ON THE DERIVATIONS AND THE RELATIVE DIFFERENTS IN COMMUTATIVE FIELDS

By

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In the previous publication [1],[1] I have, in a direct way, proved the equivalence of the definition by A. Weil and Y. Kawada of the relative different in algebraic number field with the usual one which was introduced by R. Dedekind.

Recently Prof. M. Moriya[2] has developed the theory of derivations in Noetherian ring and has given the definition of the relative different in such a commutative fields as, in its integral domains, the fundamental theorem of the multiplicative ideal theory holds. And, using the theory of $\mathfrak{p}$-adic fields, he has shown the equivalence of both definitions of the relative different in such a field.

Now, in the first step of this paper, we shall show the equivalence of both definitions without using the theory of $\mathfrak{p}$-adic fields (in the analogous way to my previous paper). In the next step, we shall show that the chain-theorem and the different-theorem hold independently of the usual one in the relative different defined by derivations.

1. Let $\mathfrak{R}$ be a commutative ring, then we consider for an ideal $\mathfrak{A}(\neq(0))$ in $\mathfrak{R}$ the residue class ring $\mathfrak{R}/\mathfrak{A}$ and define a *derivation* modulo $\mathfrak{A}$ in a subring $\mathfrak{R}'$ of $\mathfrak{R}$ as a unique mapping of $\mathfrak{R}'$ into $\mathfrak{R}/\mathfrak{A}$ with the following properties:

(1) D is a module homomorphism of $\mathfrak{R}'$ into $\mathfrak{R}/\mathfrak{A}$ i.e. for $\alpha, \beta \in \mathfrak{R}'$

$$D(\alpha+\beta)=D(\alpha)+D(\beta),$$

(2) for $\alpha, \beta \in \mathfrak{R}'$

$$D(\alpha\beta)=\beta D(\alpha)+\alpha D(\beta).$$

Now we denote by $\mathfrak{D}(\mathfrak{R}', \mathfrak{r}; \mathfrak{R}/\mathfrak{A})$ the totality of all derivations modulo $\mathfrak{A}$ in $\mathfrak{R}'$ which map every element in a subring $\mathfrak{r}$ of $\mathfrak{R}'$ onto the null element

1) The number in square brackets refer to the list of references at the end of this paper. See A. Kinohara [1].

2) See M. Moriya [1].

of $\mathfrak{R}/\mathfrak{A}$. Further we shall mean as usual by $\lambda D_1+\mu D_2(\lambda, \mu \in \mathfrak{R}, D_1, D_2 \in \mathfrak{D}(\mathfrak{A}', \mathfrak{r}; \mathfrak{R}/\mathfrak{A}))$ a derivation by

$$(\lambda D_1+\mu D_2)(\alpha) = \lambda D_1(\alpha)+\mu D_2(\alpha) \quad (\alpha \in \mathfrak{R}'),$$

then $\mathfrak{D}(\mathfrak{R}', \mathfrak{r}; \mathfrak{R}/\mathfrak{A})$ is an $\mathfrak{R}$-module.

Let k be the quotient field of the commutative integral domain $\mathfrak{o}$ in which the fundamental theorem of the multiplicative ideal theory holds, and let K be a separable extension of finite degree over k, and $\mathfrak{O}$ the ring of all integral elements in K with respect to $\mathfrak{o}$. In this paper we always consider such a relative field K/k. Then, for any ideal $\mathfrak{A}(\neq(0))$ in $\mathfrak{O}$, $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$ is $\mathfrak{O}$-module and is in general the direct sum of the finite number of $\mathfrak{O}$-cyclic group. It has been proved that for every ideal $\mathfrak{A}(\neq(0))$ in $\mathfrak{O}$ the length of the composition series of $\mathfrak{O}$-module $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$ is bounded: this length is called the *dimension* of the derivation module $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$.

Now let $d_{\mathfrak{P}}$ be a maximal dimension in the set of dimensions of the derivation modules $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)(r=0, 1, 2, \cdots)$ for every prime ideal $\mathfrak{P}(\neq(0))$ in $\mathfrak{O}$. Then we can prove that $d_{\mathfrak{P}}$ are equal to zero for almost all prime ideal $\mathfrak{P}$'s in $\mathfrak{O}$ (with the exception of finite number of $\mathfrak{P}$). Thus the ideal $\mathfrak{D}_{\mathfrak{o}}(K/k)=\prod_{\mathfrak{P}\subset\mathfrak{O}}\mathfrak{P}^{d_{\mathfrak{P}}}$ in $\mathfrak{O}$ is uniquely determined by $\mathfrak{o}$ and $\mathfrak{O}$. Hereafter we shall call the ideal $\mathfrak{D}_{\mathfrak{o}}(K/k)$ the *relative different* of K/k defined by derivations.

