THEORY OF THE SPHERICALLY SYMMETRIC SPACE-TIMES. VI. FORM-INVARIANT TENSORS UNDER GROUP OF MOTIONS AND PARALLEL TENSORS[1]

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Part I. Form-invariant tensors under group of motions

§ 1. Condition for form-invariancy

The concept of the "form-invariancy" is one of the important ideas in relativistic theories, namely it is closely connected with the equivalency of the observers.[2] In Part I, we shall obtain the general form of the form-invariant tensors under group of motions in an S_0. The group was completely determined by the present writer and all S_0's were classified into eleven types [A], [B], ⋯, [K].[3] In this section we shall obtain the general form of the condition for form-invariancy of a tensor, and then, in the following sections, we shall apply it to each individual space-time using elementary methods only.

If we denote the operator of an infinitesimal motion by $X \equiv \xi^i \partial_i$, then the vector satisfies the Killing's equation

$$Xg_{lm} + g_{im}\partial\xi^i/\partial x^l + g_{li}\partial\xi^i/\partial x^m = 0 . \qquad (i, j, \cdots = 1, \cdots, 4) .^{4)} \tag{1.1}$$

The condition for the form-invariancy of a tensor $v_{i_1 \cdots i_n}$ under X is given by[5]

$$Xv_{i_1 \cdots i_n} + v_{si_2 \cdots i_n}(\partial_{i_1}\xi^s) + \cdots + v_{i_1 \cdots i_{n-1}s}(\partial_{i_n}\xi^s) = 0 . \tag{1.2}$$

Particularly when the $v_{i_1 \cdots i_n}$ is a scalar v, (1.2) becomes

$$Xv = 0 . \tag{1.3}$$

Next in order to express (1.2) in another form we shall take any set of vectors $\overset{\alpha}{h}_i$ and $\underset{\alpha}{h}^i$ satisfying:

$$\left.\begin{aligned} &\sum_\alpha \overset{\alpha}{h}_i \overset{\alpha}{h}_j = g_{ij} , \quad \sum_\alpha \underset{\alpha}{h}^i \underset{\alpha}{h}^j = g^{ij} ; \ \sum_\alpha \overset{\alpha}{h}_i \underset{\alpha}{h}^j = \delta_i^j , \quad \overset{\alpha}{h}_i \underset{\beta}{h}^i = \delta_\beta^\alpha , \\ &\qquad\qquad\qquad\qquad\qquad (\alpha, \beta, \cdots = 1, \cdots, 4) , \end{aligned}\right\} \tag{1.4}$$

and put

$$p_\beta^\alpha = \underset{\beta}{h}^i (\xi^l \partial_l \overset{\alpha}{h}_i + \overset{\alpha}{h}_l \partial_i \xi^l) = -p_\alpha^\beta . \tag{1.5}$$

Then we shall denote by $V_{\alpha_1\cdots\alpha_n}$ the scalar components of $v_{i_1\cdots i_n}$ i.e. $V_{\alpha_1\cdots\alpha_n}=\underset{\alpha_1}{h^{i_1}}\cdots\underset{\alpha_n}{h^{i_n}}v_{i_1\cdots i_n}$. By using (1.5), (1.2) becomes

$$\left.\begin{aligned} XV_{\alpha_1\cdots\alpha_n}+p^{\rho}_{\alpha_1}V_{\rho\alpha_2\cdots\alpha_n}+\cdots+p^{\rho}_{\alpha_n}V_{\alpha_1\cdots\alpha_{n-1}\rho}=0,\\ (\rho=1,\cdots,4;\ \text{summed for }\rho),\end{aligned}\right\} \tag{1.6}$$

which can be rewritten as:

$$XV_{\alpha_1\cdots\alpha_n}+p^{\rho}_1F^1_{\rho}+p^{\rho}_2F^2_{\rho}+p^{\rho}_3F^3_{\rho}+p^{\rho}_4F^4_{\rho}=0, \tag{1.7}$$

where $F^1_1=F^2_2=F^3_3=F^4_4=0$, and F^1_{ρ}, F^2_{ρ}, $\cdots$ are the sums of the coefficients of p^{ρ}_1, p^{ρ}_2, $\cdots$ in (1.6) respectively. For example, if $(\alpha_1,\cdots,\alpha_n)=(\overbrace{11\cdots1}^{p}, \overbrace{22\cdots2}^{q}, \overbrace{33\cdots3}^{r}, \overbrace{44\cdots4}^{s})$ (p, q, r, s are the numbers of the same indices, so it holds $p, q, r, s\geqq0$ and $p+q+r+s=n$), then $F^1_{\rho}=V_{\underset{(\rho)}{(\overbrace{1\cdots1}^{p})}\overbrace{2\cdots2}^{q}\overbrace{3\cdots3}^{r}\overbrace{4\cdots4}^{s}}$ where $\underset{(\rho)}{(1\cdots1)}$ denotes the sum of p terms obtained by replacing any one of the p indices 1 by ρ.[7] When $p=0$, $F^1_{\rho}=0$. (1.7) is the equation to be solved.

Next, if v is any relative scalar of weight m, the condition for form-invariancy of v under X is given by

$$Xv+mv\partial\xi^i/\partial x^i=0; \tag{1.8}$$

which is invariant under any coordinate transformation. From (1.1) and (1.8) we know that $g\equiv|g_{ij}|$ is a form-invariant scalar of weight 2. When $v_{i_1\cdots i_n}$ is a form-invariant relative tensor of weight m, $(\sqrt{g})^{-m}v_{i_1\cdots i_n}$ is a form-invariant ordinary tensor. Hence if we get the general form of the form-invariant ordinary tensor then the general form of the form-invariant relative tensor is given by multiplying it by $(\sqrt{g})^m$, and accordingly we shall deal only with an ordinary tensor in the following.

§ 2. Space-time [A]. 1

We shall take the coordinate system in which

$$ds^2=-e^{kt}(dx^2+dy^2+dz^2)+dt^2 \tag{2.1}$$

holds, and take $\overset{\alpha}{h}_j$ as follows:

$$\overset{a}{h}_j=ie^{kt}\delta^a_j,\quad \overset{4}{h}_j=\delta^4_j,\qquad (a=1,2,3). \tag{2.2}$$

The results which will be obtained are independent of the special choice of the coordinate system and $\overset{\alpha}{h}_j$. The vectors of the operators of the group of motions in this coordinate system and their Poisson operators are given in the writer's previous papers,[8] and we shall omit them here. As is easily seen form-invariant tensors under two operators are also form-invariant under their Poisson operator. Hence we know that form-invariant tensors under $\overset{a}{T}$ and $\overset{b}{S}$ are also form-invariant under all motions of G_A (the group of the space-time [A].)[9] From (1.5), p^α_β are given by

$$\left.\begin{aligned}&\text{for } \overset{a}{T} \text{ and } U: p^\alpha_\beta = 0;\ \text{for } \overset{a}{R}: p^\alpha_\beta = -E_{a\alpha\beta4};\ \text{for } \overset{1}{S}: p^2_1 = -p^1_2 = 2k^2 y,\\ &p^3_1 = -p^1_3 = 2k^2 z,\quad p^4_1 = -p^1_4 = 2ike^{-kt},\quad \text{other } p^\alpha_\beta = 0;\ \text{etc.}\end{aligned}\right\} \tag{2.3}$$

From (1.7) and (2.3), $\partial_l V_{\alpha_1 \cdots \alpha_n} = 0$ i. e. $V_{\alpha_1 \cdots \alpha_n} = \text{const.}$ (2.4)

and $$F^\alpha_\beta = F^\beta_\alpha, \quad (\alpha \neq \beta;\ \alpha, \beta = 1, \cdots, 4), \tag{2.5}$$

are the conditions to be satisfied by the form-invariant tensors. If we use the notation introduced in §1, (2.5) becomes

(C₁) $$V_{\underset{(\beta)}{\overbrace{(\alpha\cdots\alpha)}^{p}}\ \overbrace{\beta\cdots\beta}^{q}\ \overbrace{\gamma\cdots\gamma}^{r}\ \overbrace{\delta\cdots\delta}^{s}} \overset{P}{=} V_{\overbrace{\alpha\cdots\alpha}^{p}\ \underset{(\alpha)}{\overbrace{(\beta\cdots\beta)}^{q}}\ \overbrace{\gamma\cdots\gamma}^{r}\ \overbrace{\delta\cdots\delta}^{s}},\quad \begin{pmatrix}\alpha, \beta, \gamma, \delta \neq;\\ p+q+r+s = n;\ p, q \geq 1\end{pmatrix},$$

and (C₂) $$V_{\underset{(\delta)}{\overbrace{(\alpha\cdots\alpha)}^{p}}\ \overbrace{\beta\cdots\beta}^{q}\ \overbrace{\gamma\cdots\gamma}^{r}} \overset{P}{=} 0, \quad (\alpha, \beta, \gamma, \delta \neq;\ p+q+r = n;\ p \geq 1),$$

where $\overset{P}{=}$ means that the $n!$ equations obtained by the permutation of n indices hold.

