ORTHOGONALITY RELATION IN THE ANALYSIS OF VARIANCE I

By

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The basic idea of the analysis of variance[1] consists in dividing up the sum of squares about the total mean into sums of squares according to some scheme based on the design of experiments. These sums of squares are independent quadratic forms and provide tests for various hypotheses about means. It is our purpose to investigate this decomposition from a formal point of view.

For any given classification $\mathfrak{A}$ the corresponding sum of squares $S_{\mathfrak{A}}$ between classes is a quadratic form with a projective matrix. This leads us to associate a projective operator $P_{\mathfrak{A}}$ with $S_{\mathfrak{A}}$ such that $\|P_{\mathfrak{A}}x\|^2=S_{\mathfrak{A}}$. In like manner a projective operator $P_{\mathfrak{A}\mathfrak{B}}$ is associated with an interaction variance $S_{\mathfrak{A}\mathfrak{B}}$ of type $\mathfrak{A}$, $\mathfrak{B}$. In this paper various properties of these operators will be investigated.

1. In this section we shall define classifications and projective operators associated with them. Let I be an index set of N elements, say, $1, 2, \dots, N$. Let $A, B, \dots$ stand for non void subsets of I. $\mathfrak{A}=(A_1, A_2, \dots A_n)$ is a classification of E when A_i are disjoint and $E=\sum A_i$. Each A_i is said to be an $\mathfrak{A}$- class. If $\mathfrak{A}$ consists of only one $\mathfrak{A}$-class, then we shall say that $\mathfrak{A}$ is trivial. n_A denotes the number of elements contained in A. $\mathfrak{A}$ is called regular when n_{A_i} is independent of i.

Let $x_1, x_2, \dots x_N$ be independent random variables with normal distributions having a standard deviation 1. The sum of squares $S_{\mathfrak{A}}$ between $\mathfrak{A}$-classes is given by

$$\sum\frac{(A_i)^2}{n_{A_i}}-\frac{(E)^2}{n_E},$$

where (A) denotes $\sum_{i\in A} x_i$. We consider an N-dimensional Euclidean space R^N. With each A we associate a vactor e_A whose i-th component is 1 or 0 according as $i\in A$ or not. Let P_A denote the projective operator whose

1) Cf. R. A. Fisher [1]. The numbers in square brackets refer to the list of reference at the end of this paper.

range is the subspace spanned by e_A. If we put

$$P=\sum P_{A_i}-P_E,$$

then $P_{\mathfrak{A}}$ is projective and $\|P_{\mathfrak{A}}x\|^2=S_{\mathfrak{A}}$, where x is a joint variable $(x_1, x_2, \dots, x_N)$.

2. Let ξ be a variable vector in R^N. Let $\mathfrak{A}=(A_1, \dots, A_n)$, $\mathfrak{B}=(B_1, \dots, B_m)$ be classifications of E, E' respectively. After some calculation we have

$$P_{\mathfrak{A}}P_{\mathfrak{B}}\xi=\sum_{i,j}(\xi, e_{B_j})\Phi(A_i, B_j)e_{A_i}, \tag{1}$$

where $\Phi(A_i, B_j)=\frac{n_{A_iB_j}}{n_{A_i}n_{B_j}}-\frac{n_{A_iE'}}{n_{A_i}n_{E'}}-\frac{n_{EB_j}}{n_En_{B_j}}+\frac{n_{EE'}}{n_En_{E'}}$, and (ξ, e_{B_j}) denotes the inner peoduct of ξ and e_{B_j}.

We say that $\mathfrak{A}$, $\mathfrak{B}$ are orthogonal and we write $\perp(\mathfrak{A}, \mathfrak{B})$ when $P_{\mathfrak{A}}P_{\mathfrak{B}}=0$ or equivalently $S_{\mathfrak{A}}$, $S_{\mathfrak{B}}$ are independent.[2)] It follows from (1) that $\mathfrak{A}$, $\mathfrak{B}$ are orthogonal if and only if

$$\Phi(A_i, B_j)\equiv 0. \tag{2}$$

We note that if $E=E'$ holds, $\Phi(A_i, B_j)$ reduces to $\frac{n_{A_iB_j}}{n_{A_i}n_{B_j}}-\frac{1}{n_E}$. This remark gives

THEOREM 1. *Let $\mathfrak{A}$, $\mathfrak{B}$ be classifications of E. Then $\mathfrak{A}$, $\mathfrak{B}$ are orthogonal if and only if*

$$\frac{n_{A_iB_j}}{n_{A_i}n_{B_j}}\equiv\frac{1}{n_E}. \tag{3}$$

Next consider the case $E\subset E'$. Let $\mathfrak{B}\wedge E$ stand for the classification consisting of EB_j such that $EB_j\neq 0$. It is not hard to show

THEOREM 2. *Let $\mathfrak{A}$, $\mathfrak{B}$ be classifications of E, E' respectively. If $E\subset E'$, then $P_{\mathfrak{A}}P_{\mathfrak{B}}=0$ if and only if $P_{\mathfrak{A}}P_{\mathfrak{B}\wedge E}=0$.*

3. Here we shall be concerned with the conditions under which $P_{\mathfrak{A}}$, $P_{\mathfrak{B}}$ are commutative. Given classifications $\mathfrak{A}$, $\mathfrak{B}$ of E, $\mathfrak{A}$ is a sub-classification of $\mathfrak{B}$ when any $\mathfrak{A}$-class is contained in some $\mathfrak{B}$-class. *Notation* $\mathfrak{B}\leq\mathfrak{A}$.

THEOREM 3. *Let $\mathfrak{A}$, $\mathfrak{B}$ be classifications of E. $P_{\mathfrak{A}}$, $P_{\mathfrak{B}}$ are commutative if and only if there exists a classification $\mathfrak{C}=(C_1, \dots C_l)$ of E such that $P_{\mathfrak{A}\wedge C_k}$, $P_{\mathfrak{B}\wedge C_k}$ are orthogonal. If this is the case, then $P_{\mathfrak{A}}P_{\mathfrak{B}}=P_{\mathfrak{C}}$ holds.*

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2) Cf. T. Ogasawara and M. Takahashi [1]. 4, Theorem 3.

Proof. *Necessity.* $P_{\mathfrak{A}}P_{\mathfrak{B}}=P_{\mathfrak{B}}P_{\mathfrak{A}}$ gives

(1) $$\sum_{i,j}(\xi, e_{B_j})\Phi(A_i, B_j)e_{A_i}=\sum_{i,j}(\xi, e_{A_i})\Phi(A_i, B_j)e_{B_j},$$

which implies

(2) $$\frac{n_{A_iB_{j'}}}{n_{A_i}n_{B_{j'}}}=\frac{n_{A_{i'}B_j}}{n_{A_{i'}}n_{B_j}} \quad \text{for } A_iB_j\neq 0 \text{ and } A_{i'}, B_{j'}\neq 0.$$

We write $A_i\sim A_{i'}$ if A_i, $A_{i'}$ intersect the same $\mathfrak{B}$-class. Then (2) will show that "$\sim$" is an equivalent relation. Let C_{A_i} be the union of $\mathfrak{A}$-classes equivalent to A_i. In like manner we define C_{B_j}. It is easy to verify that $C_{A_i}=C_{B_j}$ is equivalent to $A_iB_j\neq 0$. Let $\mathfrak{C}=(C_1, \ldots, C_l)$ be the set of different C_{A_i}. It is obvious that $\mathfrak{C}$ is a classification of E. Then (2) gives

(3) $$\frac{n_{A_iB_j}}{n_{A_i}n_{B_j}}=\frac{1}{n_{C_k}} \qquad \text{for } A_i, B_j\subset C_k,$$

which implies $\perp(\mathfrak{A}\wedge C_k, \mathfrak{B}\wedge C_k)$.

