T} \cdot u \leqq 2V.u$ (3°) $\widetilde{T} \cdot au = |a| \widetilde{T} \cdot u$ (4°) $V \cdot au = |a| V \cdot u$.

Case I. Let (α) be satisfied. Take any $f \varepsilon \mathfrak{B}$, and let $H_n^f = \{u;\ f(V \cdot u) \leqq n\varepsilon\}$ for any given positive number ε. Then H_n^f is closed and $E = \sum_n H_n^f$. Therefore an H_n^f contains a sphere $\|u - u_0\| \leqq \delta$. Let v be any element of E such that $\|v\| \leqq \delta$, then $f(V.v) \leqq 2n\varepsilon$. We can find a positive number δ' such that $\|u\| \leqq \delta'$ implies $\|2nu\| \leqq \delta$, whence $f(V \cdot v) \leqq \varepsilon$, that is, $f(V.u)$ is a continnous function of u. Using (1°), (2°) we see that $f(\widetilde{T} \cdot u)$ is continuous. Since $\widetilde{T} \cdot u = 0$ for $u \varepsilon D$, therefore $f(\widetilde{T} \cdot u) = 0$ for any $u \varepsilon E$ and so we obtain $\widetilde{T} \cdot u = 0$ for any $u \varepsilon E$. *Case 2.* Let (β) be satisfied. Put $H_n = \{u; \|V \cdot u\| \leqq n\varepsilon\}$. By a similar reasoning as above we can conclude that $\widetilde{T} \cdot u = 0$ for any $u \varepsilon E$.

THEOREM 8.2. Let (6.1) be satisfied. Suppose that the hypothesis of Theorem 7.2 is satisfied and also that there exists a dense subset $D \leqq E$ such that $\{T_n \cdot u\}$ (o)-converges in $\mathfrak{L}$ for any $u \varepsilon D$, then $\{T_n \cdot u\}$ (o)-converges in $\mathfrak{L}$ for any $u \varepsilon E$.

PROOF. Theorem 7.2 shows that $\{T_n \cdot u\}$ is (o)-bounded in $\mathfrak{L}$ for any $u \varepsilon E$. It is sufficient to show that $\{T_n \cdot u\}$ (o)-converges in $\mathfrak{L}$ on any positive e. By making use of Theorem 8.1 we can establish the theorem.

THEOREM 8.3. Let (β) be satisfied. Suppose that the hypothesis of Theorem 8.2 is satisfied, then $\{T_n \cdot u\}$ (o)-converges in $\mathfrak{L}$ for any $u \varepsilon E$.

PROOF. By the same reason as in the proof of the above theorem we can assume that X has a unit e. Then $\mathfrak{L}$ will be a regular Fréchet lattice by defining the quasi-norm $|||\varphi||| = \|\frac{|\varphi|}{e + |\varphi|}\|$ for any $\varphi \varepsilon \mathfrak{L}$. Hence by making use of Theorem 8.1 we can establish the theorem.

THEOREM 8.4. Let (β) be satisfied. If X has a unit e and there exists a dense subset $D \leqq E$ such that $\{T_n \cdot u\}$ (o)-converges in $\mathfrak{L}$ for $u \in D$, then there exist e_1, e_2 such that $e_1 + e_2 = e$, $e_1 \frown e_2 = 0$ and

(1) $\{T_n \cdot u\}$ (o)-converges in $\mathfrak{L}$ on e_1 for any $u \varepsilon E$,

(2) $\bigvee | T_n \cdot u | \equiv +\infty$ on e_2 except for a set of 1st category in E.

PROOF. Let A be a Boolean sub-lattice with e as unit. Let B be the totality of elements of A on which (2) holds. Then B is a σ-ideal of A. Since there exists no independent subset of A with power $\aleph_1$, B must be complete. Therefore B is a principal ideal generated by some e_2. Put $e_1 = e - e_2$. Then Teorem 8.3 shows that (1) holds.