On the other hand, the relative different $\mathfrak{D}(K/k)$ of K/k introduced by R. Dedekind, is an ideal in $\mathfrak{O}$ such that $\mathfrak{D}^{-1}(K/k)=\{\mu;\ \mu \in K\}$ with $S_{K/k}(\mu\omega) \in \mathfrak{o}$ for every element $\omega \in \mathfrak{O}$, where $S_{K/k}(\ )$ means the trace of an element in K with respect to k.

2. Now we shall study the structure of the derivation module $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$ for an arbitrary ideal $\mathfrak{A}(\neq(0))$ in $\mathfrak{O}$.

Theorem 1. *Let $\mathfrak{A}=\prod_{i=1}^{t}\mathfrak{P}_i^{e_i}$ be the prime ideal decomposition of an ideal $\mathfrak{A}$ in $\mathfrak{O}$. Then we have a direct decomposition:*

$$\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A}) \cong \sum_{i=1}^{t}\oplus\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}_i^{e_i}).$$

Proof. This follows from Y. Kawada [1] 308 Lemma 8.

Let $\mathfrak{o}^*$[1)] be the ring consisting of all integral elements in k with respect to a prime ideal $\mathfrak{p}(\subset\mathfrak{o})$. Next let $\mathfrak{O}^*$ be the ring consisting of

1) That is, $\mathfrak{o}^*=\{\gamma;\ \gamma \in k\}$ with $\gamma=u/v\ (u, v \in \mathfrak{o})$ and $\mathfrak{p}\nmid v$.

all integral elements in K with respect to a prime ideal $\mathfrak{P}$ with $\mathfrak{p}\subset\mathfrak{P}$. Hereafter we shall call $\mathfrak{o}^*$ the $\mathfrak{p}$-*integral ring* in k and $\mathfrak{O}^*$ the $\mathfrak{P}$-*integral ring* in K. Then there exists a finite minimal basis $\omega_1, \omega_2, \cdots, \omega_n$ in $\mathfrak{O}^*$ with respect to $\mathfrak{o}^*$. In the following we denote by $\mathfrak{p}^*$ and $\mathfrak{P}^*$ the prime ideal in $\mathfrak{o}^*$ and $\mathfrak{O}^*$ respectively.

Then we have the following

Theorem 2. *For an arbitrary non-negative integer r it holds that:*

$$\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\, \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^r}).$$

Proof. We can choose for every $\gamma^* \in \mathfrak{O}^*$ and an arbitrary non-negative integer ν an integral element $\gamma \in \mathfrak{O}$ such that the following congruence holds:

$$\gamma^* \equiv \gamma \,(\mathrm{mod}\ \mathfrak{P}^{*^\nu}).$$

Hence we can prove this theorem in the analogous way to A. Kinohara [1] Theorem 3.

Let $\omega \in \mathfrak{O}^*$, then it is well-known[1] that the dimension of $\mathfrak{O}^*$-module $\mathfrak{D}(\mathfrak{o}^*[\omega], \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^r})$ is maximal when $\mathfrak{D}(\mathfrak{o}^*[\omega], \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^r}) \cong \mathfrak{O}^*/(f'(\omega))_{\mathfrak{P}^*}$, where $f(x)=0$[3] in $\mathfrak{o}^*[x]$ is the canonical defining equation of an element ω and $(f'(\omega))_{\mathfrak{P}^*}$ denotes the $\mathfrak{P}^*$-component of the different of ω. Hence the maximal dimension of $\mathfrak{O}^*$-modules $\mathfrak{D}(\mathfrak{o}^*[\omega], \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^r})\,(r=0, 1, 2, \cdots)$ is $d_{\mathfrak{P}^*}$ with $(f'(\omega))_{\mathfrak{P}^*} = \mathfrak{P}^{*^{d_{\mathfrak{P}^*}}}$.

On the other hand, Prof. M. Moriya[3] has proved the following two lemmas with respect to the derivation modules.

Lemma 1. *Let $\omega_1, \omega_2, \cdots, \omega_n$ be a minimal basis in $\mathfrak{O}^*$ with respect to $\mathfrak{o}^*$, then for an arbitrary non-negative integer r, $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^r})$ is generated by the direct sum of, at most, n cyclic derivation modules in $\mathfrak{O}^*$. And, if $s \geqq N$ (N being a sufficiently large natural number), then $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^s})$ has the maximal dimension $d_{\mathfrak{P}^*}$ in the set of the dimensions of $\mathfrak{O}^*$-modules $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*\,;\, \mathfrak{O}^*/\mathfrak{P}^{*^r})\,(r=0, 1, 2, \cdots)$.*

Lemma 2. *Let $\bar{K}$ be a separable extension of finite degree over k containing K, $\bar{\mathfrak{O}}$ the ring of all integral elements in $\bar{K}$ with respect to $\mathfrak{o}$, and $\bar{\mathfrak{P}}$ a prime ideal in $\bar{\mathfrak{O}}$ with $\bar{\mathfrak{P}}^e \parallel \mathfrak{P}$[4]. Next let $\bar{\mathfrak{O}}^*$ be the ring consisting of all integral elements in $\bar{K}$ with respect to the prime ideal $\bar{\mathfrak{P}}$, and $\bar{\mathfrak{P}}^*$ the prime ideal in $\bar{\mathfrak{O}}^*$,*

1) See A. Kinohara [1] Corollary of Theorem 2.