Sufficiency. We can write $P_{\mathfrak{A}}=\sum P_{\mathfrak{A}\wedge C_k}+P_{\mathfrak{C}}$, $P_{\mathfrak{B}}=\sum P_{\mathfrak{B}\wedge C_k}+P_{\mathfrak{C}}$. Hence it is easy to see that $P_{\mathfrak{A}}P_{\mathfrak{B}}=P_{\mathfrak{C}}$.

Let $d(\mathfrak{A})$ stand for degrees of freedom of $S_{\mathfrak{A}}$. Theorem 3 gives

THEOREM 4. *Let $\mathfrak{A}$, $\mathfrak{B}$ be classifications of E. Then $P_{\mathfrak{A}}P_{\mathfrak{B}}=P_{\mathfrak{A}}$ is equivalent to $\mathfrak{A}\leqq\mathfrak{B}$. If this is the case, then $\mathfrak{A}=\mathfrak{B}$ holds if and anly if $d(\mathfrak{A})=d(\mathfrak{B})$ holds.*

Let P, Q be projective operators. We wrte $P\leqq Q$ when $PQ=P$.

In like manner the following two theorems can be proved.

THEOREM 5. *Let $\mathfrak{A}$, $\mathfrak{B}$ be classifications of E, E' respectively. Suppose that $E\subset E'$ and $E'-E\neq 0$. If there exists a $\mathfrak{B}$-class intersecting both E and $E'-E$, then $P_{\mathfrak{A}}$, $P_{\mathfrak{B}}$ are commutaive if and only if $P_{\mathfrak{A}}P_{\mathfrak{B}}=0$. Otherwise $P_{\mathfrak{A}}$, $P_{\mathfrak{B}}$ are commutative if and only if $P_{\mathfrak{A}}$, $P_{\mathfrak{B}\wedge E}$ are commutative, and when this is the case, we have $P_{\mathfrak{A}}P_{\mathfrak{B}}=P_{\mathfrak{A}}P_{\mathfrak{B}\wedge E}$.*

THEOREM 6. *Let $\mathfrak{A}$, $\mathfrak{B}$ be classifications of E, E' respectively. Suppose that $E-E'\neq 0$, $E'-E\neq 0$. Then $P_{\mathfrak{A}}$, $P_{\mathfrak{B}}$ aɹe commutative if and only if $P_{\mathfrak{A}}P_{\mathfrak{B}}=0$.*

4. In the sequel we shall be concerned with classifications of I. If, for any given $\mathfrak{A}$, $\mathfrak{B}$, any $\mathfrak{A}$-class intersects each $\mathfrak{B}$-class, then the classification consisting of classes A_iB_j will be denoted by $\mathfrak{A}\wedge\mathfrak{B}$. In like manner we define $\mathfrak{A}_1\wedge\mathfrak{A}_2\wedge\ldots\wedge\mathfrak{A}_n$ when such an intersection property holds.

DEFINITION. $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ are said to be orthogonal when

$$(1) \qquad \frac{n_{A_{i_1}^{(1)} A_{i_2}^{(2)} \ldots A_{i_n}^{(n)}}}{n_{A_{i_1}^{(1)}} n_{A_{i_2}^{(2)}} \ldots n_{A_{i_n}^{(n)}}} \equiv \frac{1}{N^{n-1}} \qquad \text{for} \quad A_i^{(\nu)} \in \mathfrak{A}_\nu$$

holds. *Notation* $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$.

DEFINITION. $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ are said to be regularly orthogonal when $\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \ldots \wedge \mathfrak{A}_n$ is well defined and regular.

It can be proved that $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are regularly orthogonal if and only if each $\mathfrak{A}_\nu$ is regular and $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$ holds. We remark that regularly orthogonal classifications corresponds to ordinary n-way classifications in the analysis of variance.

LEMMA 1. *$\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$ is equivalent to that $\perp(\mathfrak{A}_1, \mathfrak{A}_2)$, $\perp(\mathfrak{A}_1 \wedge \mathfrak{A}_2, \mathfrak{A}_3)$, ..., and $\perp(\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \ldots \wedge \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ hold.*

LEMMA 2. *$\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$ is equivaleut to that $\perp(\mathfrak{A}_1, \ldots, \mathfrak{A}_p)$, $\perp(\mathfrak{A}_{p+1}, \ldots, \mathfrak{A}_n)$* and $\perp(\mathfrak{A}_1 \wedge \ldots \wedge \mathfrak{A}_p, \mathfrak{A}_{p+1} \wedge \ldots \wedge \mathfrak{A}_n)$ hold.

LEMMA 3. *$\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$ implies $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p)$ for any $p<n$.*

LEMMA 4. *$\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$ and $\mathfrak{B}_p \leqq \mathfrak{A}_p$, $p=1, 2, \ldots, n$ imply $\perp(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n)$.*

LEMMA 5. $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n)$, $\perp(\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_q)$ and $\mathfrak{B}_1 \wedge \ldots \wedge \mathfrak{B}_q \leqq \mathfrak{A}_1 \wedge \ldots \wedge \mathfrak{A}_p$ *imply* $\perp(\mathfrak{B}_1 \ldots \mathfrak{B}_q, \mathfrak{A}_{p+1} \ldots \mathfrak{A}_n)$.

These lemmas are immediate from the orthogonality condition (1).

LEMMA 6. *Let $\perp(\mathfrak{A}, \mathfrak{B})$, $\perp(\mathfrak{A}, \mathfrak{C})$ hold. Then $P_{\mathfrak{A} \wedge \mathfrak{B}} P_{\mathfrak{A} \wedge \mathfrak{C}} = P_{\mathfrak{A}}$ if and only if $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ holds.*

Proof. It follows from Theorem 3 that $P_{\mathfrak{A} \wedge \mathfrak{B}} P_{\mathfrak{A} \wedge \mathfrak{C}} = P_{\mathfrak{A}}$ is equivalent to $\perp(A_i \wedge \mathfrak{B}, A_i \wedge \mathfrak{C})$, $i=1, 2, \ldots$, that is, $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.

When $P_{\mathfrak{A}}, P_{\mathfrak{B}}$ are commutative, $\mathfrak{C}$ as stated in Theorem 3 will be denoted by $\mathfrak{A} \vee \mathfrak{B}$. By the same way as in Lemma 6 we can prove the following two lemmas.

LEMMA 7. *Suppose that $\perp(\mathfrak{A}, \mathfrak{B})$, and $\mathfrak{B}_1, \mathfrak{B}_2 \leqq \mathfrak{B}$ hold. Then the following conditions are equivalent:*

(i) *$P_{\mathfrak{B}_1}, P_{\mathfrak{B}_2}$ are commutative.*

(ii) *$P_{\mathfrak{A} \wedge \mathfrak{B}_1}, P_{\mathfrak{B}_2}$ are commutative.*

(iii) *$P_{\mathfrak{A} \wedge \mathfrak{B}_1}, P_{\mathfrak{A} \wedge \mathfrak{B}_2}$ are commutative.*

And any one of these equivalent conditions gives $P_{\mathfrak{A} \wedge \mathfrak{B}_1} P_{\mathfrak{B}_2} = P_{\mathfrak{B}_1 \vee \mathfrak{B}_2}$ and $P_{\mathfrak{A} \wedge \mathfrak{B}_1} P_{\mathfrak{A} \wedge \mathfrak{B}_2} = P_{\mathfrak{A} \wedge (\mathfrak{B}_1 \vee \mathfrak{B}_2)}$.

LEMMA 8. *Suppose that* $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, *and* $\mathfrak{C}_1, \mathfrak{C}_2 \leq \mathfrak{C}$ *hold. Then the following conditions are equivalent:*

(i) $P_{\mathfrak{C}_1}, P_{\mathfrak{C}_2}$ *are commutative.*

(ii) $P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}_1}, P_{\mathfrak{A} \wedge \mathfrak{C}_2}$ *are commutative.*

(iii) $P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}_1}, P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}_2}$ *are commutative.*

And any one of these equivalent conditions gives $P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}_1}; P_{\mathfrak{A} \wedge \mathfrak{C}_2} = P_{\mathfrak{A} \wedge (\mathfrak{C}_1 \vee \mathfrak{C}_2)}$ *and* $P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}_1} P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}_2} = P_{\mathfrak{A} \wedge \mathfrak{B} \wedge (\mathfrak{C}_1 \vee \mathfrak{C}_2)}$.