By putting n=1, 2, 3 and 4 in (C_1) and (C_2), we can obtain V_δ=0, $V_{\alpha\beta}$=$V_{11}\Delta_{\alpha\beta}$, etc., from which we have

Theorem [2.1] *The general forms of the form-invariant scalar, vector, ⋯, and tensor of the fourth order under G_A are given by*

$$\left.\begin{aligned}v = \textit{const.},\quad v_i = 0,\quad v_{ij} = cg_{ij},\quad v_{ijk} = 0,\\ v_{ijkl} = c_1 g_{ij} g_{kl} + c_2 g_{ik} g_{jl} + c_3 g_{il} g_{jk} + c_4 \varepsilon_{ijkl},\end{aligned}\right\} \tag{2.6}$$

where c's are arbitrary constants and ε_{ijkl} is the ε tensor.

§3. Space-time [A]. 2. Solution of (C_1) and (C_2) for general n

Lemma [3.1] *If $w_{i_1 \cdots i_n}$ is form-invariant under G_A and satisfies*

$$g^{i_1 i_\lambda} w_{i_1 \cdots i_n} = 0, \quad (\lambda = 2, \cdots, n), \tag{H₁}$$

its scalar components except those given by

$$W_{\overbrace{\alpha\cdots\alpha}^{p}\ \overbrace{\beta\cdots\beta}^{q}\ \overbrace{\gamma\cdots\gamma}^{r}\ \overbrace{\delta\cdots\delta}^{s}}\ (P), \quad \begin{pmatrix}p+q+r+s = n;\ p, q, r, s \textit{ are}\\ \textit{positive odd integers}\end{pmatrix}, \tag{3.1}$$

vanish, where (P) means that the statement holds for any permutation of the indices.

Proof. Let $\bar{w}$ be any of the ${}_{n-2}C_2$ tensors of the $(n-2)$th order defined by $g^{i_\lambda i_\mu}w_{i_1\cdots i_n}$, $(\lambda, \mu=2, \cdots, n;\ \lambda \neq \mu)$, then it satisfies

$$g^{i_1 i_\lambda}\bar{w}_{i_1\cdots i_{n-2}}=0, \qquad (\lambda=2, \cdots, n-2), \tag{H$_2$}$$

and is also form-invariant under G_A. From (C_2) we have

$$W_{\alpha\widehat{\beta\cdots\beta}} \overset{P}{=} 0, \quad W_{\alpha\widehat{\beta\cdots\beta}\,\widehat{\gamma\cdots\gamma}} \overset{P}{=} 0, \qquad (\alpha, \beta, \gamma \neq), \tag{3.2}$$

where the numbers of the indices are omitted, and from this and (H_1), we have

$$\sum_\rho W_{\rho\rho\alpha\widehat{\beta\cdots\beta}} \overset{P}{=} 0, \quad \text{so} \quad W_{\alpha\alpha\alpha\widehat{\beta\cdots\beta}} \overset{P}{=} 0, \qquad (\alpha \neq \beta). \tag{3.3}$$

Then from (H_1), (H_2) and (C_1), we can prove

$$W_{\overset{5}{\widehat{\alpha\cdots\alpha}}\,\widehat{\beta\cdots\beta}} \overset{P}{=} 0, \quad W_{\alpha\alpha\beta\beta\beta\widehat{\gamma\cdots\gamma}} \overset{P}{=} 0. \tag{3.4}$$

Similarly,

$$W_{\overset{p}{\widehat{\alpha\cdots\alpha}}\,\widehat{\beta\cdots\beta}} \overset{P}{=} 0, \quad W_{\alpha\alpha\overset{p}{\widehat{\beta\cdots\beta}}\,\widehat{\gamma\cdots\gamma}} \overset{P}{=} 0, \qquad (p=\text{odd}), (\alpha, \beta, \gamma \neq). \tag{3.5}$$

In the same way we can prove that

$$W_{\overset{p}{\widehat{\alpha\cdots\alpha}}\,\widehat{\beta\cdots\beta}\,\widehat{\gamma\cdots\gamma}} \overset{P}{=} 0, \quad W_{\overset{p}{\widehat{\alpha\cdots\alpha}}\,\overset{q}{\widehat{\beta\cdots\beta}}\,\widehat{\gamma\cdots\gamma}\,\widehat{\delta\cdots\delta}} \overset{P}{=} 0, \quad \begin{pmatrix} p=\text{odd}, & q=\text{even}, \\ \alpha, \beta, \gamma, \delta \neq & \end{pmatrix}. \tag{3.6}$$

Now from (3.3) and (C_2), we have

$$W_{\beta\beta\widehat{\alpha\cdots\alpha}} \overset{Q}{=} -W_{\widehat{\alpha\cdots\alpha}}/3, \tag{3.7}$$

where $\overset{Q}{=}$ means that the equality holds for any permutation of the second, the third, $\cdots$, and the n th indices. Then using (C_1) we have $W_{\widehat{\alpha\cdots\alpha}}=0$ and $W_{\beta\beta\widehat{\alpha\cdots\alpha}} \overset{Q}{=} 0$. Repeating similar procedures we can easily prove the lemma.

By this lemma we know that when n is odd it must be that $W_{\alpha_1\cdots\alpha_n}=0$ i.e. $w_{i_1\cdots i_n}=0$. Hence using the method of the mathematical induction we can obtain

Theorem [3.2] *When n is odd any tensor of the nth order which is form-invariant under G_A vanishes identically.*

Now we shall deal with the case in which n is even. Let $n=2m$ where m is a positive integer. By making outer products of m g_{ij}'s we can construct M_m form-invariant tensors of the n th order where $M_m=2m!/(2^m\cdot m!)$. Similarly by making outer products of $(m-2)$ g_{ij}'s and one ε_{ijlm} we obtain N_m form-invariant tensors where $N_m = {}_nC_4 M_{m-2} = (2m)!/\{6\cdot 2^m\cdot(m-2)!\}$. These tensors are not necessarily independent of each other,[1c)] but in the following we shall treat them as if they were independent. As is easily seen, however, the correctness of the results is not obstructed. We shall denote the form-invariant tensor of the n th order obtained by a linear combination (constant coefficients) of these tensors by $(g, \varepsilon)_n$. They are in number

$$X_n = M_m + N_m = (2m)!\ (m^2 - m + 6)/\{6\cdot 2^m\cdot m!\}. \tag{3.8}$$

Lemma [3.3] *If we assume that a tensor of the p th order* ($p \leq n-2$) *which is form-invariant under* G_A *is of the form* $(g, \varepsilon)_p$ *and that a form-invariant tensor* $v_{i_1\cdots i_n}$ *is given, then there exists at least one tensor* $w_{i_1\cdots i_n}$ *of the form* $v_{i_1\cdots i_n} + (g, \varepsilon)_n$ *which satisfies* (H_1) *and*

$$\varepsilon^{i_1 i_\lambda i_\mu i_\nu} w_{i_1\cdots i_n} = 0, \qquad (\lambda, \mu, \nu \neq;\ \lambda, \mu, \nu = 2, \cdots, n). \tag{H_3'}$$

Proof. $(g, \varepsilon)_n$ contains X_n terms of different forms and we shall show that we can determine the X_n coefficients of these terms so as w may satisfy (H_1) and (H_3'). By putting each coefficient of the terms of different forms equal to 0, we obtain $(n-1)\ X_{n-2}$ linear equations from (H_1). Next by using the lemma [3.1] and the fact that an outer product of two ε-tensors is expressible by g_{ij}, we can show that ${}_{n-1}C_3 M_{m-2}$ equations are obtained from (H_3'). But X_n is equal to the sum of these two numbers. Hence the lemma is proved.