2) $f(x)$ is irreducible in $\mathfrak{o}^*[x]$ with the heighest cofficient 1 and $f(\omega)=0$.

3) See M. Moriya [1].

4) $\bar{\mathfrak{P}}^e \parallel \mathfrak{P}$ denotes that $\bar{\mathfrak{P}}^e | \mathfrak{P}$ and $\bar{\mathfrak{P}}^{e+1} \nmid \mathfrak{P}$.

then for an arbitrary non-negative integer r, $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*; \bar{\mathfrak{O}}^*/\bar{\mathfrak{P}}^{*^{er}})$ *coincides with* $\bar{\mathfrak{O}}^*$*-module generated by* $\mathfrak{O}^*$*-module* $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*; \mathfrak{O}^*/\mathfrak{P}^{*^r})$. *Finally, let* $d_{\mathfrak{P}^*}^{(r)}$ *and* $d_{\bar{\mathfrak{P}}^*}^{(er)}$ *be the dimension of* $\mathfrak{O}^*$*-module* $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*; \mathfrak{O}^*/\mathfrak{P}^{*^r})$ *and* $\bar{\mathfrak{O}}^*$*-module* $\mathfrak{D}(\mathfrak{O}^*, \mathfrak{o}^*; \bar{\mathfrak{O}}^*/\bar{\mathfrak{P}}^{*^{er}})$ *respectively, then*

$$d_{\bar{\mathfrak{P}}^*}^{(er)} = e d_{\mathfrak{P}^*}^{(r)} .$$

Thus, from Lemma 1 and Theorem 2 we have the following

Theorem 3. *Let* N *be a sufficiently large natural number. Then for* $s \geqq N$, $\mathfrak{O}$*-module* $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^s)$ *has the maximal dimension* $d_{\mathfrak{P}}$ *in the set of the dimensions of* $\mathfrak{O}$*-modules* $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ $(r=0, 1, 2, \cdots)$, $\mathfrak{P}\,(\neq(0))$ *being the prime ideal in* $\mathfrak{O}$.

Next we shall show the following

Theorem 4. *For an arbitrary non-negative integer* r, *the dimensions of* $\mathfrak{O}$*-modules* $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ *are equal to zero for almost all prime ideal* $\mathfrak{P}$'s$(\neq(0))$ *in* $\mathfrak{O}$ *(with exception of finite number).*

Proof. By assumption, the integral domain $\mathfrak{o}$ has the following properties;

(1) $\mathfrak{o}$ is integrally closed in the quotient field k,

(2) the maximal condition holds for every ideal in $\mathfrak{o}$,

moreover, for K/k it holds that

(3) K is a separable extension (of degree n) over k.

Let θ be an integral primitive element of K/k, and $D(\theta)$ the discriminant of θ. Then, from (1), (2) and (3), every element in $\mathfrak{O}$ can be represented[1] in the linear form of $\frac{1}{D(\theta)}, \frac{\theta}{D(\theta)}, \cdots, \frac{\theta^{n-1}}{D(\theta)}$ with cofficient in $\mathfrak{o}$. That is, $\mathfrak{O}$ is contained in the finite $\mathfrak{o}$-module $\left(\frac{1}{D(\theta)}, \frac{\theta}{D(\theta)}, \cdots, \frac{\theta^{n-1}}{D(\theta)}\right)$.

Next let $\mathfrak{o}^*$ be the ring consisting of finite sum of $\frac{\alpha}{D(\theta)^t}$ where α is an element in $\mathfrak{o}$ and t every integer, then $\mathfrak{O}$ is a subring of $\mathfrak{o}^*[\theta]$ and $\mathfrak{o}$ a subring of $\mathfrak{o}^*$. Moreover, let $\mathfrak{O}^*$ be the $\mathfrak{P}$-integral ring in K, and $\mathfrak{P}^*$ the prime ideal in $\mathfrak{O}^*$. If $\mathfrak{P} \nmid D(\theta)$, then $\mathfrak{O}^*$ contains $\mathfrak{o}^*[\theta]$ and we can choose for every $\gamma^* \in \mathfrak{o}^*[\theta]$ and an arbitrary non-negative integer ν an integral element $\gamma \in \mathfrak{O}$ such that the following congruence holds:

$$\gamma^* \equiv \gamma \pmod{\mathfrak{P}^{*^\nu}}.$$

Therefore we can prove the following relation in the analogous way to A. Kinohara [1] Theorem 3