5. In this section we shall define interaction operators associated with interaction variances. Let $\perp(\mathfrak{A}, \mathfrak{B})$. The interaction variance of type $\mathfrak{A}, \mathfrak{B}$ is, by definition, given by $S_{\mathfrak{A} \wedge \mathfrak{B}} - S_{\mathfrak{A}} - S_{\mathfrak{B}}$ to which corresponds an operator $P_{\mathfrak{A} \wedge \mathfrak{B}} - P_{\mathfrak{A}} - P_{\mathfrak{B}}$. It is a projective operator $P_{\mathfrak{A}\mathfrak{B}}$, which we call the interaction operator of type $\mathfrak{A}, \mathfrak{B}$. It is clear that $\| P_{\mathfrak{A}\mathfrak{B}} x \|^2 = S_{\mathfrak{A}\mathfrak{B}}$. Denote by $d(\mathfrak{A}\mathfrak{B})$ the degrees of freedom of $S_{\mathfrak{A}\mathfrak{B}}$, then we have[3)]

$$\begin{aligned} d(\mathfrak{A}\mathfrak{B}) &= trace\ (P_{\mathfrak{A} \wedge \mathfrak{B}} - P_{\mathfrak{A}} - P_{\mathfrak{B}}) \\ &= \{d(\mathfrak{A})+1\}\{d(\mathfrak{B})+1\} - 1 - d(\mathfrak{A}) - d(\mathfrak{B}) \\ &= d(\mathfrak{A}) d(\mathfrak{B}). \end{aligned}$$

In like manner we may define interaction operators with many factors. For example, let $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. Then the relation $S_{\mathfrak{A}\mathfrak{B}\mathfrak{C}} = S_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}} - S_{\mathfrak{A}\mathfrak{B}} - S_{\mathfrak{B}\mathfrak{C}} - S_{\mathfrak{C}\mathfrak{A}} - S_{\mathfrak{A}} - S_{\mathfrak{B}} - S_{\mathfrak{C}}$ leads us to define

$$\begin{aligned} P_{\mathfrak{A}\mathfrak{B}\mathfrak{C}} &= P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}} - P_{\mathfrak{A}\mathfrak{B}} - P_{\mathfrak{B}\mathfrak{C}} - P_{\mathfrak{C}\mathfrak{A}} - P_{\mathfrak{A}} - P_{\mathfrak{B}} - P_{\mathfrak{C}} \\ &= P_{\mathfrak{A} \wedge \mathfrak{B} \wedge \mathfrak{C}} - P_{\mathfrak{A} \wedge \mathfrak{B}} - P_{\mathfrak{B} \wedge \mathfrak{C}} - P_{\mathfrak{C} \wedge \mathfrak{A}} + P_{\mathfrak{A}} + P_{\mathfrak{B}} + P_{\mathfrak{C}}, \end{aligned}$$

which is seen to be projective by Lemma 6.

Interaction operator $P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}$, $n>1$, is said to be of order $n-1$, and is only defined for $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n)$. For the sake of completeness we say that classification operators are interaction operators of order zero.

Without difficulty we can prove

THEOREM 7. *Let* $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n)$ *hold. Then we have*

(i) $\| P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n} x \|^2 = S_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}$.

(ii) $d(\mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_n) = d(\mathfrak{A}_1) d(\mathfrak{A}_2) \cdots d(\mathfrak{A}_n)$.

(iii) $P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}$ *is a projective operator and* $P_{\mathfrak{A}_1 \wedge \mathfrak{A}_2 \wedge \cdots \wedge \mathfrak{A}_n}$ *is the sum of all interaction operators of order from* 0 *to* $n-1$ *with factors chosen*

3) Cf. T. Ogasawara and M. Takahashi [1]. 3, Theorem I.

from $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, *and these interaction operators are mutually orthogonal.*

(iv) *Let* P *be a projective operator.* $P \leqq P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}$ *is equivalent to* $PP_{\mathfrak{A}_2\wedge\mathfrak{A}_3\wedge\cdots\mathfrak{A}_n}=0$, $PP_{\mathfrak{A}_1\wedge\mathfrak{A}_3\wedge\cdots\mathfrak{A}_n}=0$, ..., $PP_{\mathfrak{A}_1\wedge\mathfrak{A}_2\wedge\cdots\mathfrak{A}_{n-1}}=0$ *and* $P\leqq P_{\mathfrak{A}_1\wedge\mathfrak{A}_2\wedge\cdots\wedge\mathfrak{A}_n}$.

(v) *In case* (iv), $P=P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}$ *if and only if trace* $(P)=d(\mathfrak{A}_1)d(\mathfrak{A}_2)\cdots d(\mathfrak{A}_n)$ *holds.*

(vi) $P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_p(\mathfrak{A}_{p+1}\wedge\cdots\wedge\mathfrak{A}_n)}$ *is the sum of all interaction operators with at least* $p+1$ *factors chosen from* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ *and containing* $\mathfrak{A}_1, \dots, \mathfrak{A}_p$.

It is noted that (ii) of this theorem implies that $P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}=0$ occurs if and only it one of $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is trivial.

THEOREM 8. $\perp(\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n)$ *and* $\mathfrak{B}_p \leqq \mathfrak{A}_p$, $p=1,2,\dots n$, *imply* $P_{\mathfrak{B}_1\mathfrak{B}_2\cdots\mathfrak{B}_n} \leqq P_{\mathfrak{A}_1\cdots\mathfrak{A}_2\cdots\mathfrak{A}_n}$.

Proof. It suffices to show that $P_{\mathfrak{A}_1\cdots\mathfrak{A}_{n-1}\mathfrak{B}_n} \leqq P_{\mathfrak{A}_1\cdots\mathfrak{A}_n}$. Since $\perp(\mathfrak{A}_1, \dots, \mathfrak{A}_{n-1}, \mathfrak{B}_n)$ holds by Lemma 4, Theorem 7 gives $P_{\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_{n-1}}P_{\mathfrak{A}_1\cdots\mathfrak{A}_{n-1}\mathfrak{B}_n}=0$. Put $\mathfrak{A}^{(i)}=\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_{i-1}\wedge\mathfrak{A}_{i+1}\wedge\cdots\mathfrak{A}_{n-1}$. Owing to Lemma 8 and Theorem 7 it is easy to see that

$$
\begin{aligned}
P_{\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_{i-1}\wedge\mathfrak{A}_{i+1}\wedge\cdots\wedge\mathfrak{A}_n}&P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_{n-1}\mathfrak{B}_n} \\
&=P_{\mathfrak{A}^{(i)}\wedge\mathfrak{A}_n}(P_{\mathfrak{A}^{(i)}\wedge\mathfrak{A}_i\wedge\mathfrak{B}_n}P_{\mathfrak{A}_1\cdots\mathfrak{A}_{n-1}\mathfrak{B}_n}) \\
&=(P_{\mathfrak{A}^{(i)}\wedge\mathfrak{A}_n}P_{\mathfrak{A}^{(i)}\wedge\mathfrak{A}_i\wedge\mathfrak{B}_n})P_{\mathfrak{A}_1\cdots\mathfrak{A}_{n-1}\mathfrak{B}_n} \\
&=P_{\mathfrak{A}^{(i)}\wedge\mathfrak{B}_n}P_{\mathfrak{A}_1\cdots\mathfrak{A}_{n-1}\mathfrak{B}_n} \\
&=0,
\end{aligned}
$$

which completes the proof by Theorem 7.

6. As R. A. Fisher observed[4], if each factor in factorial experiments is tested at two levels, any interaction is a classification. We shall show that the converse is also true. For this purpose we need

LEMMA 9. $P_{\mathfrak{C}} \leqq P_{\mathfrak{A}_1\cdots\mathfrak{A}_n}$ *implies* $d(\mathfrak{C}) \leqq d(\mathfrak{A}_1), \dots, d(\mathfrak{A}_n)$.