By the similar considerations as in the latter half of the above proof we can use

$$\varepsilon^{i_\lambda i_\mu i_\nu i_\sigma} w_{i_1\cdots i_n} = 0, \qquad (\lambda, \mu, \nu, \sigma \neq;\ \lambda, \mu, \nu, \sigma = 1, \cdots, n) \tag{H_3}$$

in place of (H_3'). In the same way we can also prove that $\bar{w}$ in the lemma [3.1] constructed with respect to the above w is a linear combination of the terms composed of one ε and $(m-3)$ g's and $\bar{w}$ satisfies (H_1) and (H_3). We shall denote this fact by (H_4).

Lemma [3.4] *If* $w_{i_1\cdots i_n}$ *is form-invariant under* G_A *and satisfies* (H_1) *and* (H_3) *it must vanish identically.*

Proof. At present the non-vanishing $W_{\alpha_1\cdots\alpha_n}$ is of the form (3.1) and we shall denote W of this type by $[p, q, r, s]$. W's of the type $[1, 1, 1, n-3]$ i.e. $W_{\alpha\beta\gamma\widehat{\delta\cdots\delta}}(P)$ are antisymmetric in (α, β, γ) by virtue of (C_1) and from (H_3) we have $W_{[\alpha\beta\gamma\delta]\widehat{\delta\cdots\delta}} \overset{P}{=} 0$. So

$$W_{\alpha\beta\gamma\overbrace{\delta\cdots\delta}^{n-3}}-W_{\alpha\beta\delta\gamma\overbrace{\delta\cdots\delta}^{n-4}}+W_{\alpha\delta\beta\gamma\overbrace{\delta\cdots\delta}^{n-4}}-W_{\delta\alpha\beta\gamma\overbrace{\delta\cdots\delta}^{n-4}}=0\,. \tag{3.8}$$

From this and (C_1) we obtain [1, 1, 1, $n-3$]=0.

In the same way from $\sum_\rho W_{\rho\rho\alpha\beta\gamma\widehat{\delta\cdots\delta}}\overset{P}{=}0$ which is obtained from (H_1) and (H_3), we have [1, 1, 3, $n-5$]=0. Then by (H_1) and (H_4) we have $\sum_\rho W_{\rho\rho\alpha\beta\gamma\widehat{\delta\cdots\delta}}\overset{P}{=}0$ from which we can show that

$$4W_{\alpha\alpha\alpha\beta\gamma\gamma\gamma\widehat{\delta\cdots\delta}}\overset{P}{=}-W_{\underset{(\alpha)}{(\gamma\gamma\gamma)}\beta\gamma\gamma\gamma\widehat{\delta\cdots\delta}}\,, \tag{3.9}$$

and

$$W_{\underset{(\alpha\alpha)}{(\gamma\gamma|\alpha\beta|\gamma\gamma\gamma)}\widehat{\delta\cdots\delta}}+W_{\underset{(\beta\beta)}{(\gamma\gamma|\alpha\beta|\gamma\gamma\gamma)}\widehat{\delta\cdots\delta}}+10W_{\gamma\gamma\alpha\beta\gamma\gamma\gamma\widehat{\delta\cdots\delta}}\overset{P}{=}0\,, \tag{3.10}$$

where $\underset{(\alpha\alpha)}{(\gamma\cdots\gamma)}$ denotes the sum of terms obtained by replacing two of parenthesized γ's by α. Then using $2W_{\alpha\underset{(\alpha\alpha)}{\overbrace{(\beta\cdots\beta)}^{q}}\widehat{\gamma\cdots\gamma}\widehat{\delta\cdots\delta}}\overset{P}{=}(q-1)\,W_{\alpha\overbrace{\beta\cdots\beta}^{q}\widehat{\gamma\cdots\gamma}\widehat{\delta\cdots\delta}}$, we obtain [1, 1, 5, $n-7$]=[1, 3, 3, $n-7$]=0. By repeating similar processes we finally obtain [p, q, r, s]=0 i.e. $w_{i_1}\cdots{}_{i_n}=0$. Q.E.D.

Using the theorem [2.1] and the lemmas given in this section we can prove the following theorem by using the mathematical induction:

Theorem [3.5] *The general form of the tensor which is form-invariant under G_A and of the n th order where n is even is given by* $(g, \varepsilon)_n$.

Hence summarizing the results, we have

Theorem [3.6] *The general form of the tensor of the n th order* ($n\geqq 0$) *which is form-invariant under G_A is given by* $(g, \varepsilon)_n$.

§4. Space-time [B]

The group of motions G_B in this space-time is composed of $\overset{a}{R}$, $\overset{a}{T}$, $\overset{a}{\bar{S}}$ and $\bar{U}$ (a=1, 2, 3), and their Poisson operators are well known.[6] As in the case of [A], form-invariant tensors under $\overset{a}{T}$ and $\overset{a}{\bar{S}}$ are form-invariant under G_B. In the coordinate system of

$$ds^2=-(dx^2+dy^2+dz^2)+dt^2\,, \tag{4.1}$$

if we take $\overset{a}{h}_j=i\delta^a_j$, $\overset{4}{h}_j=\delta^4_j$, $(a=1,2,3)$, (4.2)

$$\left.\begin{array}{l}\text{we have for } \overset{a}{T} \text{ and } \bar{U}:\ p^\alpha_\beta=0;\ \text{ for } \overset{a}{R}:\ p^\alpha_\beta=-E_{a\alpha\beta4}\,;\\ \text{and for } \overset{a}{\bar{S}}:\ p^a_4=-p^4_a=i,\ \text{ other } p^\alpha_\beta=0\,.\end{array}\right\} \tag{4.3}$$

Hence from (1.7) we have (2.4) and (2.5) again. Hence as in the case of [A], we have

Theorem [4.1] *The general form of the form-invariant tensor under G_B is given by* $(g, \varepsilon)_n$.

§ 5. Space-time [C]. 1

The group of motions G_C of [C] is composed of $\overset{a}{R}$, $\overset{a}{V}$ and $\bar{U}$, their Poisson operators are known, and form-invariant tensors under $\overset{a}{V}$ and $\bar{U}$ are form-invariant under G_C.[8] In the coordinate system of

$$ds^2 = -(1+r^2/4R^2)^{-2}\,(dx^2+dy^2+dz^2)+dt^2\,, \tag{5.1}$$

if we take $\overset{a}{h}_j = i(1+r^2/4R^2)^{-1}\delta^a_j,\quad \overset{4}{h}_j = \delta^4_j,\qquad (a,b,\cdots=1,2,3),$ (5.2)

we have for $\bar{U}$: $p^\alpha_\beta=0$; for $\overset{a}{R}$: $p^\alpha_\beta=-E_{a\alpha\beta 4}$;

for $\overset{1}{V}$: $p^1_2=-p^2_1=-y/2R^2$, $p^1_3=-p^3_1=-z/2R^2$, other $p^\alpha_\beta=0$; etc. (5.3)