1) See B. L. Van der Werden [1] 93-94.

$$\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{D}(\mathfrak{o}^*[\theta],\ \mathfrak{o}^*\,;\ \mathfrak{O}^*/\mathfrak{P}^{*^r}).$$

Let θ be an element in $\mathfrak{O}$ with the irreducible defining equation $f(x)=0$ in $\mathfrak{o}[x]$, then, since $D(\theta)\in\mathfrak{o}\,(D(\theta)\neq 0)$, $f(x)$ is also irreducible in $\mathfrak{o}^*[x]$. However, $f'(\theta)$ is not divisible by $\mathfrak{P}$ which satisfies $\mathfrak{P}\nmid D(\theta)$, because $f'(\theta)|D(\theta)$. Thus, for an arbitrary non-negative integer r, it obviously holds that:

$$\mathfrak{D}(\mathfrak{o}^*[\theta],\ \mathfrak{o}^*\,;\ \mathfrak{O}^*/\mathfrak{P}^{*^r}) \cong \mathfrak{O}^*/\mathfrak{O}^*.$$

This is, $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{O}/\mathfrak{O}$ for every prime ideal $\mathfrak{P}$ with $\mathfrak{P}\nmid D(\theta)$. Hence the dimensions of $\mathfrak{O}$-modules $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{P}^r)(r=0, 1, 2, \cdots)$ for every prime ideal $\mathfrak{P}$ with $\mathfrak{P}\nmid D(\theta)$ are equal to zero. Thus, only for the prime ideal $\mathfrak{P}$'s which satisfy $\mathfrak{P}|D(\theta)$, the dimensions of $\mathfrak{O}$-modules $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{P}^r)$ $(r=1, 2, \cdots)$ can not be equal to zero, q. e. d.

Next, from Theorem 1, 3 and 4, we have the following

Theorem 5. *The set of all the dimensions of $\mathfrak{O}$-module $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{A})$'s for all the ideal $\mathfrak{A}$'s $(\neq(0))$ of $\mathfrak{O}$ is bounded. Let its maximal dimension be d_0. Then, d_0 is uniquely determined as $d_0 = \sum_{\mathfrak{P}\subset\mathfrak{O}} d_{\mathfrak{P}}$ where $d_{\mathfrak{P}}$ is the maximal dimension in the set of the dimensions of $\mathfrak{O}$-modules $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{P}^r)$ $(r=0, 1, 2, \cdots)$ for every prime ideal $\mathfrak{P}\,(\neq(0))$ in $\mathfrak{O}$.*

Thus the ideal $\mathfrak{D}_0(K/k) = \prod_{\mathfrak{P}\subset\mathfrak{O}} \mathfrak{P}^{d_{\mathfrak{P}}}$ is uniquely determined as an ideal in $\mathfrak{O}$. This ideal $\mathfrak{D}_0(K/k)$ is called the *relative different* of K/k defined by derivations.

From Theorem 2 and Lemma 2, we obtain the following

Lemma 3. *Let $\bar{K}$ be a separable extension of finite degree over k containing K, $\bar{\mathfrak{O}}$ the ring of all integral elements in $\bar{K}$ with respect to $\mathfrak{o}$, and $\bar{\mathfrak{P}}$ a prime ideal in $\bar{\mathfrak{O}}$ with $\bar{\mathfrak{P}}^e \,\|\, \mathfrak{P}$. Then, for an arbitrary non-negative integer r, the dimension of $\bar{\mathfrak{O}}$-module $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})$ is e times that of $\mathfrak{O}$-module $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,;\ \mathfrak{O}/\mathfrak{P}^r)$.*

3. Suppose that $\mathfrak{p}$ and $\mathfrak{P}\,(\mathfrak{p}\subset\mathfrak{P})$ are some prime ideals in $\mathfrak{o}$ and $\mathfrak{O}$ respectively, and the following condition (A) is satisfied:

condition (A) *There exists an element $\eta\in\mathfrak{O}$ whose residue class modulo $\mathfrak{P}$ is a primitive element of the residue class field $\mathfrak{K}=\mathfrak{O}/\mathfrak{P}$ over $\mathfrak{k}=\mathfrak{o}/\mathfrak{p}$ and one of the prime elements of $\mathfrak{P}$ in $\mathfrak{O}$ belongs to $\mathfrak{o}[\eta]$.*

Hereafter, in such case, we shall say that K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$.