Proof. In order to prove this lemma we suppose that $d(\mathfrak{A}_1) \geqq d(\mathfrak{A}_2) \geqq \cdots \geqq d(\mathfrak{A}_n)$. $P_{\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_n} \geqq P_{\mathfrak{A}_1\cdots\mathfrak{A}_n}$ implies $\mathfrak{C} \leqq \mathfrak{A}_{\,}\wedge\cdots\wedge\mathfrak{A}_n$, so that, by Theorem 7, $P_{\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_{n-1}}P_{\mathfrak{C}}=0$, that is, $\perp(\mathfrak{A}_{\,}, \dots \mathfrak{A}_{n-1}, \mathfrak{C})$. Hence it follows that $\mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_{n-1}\wedge\mathfrak{C} \leqq \mathfrak{A}_1\wedge\cdots\wedge\mathfrak{A}_n$, so that $\{d(\mathfrak{A}_1)+1\}\cdots\{d(\mathfrak{A}_{n-1})+1\}\{d(\mathfrak{C})+1\} \leqq \{d(\mathfrak{A}_{\,})+1\}\cdots\{d(\mathfrak{A}_n)+1\}$ which implies $d(\mathfrak{C}) \leqq d(\mathfrak{A}_n)$, completing the proof.

THEOREM 9. *Suppose that no* $\mathfrak{A}_p$, $p=1, 2, \cdots, n$ *is trivial.* $P_{\mathfrak{A}_1\cdots\mathfrak{A}_n}$ *is a classification operator* $P_{\mathfrak{C}}$ *if and only if* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ *are regulerly orthogonal and* $d(\mathfrak{A}_p)=1$, $p=1, 2, \cdots, n$.

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4) Cf. R. A. Fisher [1]. 112.

Proof. *Necessity.* Theorem 7 and Lemma 9 give

$$d(\mathfrak{C}) = d(\mathfrak{A}\ )d(\mathfrak{A}_2) \cdots d(\mathfrak{A}_n) \geq d(\mathfrak{C})^n,$$

which implies $d(\mathfrak{C}) = d(\mathfrak{A}_1) = \cdots = d(\mathfrak{A}_n) = 1$ since by hypothesis, $d(\mathfrak{A}_p) > 0$, $p = 1, 2, \ldots, n$, holds. We may now set $\mathfrak{A}_p = (A_1^{(p)}, A_2^{(p)})$, $p = 1, 2, \ldots, n$ and $\mathfrak{C} = (C_1, C_2)$ where $A_1^{(1)} A_1^{(2)} \cdots A_1^{(n)} < C_1$. Then C_1 will be the union of terms with positive sign in the expansion of the formal product $(A_1^{(1)} - A_2^{(1)}) \cdots (A_1^{(n)} - A_2^{(n)})$, and C_2 the rest. Orthogonality relation in Sec. 4 (1) shows that

$$n_{A_1^{(1)}} n_{A_1^{(2)} \cdots A_1^{(n)}} = N n_{A_1^{(n)} \cdots A_1^{(n)}} = N n_{C_1 A_1^{(2)} \cdots A_1^{(n)}} = n_{C_1} n_{A_1^{(2)} \cdots A_1^{(n)}}$$

and

$$n_{A_2^{(1)}} n_{A_2^{(2)} A_1^{(3)} \cdots A_1^{(n)}} = N n_{A_2^{(1)} A_2^{(2)} A_1^{(3)} \cdots A_1^{(n)}} = N n_{C_1 A_2^{(2)} A_1^{(3)} \cdots A_1^{(n)}} = n_{C_1} n_{A_2^{(2)} A_1^{(3)} \cdots A_1^{(n)}},$$

which imply $n_{A_1^{(1)}} = n_{A_2^{(1)}}$, that is, $\mathfrak{A}_1$ is regular. The same will be true of each $\mathfrak{A}_p$.

Sufficiency. Put $\mathfrak{A}_p = (A_1^{(p)}, A_2^{(p)})$, and difine $\mathfrak{C} = (C_1, C_2)$ as above. Then it is clear that $n_{C_1} = n_{C_2} = \dfrac{N}{2}$ and $d(\mathfrak{C}) = d(\mathfrak{A}_1 \cdots \mathfrak{A}_n)$, which imply $\dfrac{n_{A_1^{(1)} \cdots A_1^{(n-1)} C_1}}{n_{A_1^{(1)}} \cdots n_{A_1^{(n-1)}} n_{C_1}} = \dfrac{1}{N^{n-1}}$ since $n_{A_1^{(1)} \cdots A_1^{(n-1)} C_1} = n_{A_1^{(1)} \cdots A_1^{(n)}} = \dfrac{1}{N^{n-1}} \times \left(\dfrac{N}{2}\right)^n = \dfrac{N}{2^n}$ holds. This and similar relations show that $P_{\mathfrak{C}}$ is orthogonal to $P_{\mathfrak{A}_2 \wedge \cdots \wedge \mathfrak{A}_n}$, $\ldots$, $P_{\mathfrak{A}_1 \wedge \cdots \wedge \mathfrak{A}_{n-1}}$. Owing to Theorem 7 we obtain $P_{\mathfrak{C}} = P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n}$.

COROLLARY. *Suppose that no $\mathfrak{A}_p$, $p = 1, 2, \ldots, n$ is trivial. If $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n}$ is a classification operator, then so is also for $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_p}$, $p < n$.*

THEOREM 10. *Suppose that $\perp(\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \mathfrak{B}, \mathfrak{C})$ holds. If $P_{\mathfrak{B}\mathfrak{C}}$ is a classification operator $P_{\mathfrak{D}}$, then $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}\mathfrak{C}} = P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{D}}$.*

Proof. We may suppose that none of $\mathfrak{B}$, $\mathfrak{C}$ is trivial. Lemma 5 implies $\perp(\mathfrak{A}_1 \cdots \mathfrak{A}_n, \mathfrak{D})$ since $\mathfrak{D} \leq \mathfrak{B} \wedge \mathfrak{C}$ holds, so that by Theorem 7 $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}} P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{D}} = P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{C}} P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{D}} = 0$. Hence also by Theorem 7 $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}\mathfrak{C}} \geq P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{D}}$ which implies $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}\mathfrak{C}} = P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{D}}$ since $d(\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}\mathfrak{C}) = d(\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{D})$ holds.

When $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n} = P_{\mathfrak{C}}$ holds, we shall say that $\mathfrak{A}_1 \cdots \mathfrak{A}_n$ is a classification. Thus the result of Theorem 10 will be written as $P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n \mathfrak{B}\mathfrak{C}} = P_{\mathfrak{A}_1 \cdots \mathfrak{A}_n (\mathfrak{B}\mathfrak{C})}$.

COROLLARY. *Suppose that $\perp(\mathfrak{A}_1 \cdots \mathfrak{A}_n, \mathfrak{B}, \mathfrak{C})$ holds and $\mathfrak{B}\mathfrak{C}$ is a classification. If $\mathfrak{C}$ is not trivial, then $P_{\mathfrak{A}_1 \mathfrak{A} \cdots \mathfrak{A}_n (\mathfrak{B}\mathfrak{C}) \mathfrak{C}} = P_{\mathfrak{A}_1 \mathfrak{A}_2 \cdots \mathfrak{A}_n \mathfrak{B}}$.*

Now assume that $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_n$ is a classification and no $\mathfrak{A}_p$ is trivial.