From (1.7) for $\bar{U}$, we have

$$V_{\alpha_1\cdots\alpha_n} = V_{\alpha_1\cdots\alpha_n}(x,y,z)\,, \tag{5.4}$$

and from (1.7) for $\overset{a}{R}$ and $\overset{a}{V}$, we can obtain:

$$\partial_a V_{\alpha_1\cdots\alpha_n}=0 \quad \text{i.e.}\quad V_{\alpha_1\cdots\alpha_n}=\text{const.} \tag{5.5}$$

and

$$F^a_{\ b} = F^b_{\ a}\,,\qquad (a,b=1,2,3)\,. \tag{5.6}$$

Conversely if V satisfies (5.5) and (5.6), it is form-invariant under G_C. (5.6) can be written as

$$(\bar{\mathrm{C}}_1)\quad V_{\underset{(b)}{(\overbrace{a\cdots a}^{p})}\overbrace{b\cdots b}^{q}\overbrace{c\cdots c}^{r}\overbrace{4\cdots 4}^{s}} = V_{\overbrace{a\cdots a}^{p}\underset{(a)}{(\overbrace{b\cdots b}^{q})}\overbrace{c\cdots c}^{r}\overbrace{4\cdots 4}^{s}}\quad \begin{pmatrix} a,b,c\neq, & p+q+r+s=n, \\ n>p,\ q\geq 1; & r,s\geq 0 \end{pmatrix}$$

$$(\bar{\mathrm{C}}_2)\quad V_{\underset{(c)}{(\overbrace{a\cdots a}^{p})}\overbrace{b\cdots b}^{q}\overbrace{4\cdots 4}^{r}} = 0\quad (a,b,c\neq,\ p+q+r=n,\ n\geq p\geq 1,\ q,r\geq 0)\,.$$

From these equations, as in § 2, we easily obtain

Theorem [5.1] *The general form of the tensor $v_{i_1\cdots i_n}$ form-invariant under G_C is given by*

$$\left.\begin{aligned} &v=\mathit{const};\quad v_i = c\lambda_i;\quad v_{ij}=c_1 g_{ij}+c_2\lambda_i\lambda_j;\\ &v_{ijk}=c_1 g_{jk}\lambda_i+c_2 g_{ik}\lambda_j+c_3 g_{ij}\lambda_k+c_4\varepsilon_{ijkl}\lambda^l+c_5\lambda_i\lambda_j\lambda_k;\\ &v_{ijkl}=(c_1 g_{ij}g_{kl}+\cdots)+\lambda^h(c_4\varepsilon_{ijkh}\lambda_l+\cdots)+(c_8 g_{ij}\lambda_k\lambda_l+\cdots)+c_{14}\lambda_i\lambda_j\lambda_k\lambda_l\,, \end{aligned}\right\} \tag{5.7}$$

corresponding to $n=0, 1, \cdots, 4$ respectively, where c's are arbitrary constants and λ_j is a form-invariant vector which is determined to within a constant multiplier and whose components are given by δ^4_j in the coordinate system of (5.1).

In v_{ijkl}, ε_{ijkl} is contained implicitly by virtue of the identity:

$$\varepsilon_{ijkl}=(\varepsilon_{ijkh}\lambda_l-\varepsilon_{ijlh}\lambda_k+\varepsilon_{iklh}\lambda_j-\varepsilon_{jklh}\lambda_i)\,\lambda^h\,. \tag{5.8}$$

§ 6. Space-time [C]. 2. Solution for general n

For brevity's sake, we shall omit the indices 4 of $V_{\alpha_1 \cdots \alpha_n}$ and put $n-s=e$ where s is the number of these omitted indices, and shall solve $(\bar{C}_1)$ and $(\bar{C}_2)$ i. e.

$$(\bar{C}_1) \quad V_{\underset{(b)}{\overbrace{(a\cdots a)}^{p}}\,\overbrace{b\cdots b}^{q}\,\overbrace{c\cdots c}^{r}} \overset{P}{=} V_{\overbrace{a\cdots a}^{p}\,\underset{(a)}{\overbrace{(b\cdots b)}^{q}}\,\overbrace{c\cdots c}^{r}}, \; (a,\, b,\, c \neq,\; p+q+r=e,\; p,\, q \geqq 1,\; r \geqq 0),$$

$$(\bar{C}_2) \quad V_{\underset{(c)}{\overbrace{(a\cdots a)}^{p}}\,\overbrace{b\cdots b}^{q}} \overset{P}{=} 0, \; (a,\, b,\, c \neq,\; p+q=e,\; p \geqq 1,\; q \geqq 0).$$

The solutions for $n=1, 2, 3$ are

$$V_a = 0, \quad V_{ab} = c\bar{\Delta}_{ab}, \quad V_{abc} = c\bar{E}_{abc}, \tag{6.1}$$

respectively where $\bar{\Delta}$ and $\bar{E}$ denote the three dimensional Δ and E.

As in the lemma [3.1] we can prove that the non-vanishing components of $W_{a_1 \cdots a_e}$ which satisfies $(\bar{C}_1)$, $(\bar{C}_2)$ and $\sum_{a_1 a_\lambda} \bar{\Delta}_{a_1 a_\lambda} W_{a_1 \cdots a_e} = 0$, $(\lambda = 2, \cdots, e)$, are of the form $[p, q, r]$ where $p+q+r=e$ and p, q, r are odd and $\geqq 1$. Hence $W_{a_1 \cdots a_e} = 0$ for even e. Then using the method of the mathematical induction as in [3.3] we can prove the following lemmas:

Lemma [6.1] *When $e=2m$, the general form of $V_{a_1 \cdots a_e}$ which satisfies $(\bar{C}_1)$ and $(\bar{C}_2)$ is given by $(\bar{\Delta})_e$ i. e. a linear combination with constant coefficients of the terms of the e th order composed of the outer products of m $\bar{\Delta}_{ab}$'s.*

Lemma [6.2] *When $e=2m+1$, the general form of $V_{a_1 \cdots a_e}$ which satisfies $(\bar{C}_1)$ and $(\bar{C}_2)$ is given by $(\bar{\Delta}, \bar{E})_e$ i. e. a linear combination with constant coefficients of the terms of e th order composed of the outer products of $(m-1)$ $\bar{\Delta}_{ab}$'s and one $\bar{E}_{abc}$.*

Using these lemmas we shall show

Theorem [6.3] *The general form of the form-invariant tensor under G_C is given by $(\lambda, g, \varepsilon)_n$ which is, by definition, a linear combination with constant coefficients of the tensors of the n th order composed of the outer products of λ_i, g_{ij} and $\varepsilon_{ijkh}\lambda^h$, and λ_i is a form-invariant vector under G_C determined uniquely to within a constant multiplier.*

Proof. When $n=0, 1, \cdots, 4$ the theorem holds by [5.1]. Assuming that it holds for $\leqq n-1$, we shall show that it holds for n also. From an arbitrary tensor $v_{i_1 \cdots i_n}$ form-invariant under G_C we construct a new form-invariant tensor $w_{i_1 \cdots i_n}$ by

$$\left.\begin{aligned} w_{i_1\cdots i_n} &= v_{i_1\cdots i_n} - \lambda_{i_1} v_{oi_2\cdots i_n} - \lambda_{i_2}(v_{i_1oi_3\cdots i_n} - \lambda_{i_1} v_{ooi_3\cdots i_n}) \\ &\quad - \lambda_{i_3}\{v_{i_1i_2oi_4\cdots i_n} - \lambda_{i_1} v_{oi_2oi_4\cdots i_n} - \lambda_{i_2}(v_{i_1ooi_4\cdots i_n} - \lambda_{i_1} v_{oooi_4\cdots i_n})\} - \cdots \end{aligned}\right\} \quad (6.2)$$

where an index o means that the index originally situated in its position has disappeared by contraction with λ^i e.g. $v_{i_1i_2oi_4\cdots i_n} = \lambda^{i_3} v_{i_1\cdots i_n}$. Hence v's with index o are of the form $(\lambda, g, \varepsilon)$ and it holds that

$$\lambda^{i_\mu} w_{i_1\cdots i_n} = 0, \qquad (\mu = 1, \cdots, n), \tag{6.3}$$

and from the form of λ^i, we have

$$W_{4\alpha_2\cdots\alpha_n} \overset{P}{=} 0, \qquad (\alpha_i = 1, \cdots, 4) \tag{6.4}$$

From the lemmas [6.1] and [6.2], we have

$$\left.\begin{aligned} W_{a_1\cdots a_n} = (\bar{\Delta})_n \text{ when } n \text{ is even}; = (\bar{\Delta}, \bar{E})_n \text{ when } n \text{ is odd,} \\ (a=1,\ 2,\ 3). \end{aligned}\right\} \quad (6.5)$$

In this coordinate system, however, it holds that

$$\Delta_{ab} - \Lambda_a\Lambda_b = \bar{\Delta}_{ab}, \quad \Delta_{4\alpha} - \Lambda_4\Lambda_\alpha = 0, \quad \sum_\gamma E_{abc\gamma}\Lambda_\gamma = \bar{E}_{abc}, \quad \sum_\gamma E_{4\alpha\beta\gamma}\Lambda_\gamma = 0. \tag{6.6}$$

Hence in the tensor form, we have

$$w_{i_1\cdots i_n} = (g_{ij})_n \text{ when } n \text{ is even}; \quad = (g_{ij} - \lambda_i\lambda_j,\ \varepsilon_{ijkl}\lambda^l)_n \text{ when } n \text{ is odd.} \tag{6.7}$$

So the theorem is obvious. Evidently $(\lambda, g, \varepsilon)_n$ is linear in ε.