Let η be an element taken in the condition (A), and θ an integral primitive element of K/k. Then we can choose a prime element π of $\mathfrak{p}$ in $\mathfrak{o}$ and an integer $e>1$ such that all conjugate elements of $\theta^*=\eta+\pi^e\theta$ with respect to k are different from one another, so that θ^* is an integral primitive element of K/k. Since $\theta^*\equiv\eta \pmod{\mathfrak{P}}$, the residue class modulo $\mathfrak{P}$ containing θ^* is a primitive element of $\mathfrak{K}/\mathfrak{k}$. If $g(\eta)(\in\mathfrak{o}[\eta])$ is a prime element of $\mathfrak{P}$, then $g(\theta^*)$ is also a prime element of $\mathfrak{P}$ in $\mathfrak{O}$ because

$$g(\theta^*)\equiv g(\eta+\pi^e\theta)\equiv g(\eta) \pmod{\mathfrak{P}^2}.$$

Thus we have the following

Lemma 4. *Let $\mathfrak{p}$ and $\mathfrak{P}(\mathfrak{p}\subset\mathfrak{P})$ be some prime ideals in $\mathfrak{o}$ and $\mathfrak{O}$ respectively, Assume that K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$. Then there exists an integral primitive element θ of K/k such that θ modulo $\mathfrak{P}$ defines a primitive element of the residue class field $\mathfrak{K}=\mathfrak{O}/\mathfrak{P}$ over $\mathfrak{k}=\mathfrak{o}/\mathfrak{p}$ and some prime element of $\mathfrak{P}$ belongs to $\mathfrak{o}[\theta]$.*

Remark. If $\mathfrak{K}$ is a separable extension over $\mathfrak{k}$, then K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$.

Lemma 5. *Let $\mathfrak{p}$ and $\mathfrak{P}$ be some prime ideals in $\mathfrak{o}$ and $\mathfrak{O}$ respectively, and $\mathfrak{p}=\mathfrak{P}^e\mathfrak{A}$ with $(\mathfrak{P},\mathfrak{A})=\mathfrak{O}$. Assume that K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$. Then there exists an integral primitive element θ with the following properties:*

i) $\theta\in\mathfrak{A}$,

ii) *for every $\omega\in\mathfrak{O}$ and an arbitrary non-negative integer r, it holds*

$$\omega=h_\omega(\theta) \pmod{\mathfrak{P}^r} \quad h_\omega(\theta)\in\mathfrak{o}[\theta].$$

Proof. Since the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$ is satisfied, we can choose an integral primitive element θ_0 of K/k for which Lemma 4 holds.

Take a prime element Π of $\mathfrak{P}$ with $(\Pi,\mathfrak{A})=\mathfrak{O}$, then we can determine an element $\xi\in\mathfrak{O}$ such that $\Pi^2\xi\equiv-\theta_0 \pmod{\mathfrak{A}}$. Clearly there exists an element $c\in\mathfrak{o}$ such that $c\equiv1 \pmod{\mathfrak{p}}$ and all conjugate elements of $\theta=\theta_0+\Pi^2\xi c$ with respect to k are different from one another. Thus it follows that $\theta\in\mathfrak{A}$.

Let $g(\theta_0)$ $(\in\mathfrak{o}[\theta_0])$ be a prime element of $\mathfrak{P}$, then $g(\theta)\in\mathfrak{o}[\theta]$ is also a prime element of $\mathfrak{P}$ in $\mathfrak{O}$. And, every representative of the residue class modulo $\mathfrak{P}$ is expressible by an element in $\mathfrak{o}[\theta]$. Thus, for every $\omega\in\mathfrak{O}$ and an arbitrary non-negative integer r, the following congruence holds:

$$\omega\equiv h_\omega(\theta) \pmod{\mathfrak{P}^r} \quad h_\omega(\theta)\in\mathfrak{o}[\theta].$$

Lemma 6. *Assume that K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$, then there exists an integral primitive element θ of K/k with the canonical defining equation $f(x)=0$ in $\mathfrak{o}[x]$ such that $(f'(\theta))_{\mathfrak{P}}$ coincides with $\mathfrak{D}(K/k)_{\mathfrak{P}}$, where $(f'(\theta))_{\mathfrak{P}}$ and $\mathfrak{D}(K/k)_{\mathfrak{P}}$ denote the $\mathfrak{P}$-component of $(f'(\theta))$ and $\mathfrak{D}(K/k)$ respectively.*

Proof. We obtain this lemma[1] for a primitive element θ such that the properties i) and ii) in Lemma 5 hold.

Theorem 6. *Assume that K/k satisfies condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$, then for an integral primitive element θ of K/k in Lemma 5, the following isomorphism holds:*

$$\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,; \mathfrak{O}/\mathfrak{P}^r) \cong \mathfrak{D}(\mathfrak{o}[\theta], \mathfrak{o}\,; \mathfrak{O}/\mathfrak{P}^r)$$

where r is an arbitrary non-negative integer.