Then repeated use of Theorem 10 and its corollary gives[5)]

$$P_{\mathfrak{A}_1\cdots\mathfrak{A}_n}=P_{(\mathfrak{A}_1\cdots\mathfrak{A}_p)(\mathfrak{A}_{p+1}\cdots\mathfrak{A}_n)},$$
$$P_{\mathfrak{A}_1\cdots\mathfrak{A}_p\mathfrak{A}_{q+1}\cdots\mathfrak{A}_n}=P_{(\mathfrak{A}_1\cdots\mathfrak{A}_p\cdots\mathfrak{A}_q)(\mathfrak{A}_{p+1}\cdots\mathfrak{A}_n)}.$$

7. Let $I=(1, 2, \dots, N)$ and $I'=(1, 2, \dots, N')$ be two index sets, and $I\times I'$ their product, that is, the set of (λ, μ), $\lambda\in I$, $\mu\in I'$. Let $\mathfrak{A}=(A_1, \dots, A_n)$ stand for any classification of I. Put $\bar{A}_i=\{(\lambda, \mu); \lambda\in A_i\}$. Then $\bar{\mathfrak{A}}=(\bar{A}_1, \dots, \bar{A}_n)$, becomes a classification of $I\times I'$, and we have $P_{\bar{\mathfrak{A}}}=P_{\mathfrak{A}}\otimes P_{I'}$, where $\otimes$ mean Kronecker product. In like manner $\bar{\mathfrak{B}}'$ is defined for any classification $\mathfrak{B}'$ of I', and $P_{\bar{\mathfrak{B}}'}=P_I\otimes P_{\mathfrak{B}'}$. It is easy to see that $\bar{\mathfrak{A}}, \bar{\mathfrak{B}}'$ are orthogonal and $P_{\bar{\mathfrak{A}}\bar{\mathfrak{B}}'}=P_{\mathfrak{A}}\otimes P_{\mathfrak{B}'}$, and $\perp(\mathfrak{B}', \mathfrak{B}'')$ implies $\perp(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}', \bar{\mathfrak{B}}'')$ and $P_{\bar{\mathfrak{A}}\bar{\mathfrak{B}}'\bar{\mathfrak{B}}''}=P_{\mathfrak{A}}\otimes P_{\mathfrak{B}'\mathfrak{B}''}$. $\otimes$ multiplication is bilinear with respect to both factors. This leads us to the following theorems.

THEOREM 11. *Suppose that* $\perp(\mathfrak{A}_1, \dots, \mathfrak{A}_n, \mathfrak{B})$. *Then* $P_{\mathfrak{B}}=\sum_1^p P_{\mathfrak{B}_i}$ *implies* $P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n\mathfrak{B}}=\sum_1^p P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}_i}$.

Proof. Since $\perp(\mathfrak{A}_1, \dots, \mathfrak{A}_n, \mathfrak{B})$, $\mathfrak{B}_i\leqq\mathfrak{B}$ and $\perp(\mathfrak{B}_i, \mathfrak{B}_j)$, $i\neq j$, hold, it follows by Lemmas 2, 4 and Theorem 7 that $\perp(\mathfrak{A}_1, \dots, \mathfrak{A}_n, \mathfrak{B}_i, \mathfrak{B}_j)$, $P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}}\geqq P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}_i}$, and $P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}_i}P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}_j}=0$ for $i\neq j$. While $d(\mathfrak{B})=\sum_1^p d(\mathfrak{B}_i)$ implies $d(\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B})=\sum d(\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}_i)$. Hence we must have the conclusion.

By the same way we can prove

THEOREM 12. *Suppose that* $\perp(\mathfrak{A}_1, \cdots, \mathfrak{A}_n, \mathfrak{B}, \mathfrak{C})$. *Then* $P_{\mathfrak{B}\mathfrak{C}}=\sum_1 P_{\mathfrak{D}_i}$ *implies* $P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{B}\mathfrak{C}}=\sum_1^p P_{\mathfrak{A}_1\cdots\mathfrak{A}_n\mathfrak{D}_i}$.

This theorem gives a method, as R. A. Fisher used[6)] in factorial experiments, to decompose an interaction into classifications.

8. In this last section we shall establish further properties of interaction operators. For the sake of simplicity we shall be concerned with interaction operators of order at most two.

THEOREM 13. *Suppose that* $\perp(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{B}_1, \mathfrak{B}_2\leqq\mathfrak{B}$ *and* $P_{\mathfrak{B}_1}, P_{\mathfrak{B}_2}$ *are commutative. Then* $P_{\mathfrak{A}\mathfrak{B}_1}P_{\mathfrak{A}\mathfrak{B}_2}=P_{\mathfrak{A}(\mathfrak{B}_1\vee\mathfrak{B}_2)}$.

Proof. By using Lemma 6 we obtain

$$\begin{aligned}P_{\mathfrak{A}\mathfrak{B}_1}P_{\mathfrak{A}\mathfrak{B}_2}&=(P_{\mathfrak{A}\wedge\mathfrak{B}_1}-P_{\mathfrak{A}}-P_{\mathfrak{B}_1})(P_{\mathfrak{A}\wedge\mathfrak{B}_2}-P_{\mathfrak{A}}-P_{\mathfrak{B}_2})\\&=P_{\mathfrak{A}\wedge\mathfrak{B}_1}P_{\mathfrak{A}\wedge\mathfrak{B}_2}-P_{\mathfrak{A}}-P_{\mathfrak{A}\wedge\mathfrak{B}_1}P_{\mathfrak{B}_2}-P_{\mathfrak{B}_1}P_{\mathfrak{A}\wedge\mathfrak{B}_2}+P_{\mathfrak{B}_1}P_{\mathfrak{B}_2}\end{aligned}$$

5) Cf. R. A. Fisher [I]. 113.
6) Cf. R. A. Fisher [1]. 122.

$$=P_{\mathfrak{A}\wedge(\mathfrak{B}_1\vee\mathfrak{B}_2)}-P_{\mathfrak{A}}-P_{\mathfrak{B}_1\vee\mathfrak{B}_2}-P_{\mathfrak{B}_1\vee\mathfrak{B}_2}+P_{\mathfrak{B}_1\vee\mathfrak{B}_2}$$
$$=P_{\mathfrak{A}(\mathfrak{B}_1\vee\mathfrak{B}_2)}.$$

THEOREM 14. *We suppose* $d(\mathfrak{A})>1$. *Then* $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ *implies* $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.

Proof. $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ is written as

(1) $$P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{A}\wedge\mathfrak{C}}-P_{\mathfrak{A}}=P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{C}}+P_{\mathfrak{B}}P_{\mathfrak{A}\wedge\mathfrak{C}}-P_{\mathfrak{B}}P_{\mathfrak{C}}.$$

After some calculation it follows that

(2) $$P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{A}\wedge\mathfrak{C}}\xi-P_{\mathfrak{A}}\xi=\sum_{i,j,k}(\xi, e_{A_iC_k})\frac{N^2}{n_{A_i}}\phi(i, j, k)e_{A_iB_j},$$

(3) $$P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{C}}\xi+P_{\mathfrak{B}}P_{\mathfrak{A}\wedge\mathfrak{C}}\xi-P_{\mathfrak{B}}P_{\mathfrak{C}}\xi$$
$$=\sum_{i,j,k}(\xi, e_{C_k})N\phi(i, j, k)e_{A_iB_j}+\sum_{i,j,k}(\xi, e_{A_iC_k})N\phi(i, j, k)e_{B_j}-$$
$$\sum_{j,k}(\xi, e_{C_k})\Phi(B_j, C_k)e_{B_j},$$

where $\phi(i, j, k)$ denote $\dfrac{N_{A_iB_jC_k}}{n_{A_i}n_{B_j}n_{C_k}}-\dfrac{1}{N^2}$. Equating the coefficients of $e_{A_iB_j}$ in (2), (3) gives

(4) $$\sum_{\nu}(\xi, e_{A_iC_\nu})\phi(i, j, \nu)\frac{N}{n_{A_i}}=\sum_{\lambda,\nu}(\xi, e_{A_\lambda C_\nu})\phi(\lambda, j, \nu)+\sum_{\nu}(\xi, e_{C_\nu})\phi(i, j, \nu)$$
$$-\sum_{\nu}(\xi, e_{C_\nu})\frac{1}{N}\Phi(B_j, C_\nu).$$

Put $\xi=e_{A_iC_k}$, then (4) gives

(5) $$\left(\frac{N}{n_{A_i}}-2\right)\phi(i, j, k)+\frac{1}{N}\Phi(B_j, C_k)=0$$

Similarly $\xi=e_{A_{i'}E_k}$, $i'\neq i$, gives

(6) $$\phi(i, j, k)+\phi(i', j, k)=\frac{1}{N}\Phi(B_j, C_k)$$

And for $i''\neq i, i'$ we obtain

(7) $$\phi(i', j, k)+\phi(i'', j, k)=\frac{1}{N}\Phi(B_j, C_k)$$

(8) $$\phi(i'', j, k)+\phi(i, j, k)=\frac{1}{N}\Phi(B_j, C_k).$$

From these equations (5)—(8) we obtain $\phi(i, j, k)=0$ so that $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, completing the proof.