§ 7. Space-times [D] and [E]

By using the similar method as those used in the preceding sections, and taking the coordinate systems in which

when the S_0 is [D]: $ds^2 = -e^{2g(t)}(dx^2 + dy^2 + dz^2) + dt^2$, $(\dot{g} \neq \text{const.})$, (7.1)

and when the S_0 is [E]: $ds^2 = -F(dx^2 + dy^2 + dz^2) + dt^2$, (7.2)

$$(F = e^{2g(t)}(1 + r^2/4R^2)^{-2}, \quad e^{2g}\dot{g} \neq 1/R^2, \ \dot{g} \neq 0),$$[12]

hold respectively, we can easily obtain the following theorem:

Theorem [7.1] *The general form-invariant tensor under G_D or G_E is given by $(\lambda, g, \varepsilon; \rho)_n$, where ρ is a form-invariant scalar, λ_i is a form-invariant vector determined uniquely to within a multiplier $\varphi(\rho)$ (φ being arbitrary function) and $(\lambda, g, \varepsilon; \rho)_n$ is the same notation as $(\lambda, g, \varepsilon)_n$ with a proviso that the coefficients of the linear combination are arbitrary functions of ρ.*

In the coordinate system of (7.1) or (7.2) ρ and λ_j are given by $\rho=\rho(t)$ and $\lambda_j=\delta_j^4\lambda_4(t)$.

§8. Spherically symmetric tensors. Space-times [F], ⋯ , [K]

The line element of an S_0 is reducible to the form

$$ds^2 = -A(r, t)\,dr^2 - B(r, t)(d\theta^2+\sin^2\theta d\phi^2)+C(r, t)\,dt^2, \qquad (8.1)$$

by taking a s.s. coordinate system and (8.1) is form-invariant under the group of rotations G_3. We shall call a tensor form-invariant under G_3 *spherically symmetric* and first shall determine the general form of this s.s. tensor. If we take

$$\overset{1}{h}_j = i\sqrt{A}\,\delta_j^1, \quad \overset{2}{h}_j = i\sqrt{B}\,\delta_j^2, \quad \overset{3}{h}_j = i\sqrt{B}\sin\theta\,\delta_j^3, \quad \overset{4}{h}_j = \sqrt{C}\,\delta_j^4, \qquad (8.2)$$

in this coordinate system, we have

$$\left.\begin{array}{lll} \text{for } \overset{1}{R} \equiv -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi: & p_2^3 = -p_3^2 = \cos\phi/\sin\theta, & \text{other } p_\beta^\alpha = 0, \\ \text{for } \overset{2}{R} \equiv \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi: & p_2^3 = -p_3^2 = \sin\phi/\sin\theta, & \text{other } p_\beta^\alpha = 0, \\ \text{and for } \overset{3}{R} \equiv \partial_\phi: & p_\beta^\alpha = 0. & \end{array}\right\} \qquad (8.3)$$

Then, if $v_{i_1\cdots i_n}$ is s.s., from (1.7) we have

$$V_{x_1\cdots x_n} = V_{x_1\cdots x_n}(r, t), \quad L_1(V) \equiv F_3^2 - F_2^3 = 0, \qquad (8.4)$$

as the condition to be satisfied by $v_{i_1\cdots i_n}$. By putting $n=0$ and 1 we have

$$v = v(r, t), \quad v_i = a(r, t)\,\lambda_i + b(r, t)\,\mu_i, \qquad (8.5)$$

where a and b are arbitrary functions, and λ_i and μ_i are unit vectors mutually orthogonal and are given by

$$\lambda_j = \sqrt{C}\,\delta_j^4, \quad \mu_j = i\sqrt{A}\,\delta_j^1, \qquad (8.6)$$

in this coordinate system. (8.5) is the general form of the s.s. scalar and vector.[13]

If we put

$$\left.\begin{array}{l} w_{i_1\cdots i_n} = v_{i_1\cdots i_n} - \lambda_{i_1} v_{oi_2\cdots i_n} - \lambda_{i_2}(v_{i_1 o i_3\cdots i_n} - \lambda_{i_1} v_{ooi_3\cdots i_n}) - \cdots, \\ u_{i_1\cdots i_n} = w_{i_1\cdots i_n} - \mu_{i_1} w_{\Delta i_2\cdots i_n} - \mu_{i_2}(w_{i_1\Delta i_3\cdots i_n} - \mu_{i_1} w_{\Delta\Delta i_3\cdots i_n}) - \cdots, \end{array}\right\} \qquad (8.7)$$

where an index Δ is the notation similar to the index o concerning μ_i instead of λ_i. Then $u_{i_1\cdots i_n}$ is s.s. and it holds that

$$\lambda^{i_\sigma} u_{i_1 \cdots i_n} = 0\,, \quad \mu^{i_\sigma} u_{i_1 \cdots i_n} = 0\,, \qquad (\sigma = 1, \cdots, n)\,. \tag{8.8}$$

Hence we have:

$$U_{1\alpha_2 \cdots \alpha_n} \overset{P}{=} 0\,, \quad U_{4\alpha_2 \cdots \alpha_n} \overset{P}{=} 0\,, \qquad (\alpha_i = 1, \cdots, 4), \tag{8.9}$$

and the second equation of (8.4) becomes

$$(\bar{\bar{\mathrm{C}}}_1) \quad U_{\underset{(\beta)}{\overbrace{(\alpha\cdots\alpha)}^{p}}\, \overbrace{\beta\cdots\beta}^{q}} \overset{P}{=} U_{\overbrace{\alpha\cdots\alpha}^{p}\, \underset{(\alpha)}{\overbrace{(\beta\cdots\beta)}^{q}}}\,; \quad (\bar{\bar{\mathrm{C}}}_2) \quad U_{\underset{(\beta)}{\overbrace{(\alpha\cdots\alpha)}}} \overset{P}{=} 0\,, \quad \begin{pmatrix} p,\, q \geqq 1,\ \alpha \neq \beta, \\ \alpha,\, \beta = 2,\ 3 \end{pmatrix},$$

i. e. two dimensional (C_1) and (C_2) respectively. Then in the same way as in § 3 we can easily prove that

$$U_{\alpha_1 \cdots \alpha_n} = (\bar{\bar{\Delta}}, \bar{\bar{E}}\,;\ \rho\,(r,\, t))_n\,, \qquad (\alpha_i = 2,\ 3)\,, \tag{8.10}$$

where $\bar{\bar{\Delta}}$ and $\bar{\bar{E}}$ are two dimensional (i. e. in (θ, ϕ)-space) Δ and E. Hence by using $g_{ij} - \lambda_i \lambda_j - \mu_i \mu_j$ and $\varepsilon_{ijkl} \lambda^k \mu^l$ (which coincide with $\bar{\bar{g}}$ and $\bar{\bar{\varepsilon}}$ respectively in the (θ, ϕ)-space), we can prove

$$v_{i_1 \cdots i_n} = (\lambda,\, \mu,\, g,\, \varepsilon\,;\ \rho)_n\,, \quad \rho = \rho\,(r,\, t)\,, \tag{8.11}$$

where $(\lambda, \cdots; \rho)_n$ is the similar notation as those in the preceding theorems. Hence we have

Theorem [8.1] *The general form of the s. s. tensor is given by* $(\lambda, \mu, g, \varepsilon\,; \rho)_n$, *where* ρ *is a s. s. scalar and* λ_i *and* μ_i *are s. s. unit vectors mutually orthogonal.*

In the coordinate system of (8.1), $\rho = \rho\,(r, t)$ and an example of the pair of λ_i and μ_i is given by (8. 6).