Proof. See A. Kinohara [1] Theorem 3.

Thus we can easily show the following

Lemma 7. *If K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$, then the $\mathfrak{O}$-modules $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,; \mathfrak{O}/\mathfrak{P}^r)$ $(r=0, 1, 2, \cdots)$ are always $\mathfrak{O}$-cyclic group, and for the maximal dimension $d_{\mathfrak{P}}$ of $\mathfrak{O}$-modules $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}\,; \mathfrak{O}/\mathfrak{P}^r)$, $\mathfrak{P}^{d_{\mathfrak{P}}}$ is $(f'(\theta))_{\mathfrak{P}}$ where $f(x)=0$ in $\mathfrak{o}[x]$ is the canonical defining equation of an element θ taken in Theorem 6 and $(f'(\theta))_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-component of $(f'(\theta))$.*

Hence we obtain the following

Theorem 7. *If K/k satisfies the condition (A) on $\mathfrak{p}$ and $\mathfrak{P}$, then we have $\mathfrak{D}_0(K/k)_{\mathfrak{P}}=\mathfrak{D}(K/k)_{\mathfrak{P}}$.*

Theorem 8. *Let $k=K_0\subset K_1\subset \cdots \subset K_m$ be a separable extension series of finite degree over k, $\mathfrak{O}_j$ $(j=1, 2, \ldots, m)$ the ring of all integral elements in K_j with respect to $\mathfrak{o}$, and $\mathfrak{P}_j$ the prime ideal in $\mathfrak{O}_j$ with $\mathfrak{P}_j\subset\mathfrak{P}_{j+1}$. If K_{i+1}/K_i $(i=0, 1, \ldots, m-1)$ satisfies the condition (A) on $\mathfrak{P}_i$ and $\mathfrak{P}_{i+1}$, then*

$$\mathfrak{D}_0(K_m/K_0)_{\mathfrak{P}_m} = \prod_{i=0}^{m-1} \mathfrak{D}_0(K_{i+1}/K_i)_{\mathfrak{P}_{i+1}}$$

where $\mathfrak{D}_0(K_{i+1}/K_i)_{\mathfrak{P}_{i+1}}$ denotes the $\mathfrak{P}_{i+1}$-component of the relative different of K_{i+1}/K_i by derivations.

Proof. This theorem is trivial for $m=1$. By assuming the relation $\mathfrak{D}_0(K_{m-1}/K_0)_{\mathfrak{P}_{m-1}}= \prod_{i=0}^{m-2}\mathfrak{D}_0(K_{i+1}/K_i)_{\mathfrak{P}_{i+1}}$ we shall prove

$$(\mathrm{A})\quad \mathfrak{D}_0(K_m/K_0)_{\mathfrak{P}_m} = \mathfrak{D}_0(K_{m-1}/K_0)_{\mathfrak{P}_{m-1}}\mathfrak{D}_0(K_m/K_{m-1})_{\mathfrak{P}_m}.$$

Put $\mathfrak{D}_0(K_m/K_0)_{\mathfrak{P}_m}=\mathfrak{P}_m^{\sigma}$, $\mathfrak{D}_0(K_{m-1}/K_0)_{\mathfrak{P}_{m-1}}=\mathfrak{P}_{m-1}^{\sigma}$, $\mathfrak{D}_0(K_m/K_{m-1})_{\mathfrak{P}_m}=\mathfrak{P}_m^{h}$ and $\mathfrak{P}_m^{e} \,\|\, \mathfrak{P}_{m-1}$. Then, by Lemma 3, (A) is equivalent to

1) See E. Hecke [1] 135-136 Hilfssatz C.

(B) $H=h+eg$.

In order to prove (B), we have to show that there exists a natural number r such that the maximal dimension of $\mathfrak{O}_m$-modules $\mathfrak{D}(\mathfrak{O}_m, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$, $\mathfrak{D}(\mathfrak{O}_m, \mathfrak{O}_{m-1}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$ and $\mathfrak{D}(\mathfrak{O}_{m-1}, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$ is H, h and eg respectively, and

$$\text{(C)}\quad \frac{\mathfrak{D}(\mathfrak{O}_m, \mathfrak{o} \quad ; \mathfrak{O}_m/\mathfrak{P}_m^{er})}{\mathfrak{D}(\mathfrak{O}_m, \mathfrak{O}_{m-1}; \mathfrak{O}_m/\mathfrak{P}_m^{er})} \simeq \mathfrak{D}(\mathfrak{O}_{m-1}, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$$

holds. Considering the structure of the above derivation modules as $\mathfrak{O}_m$-module, those dimensions are all finite. Hence there exists the positive integer $r \geqq r_1$ such that the former part of the statement is satisfied.