From the proof of the above theorem it is easy to see that when $d(\mathfrak{A})=1$, $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ is equivalent to the following

$$\left(\frac{N}{n_{A_1}}-2\right)\phi(1, j, k)+\frac{1}{N}\Phi(B_j, C_k)=0,$$
$$\left(\frac{N}{n_{A_2}}-2\right)\phi(2, j, k)+\frac{1}{N}\Phi(B_j, C_k)=0.$$

Hence we have

THEOREM 15. *Suppose that* $\perp(\mathfrak{A}, \mathfrak{B})$ *and* $\mathfrak{B}_1, \mathfrak{B}_2 \leqq \mathfrak{B}$. *A necessary and sufficient condition for* $P_{\mathfrak{A}\mathfrak{B}_1}P_{\mathfrak{A}\mathfrak{B}_2}=0$ *is that* $\perp(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ *holds, or* $\perp(\mathfrak{B}_1, \mathfrak{B}_2)$ *and* $\mathfrak{A}$ *is regular and* $d(\mathfrak{A})=1$.

THEOREM 16. *Let* $d(\mathfrak{A})=1$. *If* $\mathfrak{A}$ *is regular, then* $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ *holds if and only if* $\perp(\mathfrak{B}, \mathfrak{C})$ *holds. If* $\perp(\mathfrak{B}, \mathfrak{C})$ *holds, then* $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ *holds if and only if* $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *or* $\mathfrak{A}$ *is regular.*

THEOREM 17. *Let* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *be not trivial. Then* $P_{\mathfrak{A}\mathfrak{B}}, P_{\mathfrak{B}\mathfrak{C}}, P_{\mathfrak{C}\mathfrak{A}}$ *are mutually orthogonal if and only if* $\perp(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *or* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *are regular and* $d(\mathfrak{A})=d(\mathfrak{B})=d(\mathfrak{C})=1$.

THEOREM 18. *Suppose that* $\mathfrak{A}\wedge\mathfrak{B}=\mathfrak{A}\wedge\mathfrak{C}$ *holds and none of* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ *is trivial. Then* $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ *is equivalent to* $P_{\mathfrak{A}}=P_{\mathfrak{B}\mathfrak{C}}$.

Proof. *Necessity.* $\mathfrak{A}\wedge\mathfrak{B}=\mathfrak{A}\wedge\mathfrak{C}$ implies $d(\mathfrak{B})=d(\mathfrak{C})$. While by definition $P_{\mathfrak{A}\wedge\mathfrak{B}}=P_{\mathfrak{A}\wedge\mathfrak{C}}=P_{\mathfrak{A}}+P_{\mathfrak{C}}+P_{\mathfrak{A}\mathfrak{C}}$ which implies $P_{\mathfrak{A}\mathfrak{B}}\leqq P_{\mathfrak{C}}$ since $P_{\mathfrak{A}}P_{\mathfrak{A}\mathfrak{B}}=P_{\mathfrak{A}\mathfrak{C}}P_{\mathfrak{A}\mathfrak{B}}=0$. Hence, it follows from $d(\mathfrak{A}\mathfrak{B})\geqq d(\mathfrak{C})$ that $P_{\mathfrak{C}}=P_{\mathfrak{A}\mathfrak{B}}$ or equivalently $P_{\mathfrak{A}}=P_{\mathfrak{B}\mathfrak{C}}$.

Sufficiency. $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=P_{\mathfrak{C}}P_{\mathfrak{A}\mathfrak{C}}=0.$

COROLLARY. *Suppose that* $\mathfrak{A}\wedge\mathfrak{B}\leqq\mathfrak{A}\wedge\mathfrak{C}$. *Then* $P_{\mathfrak{A}\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ *is equivalen to* $P_{\mathfrak{A}\mathfrak{B}}\leqq P_{\mathfrak{C}}$.

THEOREM 19. *Let* $\mathfrak{A}$ *be not trivial. Then* $P_{\mathfrak{A}\mathfrak{B}}\geqq P_{\mathfrak{A}\mathfrak{C}}$ *implies* $\mathfrak{B}\geqq\mathfrak{C}$.

Proof. $P_{\mathfrak{A}\mathfrak{B}}\geqq P_{\mathfrak{A}\mathfrak{C}}$ implies $P_{\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ and $P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=P_{\mathfrak{A}\mathfrak{C}}$. Now $P_{\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=0$ is equivalent to $P_{\mathfrak{B}}P_{\mathfrak{A}\wedge\mathfrak{C}}=P_{\mathfrak{B}}P_{\mathfrak{C}}$, or

$$(1) \qquad \sum_{i,k}(\xi, e_{A_iC_k})\frac{n_{A_iB_jC_k}}{n_{B_j}n_{A_iC_k}}=\sum_k(\xi, e_{C_k})\frac{n_{B_jC_k}}{n_{B_j}n_{C_k}} \quad \textit{for any } B_j,$$

which implies

$$\frac{n_{A_iB_jC_k}}{n_{B_j}n_{A_iC_k}}=\frac{n_{B_jC_k}}{n_{B_j}n_{C_k}} \qquad \textit{for any } A_i, B_j, C_k,$$

that is,

$$(2) \qquad n_{A_iB_jC_k}=\frac{1}{N}n_{A_i}n_{B_jC_k}.$$

This shows that $B_jC_k \neq 0$ implies $A_iB_jC_k \neq 0$. While $P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{A}\mathfrak{C}}=P_{\mathfrak{A}\mathfrak{C}}$ is written as $P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{A}\wedge\mathfrak{C}}-P_{\mathfrak{A}\wedge\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{A}\wedge\mathfrak{C}}-P_{\mathfrak{C}}$, or

$$(3)\qquad \sum_{\nu}(\xi,\ e_{A_iC_\nu})\frac{n_{A_iB_jC_\nu}}{n_{A_iB_j}n_{A_iC_\nu}}-\sum_{\nu}(\xi,\ e_{C_\nu})\frac{n_{A_iB_jC_\nu}}{n_{A_iB_j}n_{C_\nu}}$$
$$=(\xi,\ e_{A_iC_k})\frac{1}{n_{A_iC_k}}-(\xi,\ e_{C_k})\frac{1}{n_{C_k}}\qquad \textit{for any } A_iB_jC_k\neq 0\,.$$

If we put $\xi=e_{A_iC_k}$ in (3) and make use of (2), then it follows that $\left(\frac{N}{n_{A_i}}-1\right)\frac{n_{B_jC_k}}{n_{B_j}n_{C_k}}=\left(\frac{N}{n_{A_i}}-1\right)\frac{1}{n_{C_k}}$, or $n_{B_jC_k}=n_{B_j}$, that is, $B_j\subset C_k$, completing the proof.