G_G, G_H and G_K coincide with G_3. Hence we have

Theorem [8. 2] *The general form of the form-invariant tensor under* G_G *or* G_H *or* G_K *is given by* $(\lambda, \mu, g, \varepsilon\,; \rho)_n$ *in* [8. 1].

Further using this result we can obtain

Theorem [8. 3] *The general form of the form-invariant tensor under* G_F *or* G_{J_1} *or* G_{J_2} *is given by* $(\lambda, \mu, g, \varepsilon\,; \rho)_n$ *where* ρ *is a form-invariant scalar and* λ_i *and* μ_i *are mutually orthogonal form-invariant vectors under respective groups.*

In a s. s. coordinate system λ_i and μ_i are given by (8. 6), and ρ in the case of G_F is given by $\rho = \rho\,(r)$. When the S_0 is $[J_1]$ or $[J_2]$, if we take the coordinate system in which the line element of (r, t)-space is given by $-A(t)\,dr^2 + dt^2$ or $-dr^2 + C(r)\,dt^2$, ρ is given by $\rho = \rho\,(t)$ or $\rho = \rho\,(r)$ respectively.

The space-time $[I]$ is composed of two two dimensional spaces i.e. (r, t)-space and (θ, ϕ)-space and both spaces are of constant curvature. Hence applying [8.1] to these spaces we have

Theorem [8.4] *The general form of the form-invariant tensor under G_I is given by $(g, \varepsilon)_n$.*

By the investigations hitherto developed we have succeeded in determining the general forms of the form-invariant tensors in any S_0 starting with the study of the tensors in the space-time [A]. But we may adopt the converse process, namely we may start with the determination of the general form of the s.s. tensors and then proceed to the case of each individual space-time.

Lastly we add that the curvature tensor of each space-time is form-invariant under respective group of motions and this is also easily seen from the fundamental equations for an S_0.

§ 9. Form-invariant tensors under sub-groups

In this section, in view of the physical application, we shall treat the form-invariant tensors under sub-groups of motions which keep the spatial origin ($x=y=z=0$) invariant, in the space-time S_I. This property is characterized by $\xi^r=0$ in any s.s. coordinate system and not necessarily independent of the coordinate system. We shall consider the problem in the coordinate system in which $B=r^2$ holds and denote such a sub-group by G'. Then we have $G_A'=(\overset{a}{R}$ and $U)$; $G_B'=G_C'=G_F'=(\overset{a}{R}$ and $\bar{U})$; $G_D'=G_E'=G_G'=G_H'=(\overset{a}{R})$. Hence we have

Theorem [9.1] *The general form of the form-invariant tensors under G_A' or $\cdots$ or G_H' is given by $(\lambda, \mu, g, \varepsilon; \rho)_n$ where ρ, λ_i and μ_i are the same notations as those in the preceding theorems and obtained by solving (1.7) for $n=0$ and 1.*

With respect to another sub-groups we can also determine the general forms of the form-invariant tensors in the same way.

Part II. Parallel tensors in an S_0

§ 10. Condition for the parallelism of tensors

In this part we shall determine the general form of the parallel relative tensor which is defined by

$$\nabla_j v_{i_1 \cdots i_n} = 0\,. \tag{10.1}$$

When $v_{i_1 \cdots i_n}$ is an ordinary tensor, multiplying (10.1) by $\underset{\beta}{h^j}\underset{\alpha_1}{h^{i_1}} \cdots \underset{\alpha_n}{h^{i_n}}$, we have

$$\underset{\beta}{h^s}\nabla_s V_{\alpha_1 \cdots \alpha_n} + \sum_\rho \{\gamma_{\rho\alpha_1\beta} V_{\rho\alpha_2 \cdots \alpha_n} + \cdots + \gamma_{\rho\alpha_n\beta} V_{\alpha_1 \cdots \alpha_{n-1}\rho}\} = 0\,, \tag{10.2}$$

or
$$\underset{\beta}{h^s}\nabla_s V_{\alpha_1 \cdots \alpha_n} + \sum_\rho \{\gamma_{\rho_1\beta} F^1_\rho + \cdots + \gamma_{\rho_4\beta} F^4_\rho\} = 0\,, \tag{10.3}$$

where
$$\gamma_{\alpha\beta\gamma} = (\nabla_i \overset{\alpha}{h}_j)\, \underset{\beta}{h^j} \underset{\gamma}{h^i} = -\gamma_{\beta\alpha\gamma} \tag{10.4}$$

is the coefficient of rotation. Specially when v is a scalar, we have v=const.

When v is a parallel relative scalar of weight m, from $\nabla_i v=0$ we have $v=(\sqrt{g})^m \times$const. Hence if the solution of (10·3) is known, the general form of the parallel relative tensor is given by multiplying it by $(\sqrt{g})^m$ as in the case of form-invariant relative tensor and accordingly we shall deal only with the ordinary tensors again. Of course the results to be obtained are independent of the special choice of the coordinate system and $\underset{\alpha}{h^i}$ again.

§ 11. Parallel tensors in S_1

For an S_0, if we take a s.s. coordinate system in which (8.1) holds and choose $\underset{\alpha}{h^i}$ as in (8.2), we have

$$\left.\begin{aligned} &\gamma_{212} = \gamma_{313} = iB'/2B\sqrt{A}\,, \quad \gamma_{242} = \gamma_{343} = -\dot{B}/2B\sqrt{C}\,, \quad \gamma_{323} = i\cot\theta/\sqrt{B}\,, \\ &\gamma_{141} = -\dot{A}/2A\sqrt{C}\,, \quad \gamma_{414} = iC'/2C\sqrt{A}\,, \quad (\gamma_{\alpha\beta\gamma} = -\gamma_{\beta\alpha\gamma}), \quad \text{other } \gamma\text{'s} = 0\,. \end{aligned}\right\} \tag{11.1}$$

When the S_0 is an S_I, by taking the coordinate system in which $B=r^2$ holds, the above γ's become

$$\left.\begin{aligned} &\gamma_{212} = \gamma_{313} = i/r\sqrt{A}\,, \ \gamma_{323} = i\cot\theta/r\,, \ \gamma_{141} = -\dot{A}/2A\sqrt{C}\,, \ \gamma_{414} = iC'/2C\sqrt{A}\,, \\ &(\gamma_{\alpha\beta\gamma} = -\gamma_{\beta\alpha\gamma}), \quad \text{other } \gamma\text{'s} = 0\,. \end{aligned}\right\} \tag{11.2}$$

and (10.3) becomes:

$$\left.\begin{aligned} &\text{(a) } \partial_r V + (i\dot{A}/2\sqrt{AC})\, M_1(V) = 0\,, \quad \text{(b) } \partial_t V + (iC'/2\sqrt{AC})\, M_1(V) = 0\,, \\ &\text{(c) } \partial_\theta V - (1/\sqrt{A})\, L_3(V) = 0\,, \quad \text{(d) } \partial_\phi V + (\sin\theta/\sqrt{A}) L_2(V) - \cos\theta L_1(V) = 0\,, \end{aligned}\right\} \tag{11.3}$$

where

$$L_a(V) = \sum_{b,c} E_{abc} F^b_c, \quad M_a(V) = F^a_4 - F^4_a, \qquad (a, b, c = 1, 2, 3). \tag{11.4}$$

Hence we have only to solve (11.3) with (11.4). Before doing this, however, we shall show the following lemmas concerning the operators L_a and M_a:

Lemma [11.1] *The following identities hold:*

(i) $(L_aL_b - L_bL_a)(V) = \sum_c E_{abc}L_c(V)$; (ii) $(L_aM_b - M_bL_a)(V) = \sum_c E_{abc}M_c(V)$;
(iii) $(M_aM_b - M_bM_a)(V) = \sum_c E_{abc}L_c(V)$.