Let us take a derivation $\overline{D} \in \mathfrak{D}(\mathfrak{O}_m, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}^{er})$, then it induces a derivation $\varphi(\overline{D})=D' \in \mathfrak{D}(\mathfrak{O}_{m-1}, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}^{er})$, by applying $\overline{D}$ to $\mathfrak{O}_{m-1}$. The kernel of the mapping φ is clearly $\mathfrak{D}(\mathfrak{O}_m, \mathfrak{O}_{m-1}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$. Hence we have an isomorphic mapping from the left hand side of the relation (C) into the right.

We can see that this mapping is an onto-mapping, if for every derivation $D \in \mathfrak{D}(\mathfrak{O}_{m-1}, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$ we have an extension $D^* \in \mathfrak{D}(\mathfrak{O}_m, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$ of D. However, since K_m/K_{m-1} satisfies the condition (A) on $\mathfrak{P}_{m-1}$ and $\mathfrak{P}_m$, we can take $\theta (\in \mathfrak{O}_m)$ such that $\mathfrak{D}(\mathfrak{O}_m, \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er}) \simeq \mathfrak{D}(\mathfrak{O}_{m-1}[\theta], \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$.

By A. Weil [1] 12 Proposition 15, the condition that D has an extension $D^{**} \in \mathfrak{D}(\mathfrak{O}_{m-1}[\theta], \mathfrak{o}; \mathfrak{O}_m/\mathfrak{P}_m^{er})$ is that the congruence

$$f^D(\theta)+f'(\theta)\lambda \equiv 0 \pmod{\mathfrak{P}_m^{er}}$$

has a solution $\lambda \in \mathfrak{O}_m$, where $f(x)=x^t+a_1x^{t-1}+\cdots+a_t (\in \mathfrak{O}_{m-1}[x])$ is the canonical defining polynomial of an element θ and $f^D(\theta)=D(a_1)\theta^{t-1}+\cdots+D(a_t)$.

Let $g_i(x) \in \mathfrak{o}[x]$ $(i=1, 2, \cdots, t)$ be the canonical defining polynomial of the element a_i, then

$$D(g_i(a_i)) = g_i^D(a_i)+g_i'(a_i)D(a_i) \equiv g_i'(a_i)D(a_i) \equiv 0 \pmod{\mathfrak{P}_m^{er}}.$$

Now let $\mathfrak{P}_m^s$ be the (set-theoretically) smallest $\mathfrak{P}_m$-component of $(g'(a_i))$ $(1 \leqq i \leqq t)$, then it follows:

$$D(a_i) \equiv 0 \pmod{\mathfrak{P}_m^{er-s}}.$$

Hence we have $f^D(\theta) \equiv 0 \pmod{\mathfrak{P}_m^{er-s}}$. Next, since $(f'(\theta))_{\mathfrak{P}_m}=\mathfrak{D}(K_m/K_{m-1})_{\mathfrak{P}_m}=\mathfrak{P}_m^h$, then we have a solution $\lambda \in \mathfrak{O}_m$ if we take $r \geqq r_2 = \left[\frac{h+s}{e}\right]$[1]$+1$. Thus the above mapping φ is an onto-mapping for $r \geqq \text{Max.}(r_1, r_2)$, q.e.d.

1) [ ] denotes the Gauss symbol.

4. Let N be a separable and normal extension of finite degree over k, $\bar{\mathfrak{O}}$ the ring of all integral elements in N with respect to $\mathfrak{o}$, and $\bar{\mathfrak{P}}$ a prime ideal in $\bar{\mathfrak{O}}$. Then we can take the following series of the fields:

$$k_Z \subset k_T \subset N_0 \subset N_1 \subset \cdots\cdots \subset N_m = N$$

where k_Z, k_T and N_0 denote the decomposition field, inertia field and first ramification field of N/k with respect to $\bar{\mathfrak{P}}$ respectively, and N_{i+1} is an extension of p-th degree over N_i where p is the characteristic of the residue class field $\mathfrak{k}=\mathfrak{o}/\mathfrak{p}$ and $i=0, 1, \ldots, m-1$.