COROLLARY. *If $\mathfrak{A}$ is not trivial, then $P_{\mathfrak{A}\mathfrak{B}}=P_{\mathfrak{A}\mathfrak{C}}$ implies $\mathfrak{B}=\mathfrak{C}$.*

THEOREM 20. *Let $\mathfrak{A}$, $\mathfrak{B}$ be not trivial and suppose that $\mathfrak{A}\wedge\mathfrak{B}=\mathfrak{C}\wedge\mathfrak{D}$ holds for the case $d(\mathfrak{A})=d(\mathfrak{B})=1$. Then $P_{\mathfrak{A}\mathfrak{B}}=P_{\mathfrak{C}\mathfrak{D}}$ implies $\mathfrak{A}=\mathfrak{C}$, $\mathfrak{B}=\mathfrak{D}$ or $\mathfrak{A}=\mathfrak{D}$, $\mathfrak{B}=\mathfrak{C}$.*

Proof. If $A_1B_1C_1D_1\neq 0$ holds, then equating the coefficients of $e_{A_1B_1C_1D_1}$ in both sides of $P_{\mathfrak{A}\mathfrak{B}}\xi=P_{\mathfrak{C}\mathfrak{D}}\xi$ gives

$$(1)\qquad (\xi,\ e_{A_1B_1})\left(\frac{N}{n_{A_1}}-1\right)\left(\frac{N}{n_{B_1}}-1\right)+(\xi,\ e_{(A_1+B_1)^c})$$
$$=(\xi,\ e_{C_1D_1})\left(\frac{N}{n_{C_1}}-1\right)\left(\frac{N}{n_{D_1}}-1\right)+(\xi,\ e_{(C_1+D_1)^c}),$$

$$(2)\qquad (\xi,\ e_{A_1-B_1})\left(\frac{N}{n_{A_1}}-1\right)+(\xi,\ e_{B_1-A_1})\left(\frac{N}{n_{B_1}}-1\right)$$
$$=(\xi,\ e_{C_1-D_1})\left(\frac{N}{n_{C_1}}-1\right)+(\xi,\ e_{D_1-C_1})\left(\frac{N}{n_{D_1}}-1\right).$$

Then (2) implies $n_{A_1}=n_{C_1}$, $n_{B_1}=n_{D_1}$ or $n_{A_1}=n_{D_1}$, $n_{B_1}=n_{C_1}$, so that $n_{A_1B_1}=n_{C_1D_1}$ holds.

Case 1. One of $\mathfrak{A}$, $\mathfrak{B}$ is not regular.

We may suppose that $\mathfrak{B}$ is not regular. Consequently for any A_1 we can choose B_1, C_1, D_1 such that $A_1B_1C_1D_1\neq 0$ and $n_{A_1}\neq n_{B_1}$. In order to prove the theorem we may suppose that $n_{A_1}=n_{C_1}$, $n_{B_1}=n_{D_1}$. Then (2) gives $A_1-B_1=C_1-D_1$, $B_1-A_1=D_1-C_1$. If $\left(\frac{N}{n_{A_1}}-1\right)\left(\frac{N}{n_{B_1}}-1\right)\neq 1$ holds, then (1) gives $A_1B_1=C_1D_1$. Hence we must have $A_1=C_1$, $B_1=D_1$. Next consider the case $\left(\frac{N}{n_{A_1}}-1\right)\left(\frac{N}{n_{B_1}}-1\right)=1$. Since $A_1B_1=C_1D_1$ implies $A_1=C_1$,

$B_1 \neq D_1$, it is sufficient to consider that $A_1B_1 \neq C_1D_1$. Now $n_{A_1B_1}=n_{C_1D_1}$ implies $A_1B_1-C_1D_1 \neq 0$, $C_1D_1-A_1B_1 \neq 0$. Choose A_2, B_2 such that $A_2B_2(C_1D_1-A_1B_1) \neq 0$. It is easy to see that $A_1A_2=B_1B_2=0$. By the same argument we have $n_{A_2}=n_{D_1}$, $n_{B_2}=n_{C_1}$ and $A_2-B_2=D_1-C_1$, $B_2-A_2=C_1-D_1$. Then $\left(\frac{N}{n_{A_1}}-1\right)\left(\frac{N}{n_{A_2}}-1\right)=1$ implies $n_{A_1}+n_{A_2}=N$, so that $\mathfrak{A}=(A_1, A_2)$, $\mathfrak{B}=(B_1, B_2)$. Similarly we define C_2, D_2 and we shall have $A_1-B_1=D_2-C_2$, $B_1-A_1=C_2-D_2$ and $\mathfrak{C}=(C_1, C_2)$, $\mathfrak{D}=(D_1, D_2)$. It follows from these that $A_1B_2=C_1D_2$ and $A_2B_1=C_2D_1$. Since $A_1B_1-C_1D \neq_1 0$. $C_1D_1-A_1B_1 \neq 0$ imply $A_1B_1C_2D_2 \neq 0$, $A_2B_2C_1D_1 \neq 0$ and $\mathfrak{A} \wedge \mathfrak{B}=\mathfrak{C} \wedge \mathfrak{D}$ holds, we must have $A_1B_1=C_2D_2$, $A_2B_2=C_1D_1$. Then it follows that $A_1=A_1B_1+A_1B_2=C_2D_2+C_1D_2=D_2$ and $B_1=B_1A_1+B_1A_2=C_2D_2+C_2D_1=C_2$. Hence we obtain $\mathfrak{A}=\mathfrak{D}$ and $\mathfrak{B}=\mathfrak{C}$.

Thus any $\mathfrak{A}$-class is a $\mathfrak{C}$ or $\mathfrak{D}$-class. Since any $\mathfrak{C}$-class intersects each $\mathfrak{D}$-class, we must have $\mathfrak{A}=\mathfrak{C}$ or $\mathfrak{A}=\mathfrak{D}$. Then corollary of the preceeding theorem shows that $\mathfrak{A}=\mathfrak{C}$, $\mathfrak{B}=\mathfrak{D}$ or $\mathfrak{A}=\mathfrak{D}$, $\mathfrak{B}=\mathfrak{C}$.

Case 2. Both $\mathfrak{A}$ and $\mathfrak{B}$ are regular and $d(\mathfrak{A}) \neq d(\mathfrak{B})$.
The proof for this case will be done as for Case 1 and so omitted.

Case 3. Both $\mathfrak{A}$ and $\mathfrak{B}$ are regular and $d(\mathfrak{A})=d(\mathfrak{B})=1$. We may put $\mathfrak{A}=(A_1, A_2)$, $\mathfrak{B}=(B_1, B_2)$, $\mathfrak{C}=(C_1, C_2)$, $\mathfrak{D}=(D_1, D_2)$ and assume $A_1B_1C_1D_1 \neq 0$. Then (1), (2) give $A_1B_1+A_2B_2=C_1D_1+C_2D_2$. One of A_1B_1, A_2B_2 coincides with one of C_1D_1, C_2D_2, since $\mathfrak{A} \wedge \mathfrak{B}=\mathfrak{C} \wedge \mathfrak{D}$. The same is true of pairs A_1B_2, A_2B_1; C_1D_2, C_2D_1. Thus we obtain that $\mathfrak{A}=\mathfrak{C}$, $\mathfrak{B}=\mathfrak{D}$ or $\mathfrak{A}=\mathfrak{D}$, $\mathfrak{B}=\mathfrak{C}$.