Proof. By the definition of L_a

$$\begin{aligned} L_2L_3(V) \overset{P}{=} \{ & V\underset{(2)}{\overbrace{(1\cdots1)}}\,\overbrace{2\cdots2}\,\underset{(1)}{\overbrace{(3\cdots3)}}\,\overbrace{4\cdots4} - V\overbrace{1\cdots1}\,\underset{(1)}{\overbrace{(2\cdots2)}}\,\underset{(1)}{\overbrace{(3\cdots3)}}\,\overbrace{4\cdots4}\} - \{V\underset{(2)(3)}{\overbrace{(1\cdots1)}}\,\overbrace{2\cdots2}\,\overbrace{3\cdots3}\,\overbrace{4\cdots4} \\ & - V\underset{(3)}{\overbrace{(1\cdots1)}}\,\underset{(1)}{\overbrace{(2\cdots2)}}\,\overbrace{3\cdots3}\,\overbrace{4\cdots4} - V\overbrace{1\cdots1}\,\underset{(3)}{\overbrace{(2\cdots2)}}\,\overbrace{3\cdots3}\,\overbrace{4\cdots4}\}, \end{aligned} \tag{11.5}$$

where $\underset{(\beta)(\gamma)}{\overbrace{(\alpha\cdots\alpha)}^{s}}$, $(\alpha \neq \beta,\ \alpha \neq \gamma$; s is omitted in (11.5) for brevity's sake) is the sum of the $s(s-1)$ terms obtained by operating $\underset{(\gamma)}{\overbrace{(\alpha\cdots\alpha)}^{s-1}}$ to $\underset{(\beta)}{\overbrace{(\alpha\cdots\alpha)}^{s}}$ and is symmetric in β and γ when $s \geqq 2$, and is equal to 0 when $s=0$ or 1. Using the similar relations we can easily prove the identities.

As a corollary of this lemma we have

Lemma [11.2] $M_a(V)=0$, $(a=1, 2, 3)$, *is equivalent to* $F^\alpha_\beta = F^\beta_\alpha$.

Now we shall solve (11.3). The condition for integrability is

(e) $(\dot{a} - b')M_1(V)=0$, (f) $(a/\sqrt{A})M_2(V) - (1/\sqrt{A})' L_3(V)=0$,
(g) $(a/\sqrt{A})M_3(V) + (1/\sqrt{A})' L_2(V)=0$, (h) $(b/\sqrt{A})M_2(V) - (1/\sqrt{A})^{\cdot} L_3(V)=0$,
(i) $(b/\sqrt{A})M_3(V) + (1/\sqrt{A})^{\cdot} L_2(V)=0$, (j) $(1-A)L_1(V)=0$,

where $a = -i\dot{A}/2\sqrt{AC}$ and $b = -iC'/2\sqrt{AC}$.[14]

(I) When $A=1$ and $C'=0$. In this case by a transformation of t, we have $C=1$ i.e. Minkowski metric (the space-time is [B]) and it is evident that the solution of the original equation (10.1) for (4.1) and (4.2) is given by

$$v_{i_1} \cdots {}_{i_n} = \text{arbitrary const.} = \sum_\alpha V_{\alpha_1} \cdots {}_{\alpha_n} \overset{\alpha_1}{h}_{i_1} \cdots \overset{\alpha_n}{h}_{i_n}, \tag{11.6}$$

where $V_{\alpha_1} \cdots {}_{\alpha_n}$ is constant and $\overset{\alpha}{h}_i$ are four parallel unit vectors mutually orthogonal. Hence by putting $\overset{\alpha}{h}_i$ as $\overset{\alpha}{\lambda}_i$, we have

Theorem [12.3] *In the space-time [B] parallel tensors are of the form* $(\overset{1}{\lambda}, \overset{2}{\lambda}, \overset{3}{\lambda}, \overset{4}{\lambda})_n$ *where* $\overset{\alpha}{\lambda}_i$ *are four parallel unit vectors mutually orthogonal.*

Though g and ε do not appear in the above expression, they are contained implicitly by virtue of the identities $g_{ij}=\sum_{\alpha}\overset{\alpha}{h}_i\overset{\alpha}{h}_j$ and $\varepsilon_{ijkl}=4!\ \overset{1}{h}_{[i}\overset{2}{h}_j\overset{3}{h}_k\overset{4}{h}_{l]}$.

(II) When $A\neq 1$ and $C'=0$. (II_1) When $\dot{A}=0$. By a transformation of t, we have

$$ds^2=-A(r)\,dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+dt^2. \tag{11.7}$$

We shall denote by [L] the space-time defined by this (11.7). Then from (a), $\cdots$ (d), we easily obtain (5.5) and (5.6). Hence we have

Theorem [11.4] *In* [L], *the general form of the parallel tensor is given by* $(\lambda, g, \varepsilon)_n$ *where* λ_i *is a parallel unit vector determined uniquely to within a constant multiplier.*

$\lambda_i=\delta_i^4$ in the coordinate system of (11.7). The space-time of Einstein universe [C] belongs to this [L].

In the case where (II_2) $\dot{A}\neq 0$, or (III) $A=1$ and $C'\neq 0$, or (IV) $A\neq 1$ and $C'\neq 0$, using the lemmas [12.1] and [12.2], we can easily obtain (2.4) and (2.5). Consequently we have $V=(\Delta, E)_n$ i.e. $v=(g, \varepsilon)_n$. Hence we have

Theorem [11.5] *In the space-time* S_I *excluding* [B] *and* [L], *the general form of the parallel tensor is given by* $(g, \varepsilon)_n$.

We shall denote this S_I by [M]. Hence when the S_I is an [M], there exists no parallel tensor whose order is odd.

§ 12. Parallel tensors in S_{II}

Substituting B=const. into (11.1), we have

$$\left.\begin{array}{l}\gamma_{323}=i\cot\theta/\sqrt{B}\,,\quad \gamma_{141}=-\dot{A}/2A\sqrt{C}=a_1\,,\quad \gamma_{414}=iC'/2C\sqrt{A}=-b_1\,,\\ \gamma_{\alpha\beta\gamma}=-\gamma_{\beta\alpha\gamma}\,,\quad \text{other}\ \ \gamma\text{'s}=0\,.\end{array}\right\} \tag{12.1}$$

So (10.3) becomes

$$\partial_r V=a_1M_1(V),\quad \partial_t V=b_1M_1(V),\quad \partial_\theta V=0,\quad \partial_\phi V=(r\cos\theta/\sqrt{B})\,L_1(V). \tag{12.2}$$

From this we have $\quad V=V(r, t)$ and $L_1(V)=0$, (12.3)

and the condition for integrability of the first two equations is

$$(\dot{a}_1-b_1')\,M_1(V)=0,\quad \text{i.e.}\quad K_{\dot{1}\dot{4}}{}^{14}M_1(V)=0. \tag{12.4}$$

(I) When $\xi\equiv K_{\dot{1}\dot{4}}{}^{14}=0$. We shall denote this S_{II} by [N]. In [N], since (r, t)-space is flat, we have

$$ds^2 = -dr^2 - B(d\theta^2 + \sin^2\theta d\phi^2) + dt^2, \qquad (B = \text{const.}), \qquad (12.5)$$

by a suitable transformation of (r, t). Then as in §8, using $L_1(V)=0$, we have

Theorem [12.1] *In* [N], *the general form of the parallel tensor is given by* $(\lambda, \mu, g, \varepsilon)_n$ *where* λ_i *and* μ_i *are parallel unit vectors mutually orthogonal.*

In the coordinate system of (12.5) we can take $\lambda_j = \delta_j^4$ and $\mu_j = \delta_j^1$.