Case 4. Both $\mathfrak{A}$ and $\mathfrak{B}$ are regular and $d(\mathfrak{A})=d(\mathfrak{B})>1$.
Since $\left(\frac{N}{n_{A_i}}-1\right)\left(\frac{N}{n_{B_j}}-1\right)>1$ for any A_j, B_j, it follows from (1), (2) that $A_1B_1=C_1D_1$, $d(\mathfrak{A})=d(\mathfrak{B})=d(\mathfrak{C})=d(\mathfrak{D})$ and $\mathfrak{C}$, $\mathfrak{D}$ are regular. Hence we obtain $\mathfrak{A} \wedge \mathfrak{B}=\mathfrak{C} \wedge \mathfrak{D}$, which implies

$$(3) \qquad P_{\mathfrak{A}}+P_{\mathfrak{B}}=P_{\mathfrak{C}}+P_{\mathfrak{D}}.$$

If any $\mathfrak{A}$-class intersects each $\mathfrak{C}$-class, then orthogonality condition will hold for $\mathfrak{A}$, $\mathfrak{C}$. And multiplying both sides of (3) by $P_{\mathfrak{A}}$ will give $P_{\mathfrak{A}}=P_{\mathfrak{A}}P_{\mathfrak{D}}$, that is, $\mathfrak{A} \leqq \mathfrak{D}$. Then $d(\mathfrak{A})=d(\mathfrak{D})$ implies $\mathfrak{A}=\mathfrak{D}$ $\mathfrak{B}=\mathfrak{C}$. Thus we may assume that there exist A_i and C_k such that $A_iC_k=0$. Take C_ν intersecting A_i, then there exist B_j, D_μ such that $A_iB_jC_\nu D_\mu \neq 0$. The equation (1) for these classes gives $A_iB_j=C_\nu D_\mu$ and $A_i+B_j=C_\nu+D_\mu$. First consider that $A_jC_\nu-A_iB_j \neq 0$ holds. Then there exist $B_{j'}$, $D_{\mu'}$ such that $B_{j'}D_{\mu'}(A_iC_\nu-A_iB_j) \neq 0$. Hence it follows from $A_iB_{j'}=C_\nu D_{\mu'}$ that $B_jB_{j'}=$

$D_\mu D_{\mu'}=0$ and $A_i+B_{j'}=C_\nu+D_{\mu'}$, so that we obtain $A_i=(A_i+B_j)(A_i+B_{j'})=(C_\nu+D_\mu)(C_\nu+D_{\mu'})=C_\nu$. Next suppose that $A_iC_\nu\neq 0$ would imply an existence of B_j such that $A_iB_j=A_iC_\nu$. Then it follows from the relation $n_{A_iC_\nu}=n_{A_iB_j}=\frac{N}{\{d(\mathfrak{A})+1\}^2}$ that $n_{A_i}=\sum_{\nu\neq k} n_{A_iC_\nu}\leq\frac{Nd(\mathfrak{A})}{\{d(\mathfrak{A})+1\}^2}<\frac{N}{d(\mathfrak{A})+1}=n_{A_i}$, which is a contradiction. Thus we have shown that A_i is a $\mathfrak{C}$-class. Therefore any $\mathfrak{A}$-class does not intersect some $\mathfrak{C}$-class. This implies by the above argument that any $\mathfrak{A}$-class is a $\mathfrak{C}$-class, that is, $\mathfrak{A}=\mathfrak{C}$. Then (3) implies $\mathfrak{B}=\mathfrak{D}$.

THEOREM 21. *Let $\mathfrak{A}$, $\mathfrak{B}$ be not trivial. Then $P_{\mathfrak{A}\mathfrak{B}}=P_{\mathfrak{A}\mathfrak{C}\mathfrak{D}}$ implies $\mathfrak{B}=\mathfrak{C}\mathfrak{D}$.*

Proof. For any given C_1, D_1 we take A_1, B_1 such that $A_1B_1C_1D_1\neq 0$. Then taking account of the coefficients of $e_{A_1B_1C_1D_1}$ in both sides of $P_{\mathfrak{A}\mathfrak{B}}\xi=P_{\mathfrak{A}\mathfrak{C}\mathfrak{D}}\xi$ we obtain

$$\begin{aligned}&(\xi,\ e_{A_1-B_1})\left(\frac{N}{n_{A_1}}-1\right)+(\xi,\ e_{B_1-A_1})\left(\frac{N}{n_{B_1}}-1\right)\\ (1)\quad &=(\xi,\ e_{A_1C_1-D_1})\left(\frac{N}{n_{A_1}}-1\right)\left(\frac{N}{n_{C_1}}-1\right)+(\xi,\ e_{A_1D_1-C_1})\left(\frac{N}{n_{A_1}}-1\right)\left(\frac{N}{n_{D_1}}-1\right)\\ &\quad+(\xi,\ e_{C_1D_1-A_1})\left(\frac{N}{n_{C_1}}-1\right)\left(\frac{N}{n_{D_1}}-1\right).\end{aligned}$$

Hence we obtain

$$(\xi,\ e_{A_1-B_1})=(\xi,\ e_{A_1C_1-D_1})\left(\frac{N}{n_{C_1}}-1\right)+(\xi,\ e_{A_1D_1-C_1})\left(\frac{N}{n_{D_1}}-1\right),$$

which implies $\frac{N}{n_{C_1}}=\frac{N}{n_{D_1}}=2$, so that $\mathfrak{C}$, $\mathfrak{D}$ are regularly orthogonal and $d(\mathfrak{C})=d(\mathfrak{D})=1$. Then Theorem 9 shows that $\mathfrak{C}\mathfrak{D}$ is a classification. By making use of Corollary of Theorem 19 we obtain $\mathfrak{B}=\mathfrak{C}\mathfrak{D}$, completing the proof.

THEOREM 22. *$P_{\mathfrak{A}}=P_{\mathfrak{B}}+P_{\mathfrak{C}}$ implies that $\mathfrak{B}$ or $\mathfrak{C}$ is trivial.*

Proof. $\mathfrak{B}$, $\mathfrak{C}\leq\mathfrak{A}$ implies $\mathfrak{B}\wedge\mathfrak{C}\leq\mathfrak{A}$. Hence we obtain that $P_{\mathfrak{A}}\geq P_{\mathfrak{B}\wedge\mathfrak{C}}=P_{\mathfrak{B}}+P_{\mathfrak{C}}+P_{\mathfrak{B}\mathfrak{C}}\geq P_{\mathfrak{A}}$. Therefore it follows that $P_{\mathfrak{B}\mathfrak{C}}=0$ that is, $\mathfrak{B}$ or $\mathfrak{C}$ is trivial.

By using Theorem 19 we have

COROLLARY. *Let $\mathfrak{A}$ be not trivial. Then $P_{\mathfrak{A}\mathfrak{B}}=P_{\mathfrak{A}\mathfrak{B}_1}+P_{\mathfrak{A}\mathfrak{B}_2}$ implies that $\mathfrak{B}_1$ or $\mathfrak{B}_2$ is trivial.*

THEOREM 23. *Let $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ be not trivial. Then $P_{\mathfrak{A}}=P_{\mathfrak{A}_1}+P_{\mathfrak{A}_2}+P_{\mathfrak{A}_3}$ implies that $\mathfrak{A}=\mathfrak{A}_1\wedge\mathfrak{A}_2$ and $\mathfrak{A}_3=\mathfrak{A}_1\mathfrak{A}_2$.*

Proof. we may suppose that $d(\mathfrak{A}_1)\geqq d(\mathfrak{A}_2)\geqq d(\mathfrak{A}_3)$ holds. Since $\mathfrak{A}_1, \mathfrak{A}_2\leqq\mathfrak{A}$ holds, it follows that $\mathfrak{A}_1\wedge\mathfrak{A}_2\leqq\mathfrak{A}$, so that we obtain $P_{\mathfrak{A}}\geqq P_{\mathfrak{A}_1}+P_{\mathfrak{A}_2}+P_{\mathfrak{A}_1\mathfrak{A}_2}$ which implies $P_{\mathfrak{A}_3}\geqq P_{\mathfrak{A}_1\mathfrak{A}_2}$. Then we must have $\mathfrak{A}_3=\mathfrak{A}_1\mathfrak{A}_2$.

By using Theorem 19 we have

Corollary. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ be not trivial. Then $P_{\mathfrak{A}\mathfrak{B}}=P_{\mathfrak{A}\mathfrak{B}_1}+P_{\mathfrak{A}\mathfrak{B}_2}+P_{\mathfrak{A}\mathfrak{B}_3}$ implies that $\mathfrak{B}=\mathfrak{B}_1\wedge\mathfrak{B}_2$ and $\mathfrak{B}_3=\mathfrak{B}_1\mathfrak{B}_2$ hold.*

References.

R. A. Fisher [I], *The Design of Experiments*, Fourth ed., Edinburgh and London 1947.

T. Ogasawara and M. Takahashi [I], *Independence of Quadratic Quantities in a Normal System*, this journal, **15** (1951), 1—9.