(II) When $K_{14}{}^{14} \neq 0$. From (12.4), we have $M_1(V)=0$, so V=const. again, and the equations to be solved are $L_1(V)=M_1(V)=0$. Obviously the solutions of these equations in (r, t)- and (θ, ϕ)-spaces are given by $(\bar{\Delta}, \bar{E})$ and $(\bar{\bar{\Delta}}, \bar{\bar{E}})$ respectively, where $\bar{\Delta}, \bar{E}$; $\bar{\bar{\Delta}}, \bar{\bar{E}}$ are two dimensional Δ and E in the respective spaces. So, expressing this S_{II} by [O], we have

Theorem [12.2] *In* [O], *the general form of the parallel tensor is given by* $(\bar{g}, \bar{\bar{g}}, \bar{\varepsilon}, \bar{\bar{\varepsilon}})_n$ *where* $\bar{g}_{ij}$ *and* $\bar{\bar{g}}_{ij}$ *are symmetric parallel tensors linearly independent, and* $\bar{\varepsilon}_{ij}$ *and* $\bar{\bar{\varepsilon}}_{ij}$ *are anti-symmetric parallel tensors linearly independent.*

In the coordinate system of (8.1), a set of these tensors is given by

$$\left.\begin{array}{l} \bar{g}_{11} = -hA, \ \bar{g}_{44} = hC, \ \text{other } \bar{g}_{ij} = 0; \quad \bar{\bar{g}}_{22} = h'B, \ \bar{\bar{g}}_{33} = h'B\sin^2\theta, \ \text{other } \bar{\bar{g}}_{ij} = 0; \\ \bar{\varepsilon}_{14} = -\bar{\varepsilon}_{41} = ik\sqrt{AC}, \ \text{other } \bar{\varepsilon}_{ij} = 0; \quad \bar{\bar{\varepsilon}}_{23} = -\bar{\bar{\varepsilon}}_{32} = k'B\sin\theta, \ \text{other } \bar{\bar{\varepsilon}}_{ij} = 0, \end{array}\right\} \qquad (12.6)$$

where h, h', k and k' are arbitrary constants, and from this we have

$$\left.\begin{array}{l} g_{ij} = \bar{g}_{ij}/h - \bar{\bar{g}}_{ij}/h', \\ \varepsilon_{ijkl} = \{(\bar{\varepsilon}_{ij}\bar{\bar{\varepsilon}}_{kl} + \bar{\varepsilon}_{kl}\bar{\bar{\varepsilon}}_{ij}) - (\bar{\varepsilon}_{ik}\bar{\bar{\varepsilon}}_{jl} + \bar{\varepsilon}_{jl}\bar{\bar{\varepsilon}}_{ik}) + (\bar{\varepsilon}_{il}\bar{\bar{\varepsilon}}_{jk} + \bar{\varepsilon}_{jk}\bar{\bar{\varepsilon}}_{il})\}/kk'. \end{array}\right\} \qquad (12.7)$$

§ 13. Conclusion of Part II

Putting together the results of § 11 and § 12 we know that from the standpoint of the parallel tensors, S_0's are classified into the following five types: [B], [L], [M]; [N], [O], and the numbers of the linealy independent parallel vectors are 4, 1, 0; 2, 0 respectively. The first three belong to S_I and the last two to S_{II}. If we use c.s. of the S_0 we can easily express this classification in invariant form in several ways. An example is given by

$$S_0 \begin{cases} S_I \begin{cases} \overset{1}{\rho} = \overset{2}{\rho} = \overset{3}{\rho} = \overset{4}{\rho} = 0 & \cdots\cdots\cdots\cdots [\text{B}] \\ \text{other ones} \begin{cases} \text{when a c.s. whose } \sigma = \bar{\sigma} = \bar{\kappa} = 0 \text{ exists}^{15)} & \cdots\cdots\cdots\cdots [\text{L}] \\ \text{other ones} & \cdots\cdots\cdots\cdots [\text{M}] \end{cases} \end{cases} \\ S_{II} \begin{cases} \overset{1}{\rho} + \overset{2}{\rho} = 0 & \cdots\cdots\cdots\cdots [\text{N}] \\ \overset{1}{\rho} + \overset{2}{\rho} \neq 0 & \cdots\cdots\cdots\cdots [\text{O}] \end{cases} \end{cases}$$

We can also obtain easily the relation between the above classification and that obtained in (II) from the standpoint of the group of motions.

Lastly, we add that we can generalize the results obtained in this paper to the case of the n dimensional s. s. space-time.

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Notes

1) This paper is a continuation of (**I**) H. Takeno, Journ. Math. Soc. Japan, **3** (1951), 317; (**II**) ——, this Journal, **16** 1952), 67; (**III**) ——, this Journal, **16** (1952), 291; (**IV**) ——, this Journal, **16** (1952), 299; (**V**) ——, this Journal, **16** (1953), 497. See also H. Takeno, Prog. Theor. Phys., **7** (1952), 317. The same notations as in these papers are used throughout the present paper.

2) Y. Ueno and H. Takeno, Prog. Theor. Phys. **7** (1952), 291.

3) See (**II**).

4) Throughout the present paper $i, j, k, \cdots (=1, \cdots, 4)$ and $\alpha, \beta, \cdots, \rho\ (=1, \cdots, 4)$ denote tensor and scalar indices respectively.

5) Since the form-invariancy of a tensor under motion is preserved by the raising or lowering of the indices, we can take covariant tensor without losing the generality.

6) Throughout the present paper we shall denote the scalar components of tensors by using the corresponding capital letters.

7) For example,

$$V_{\underset{(2)}{\overbrace{(1\cdots1)}^{p}}2\cdots} = V_{\overbrace{21\cdots1}^{p}2\cdots} + V_{\overbrace{121\cdots1}^{p}2\cdots} + \cdots + V_{\overbrace{11\cdots12}^{p}2\cdots}$$

8) (**II**); H. Takeno, this Journal, **11** (1942), 201. The same notations as those in these papers are used in the present paper.

9) $(\overset{a}{T}, \overset{b}{S}) = -2k^2 E_{abc4}\overset{c}{R} + 2k\Delta_{ab}U$, where E_{abcd} and Δ_{ab} are scalar components of tensors ε_{ijkl} (not tensor density; $\varepsilon_{1234} = \sqrt{-g}$) and δ_i^j respectively.

10) For example the following identities hold:

$$g_{ij}\varepsilon_{lmpq} = \{g_{l(i}\varepsilon_{j)mpq} - g_{m(i}\varepsilon_{j)lpq}\} + \{g_{p(i}\varepsilon_{j)qlm} - g_{q(i}\varepsilon_{j)plm}\},$$
$$g_{l[i}\varepsilon_{j]mpq} - g_{m[i}\varepsilon_{j]lpq} = g_{p[l}\varepsilon_{m]qij} - g_{q[l}\varepsilon_{m]pij} = g_{i[p}\varepsilon_{q]jlm} - g_{j[p}\varepsilon_{q]ilm}$$

11) $\varepsilon_{i_1 \cdots i_4}\varepsilon_{j_1 \cdots j_4} = 4!\ g_{[i_1[j_1} \cdots g_{i_4]j_4]}$.

12) H. Takeno, this Journal **11** (1942), 229.

13) The invariant definition of a s. s. scalar using c. s. and introduced in (**I**) is somewhat different from that given here. In the standard coordinate system for the c. s., however, both coincide with each other.

14) If we express these equations by using the components of the curvature tensor, we have;

$$\text{(e) } \xi M_1 = 0, \quad \text{(f) } i\sqrt{C}\,\gamma M_2 - \sqrt{A}\,\alpha L_3 = 0, \quad \text{(g) } i\sqrt{C}\,\gamma M_3 + \sqrt{A}\,\alpha L_2 = 0,$$
$$\text{(h) } i\sqrt{A}\,\beta M_2 + \sqrt{C}\,\gamma L_3 = 0, \quad \text{(i) } i\sqrt{A}\,\beta M_3 - \sqrt{C}\,\gamma L_2 = 0, \quad \text{(j) } \eta L_1 = 0,$$

where the operand V is omitted for brevity's sake.

15) Strictly speaking this means '[L] is characterized by the existence of a c. s. whose $\sigma = \bar{\sigma} = \bar{\kappa} = 0$ and when this condition is satisfied this relation holds for all c. s.'. The condition $\sigma = \bar{\sigma} = \bar{\kappa} = 0$ can be replaced by $\overset{4}{\rho} = \overset{4}{\rho}(F)$, $\overset{3}{\rho} + 2\overset{4}{\rho} = 0$ where F is the scalar introduced in (**I**).