We consider the series of the rings of all integral elements in the above series of the fields with respect to $\mathfrak{o}$, that is

$$\mathfrak{o}_Z \subset \mathfrak{o}_T \subset \mathfrak{O}_0 \subset \mathfrak{O}_1 \subset \cdots\cdots \subset \mathfrak{O}_m = \bar{\mathfrak{O}}\,.$$

Let $\mathfrak{p}_Z \subset \mathfrak{p}_T \subset \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_m = \bar{\mathfrak{P}}$ be the series of the prime ideals in the above series of the rings. Moreover, we shall denote by $\mathfrak{k}_Z \subset \mathfrak{k}_T \subset \mathfrak{K}_0 \subset \mathfrak{K}_1 \subset \cdots \subset \mathfrak{K}_m = \bar{\mathfrak{K}}$ the series of the residue class fields $\mathfrak{o}_Z/\mathfrak{p}_Z \subset \mathfrak{o}_T/\mathfrak{p}_T \subset \mathfrak{O}_0/\mathfrak{P}_0 \subset \mathfrak{O}_1/\mathfrak{P}_1 \subset \cdots \subset \mathfrak{O}_m/\mathfrak{P}_m = \bar{\mathfrak{O}}/\bar{\mathfrak{P}}$.

Then we can prove the following

Theorem 9. i) $\mathfrak{D}_0(k_Z/k)_{\mathfrak{p}_Z} = \mathfrak{D}(k_Z/k)_{\mathfrak{p}_Z} = \mathfrak{o}_Z$;

ii) $\mathfrak{D}_0(k_T/k_Z)_{\mathfrak{p}_T} = \mathfrak{D}(k_T/k_Z)_{\mathfrak{p}_T} = \mathfrak{o}_T$;

iii) $\mathfrak{D}_0(N_0/k_T)_{\mathfrak{P}_0} = \mathfrak{D}(N_0/k_T)_{\mathfrak{P}_0} = \mathfrak{P}_0^{e_0-1}$

where $e_0=[N_0 : k_T]$[1];

iv) $\mathfrak{D}(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} = \mathfrak{D}(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} = (\varphi'(\alpha))_{\mathfrak{P}_{i+1}}$ $(i=0, 1, \ldots, m-1)$, *where α is some integral primitive element of N_{i+1}/N_i such that $(\varphi'(\alpha))_{\mathfrak{P}_{i+1}}$-the $\mathfrak{P}_{i+1}$-component of the different of α- is always divisible by $\mathfrak{P}_{i+1}$*;

v) $\mathfrak{D}_0(N/k)_{\bar{\mathfrak{P}}} = \mathfrak{D}(N/k)_{\bar{\mathfrak{P}}}$.

Proof. i) In this case, $(\mathfrak{k}_Z : \mathfrak{k})=1$, and $\mathfrak{p}$ is unramified in $\mathfrak{o}_Z$. Then from Lemma 4 and 5 we can take an integral primitive element θ with $\mathfrak{p}_Z^{r+1} | \theta - c$ such that the following isomorphisms hold:

$$\mathfrak{D}(\mathfrak{o}_Z, \mathfrak{o}; \mathfrak{o}_Z/\mathfrak{p}_Z^r) \cong \mathfrak{D}(\mathfrak{o}[\theta], \mathfrak{o}; \mathfrak{o}_Z/\mathfrak{p}_Z^r) \qquad (r = 0, 1, 2 \cdots)$$

where $c \in \mathfrak{o}$ is the representative of the primitive element of $\mathfrak{k}_Z/\mathfrak{k}$.

Let us take a prime element π of $\mathfrak{p}_Z$ in $\mathfrak{o}_Z$, then since $\mathfrak{p}_Z^{r+1} | \theta - c$, $\theta - c$ is expressible in the form:

$$\theta - c = \pi^{r+1} \frac{\mathfrak{a}}{\mathfrak{b}} \qquad (\mathfrak{b}, \mathfrak{p}_Z) = \mathfrak{o}_Z$$

1) $[N_0 : k_T]$ denotes the degree of N_0/k_T.

where $\mathfrak{a}$, $\mathfrak{b}$ are ideals in $\mathfrak{o}_Z$. Obviously, we can choose an ideal $\mathfrak{c}$ prime to $\mathfrak{p}_Z$ such that $\mathfrak{b}\mathfrak{c}$ is a principal ideal (β) in $\mathfrak{o}_Z$. Therefore we can put for some $\alpha \in \mathfrak{o}_Z$

$$\theta - c = \pi^{r+1}\frac{\alpha}{\beta} \qquad (\beta, \mathfrak{p}_Z) = \mathfrak{o}_Z .$$

Then for $D \in \mathfrak{D}(\mathfrak{o}_Z, \mathfrak{o};\ \mathfrak{o}_Z/\mathfrak{p}_Z^r)$ we have:

$$D(\beta(\theta - c)) = (\theta - c)D(\beta) + \beta D(\theta) = (r+1)\pi^r \alpha D(\pi) + \pi^{r+1}D(\alpha) \equiv 0 \pmod{\mathfrak{p}_Z^r},$$

since $\theta - c \equiv 0 \pmod{\mathfrak{p}_Z^{r+1}}$ and $(\beta, \mathfrak{p}_Z) = \mathfrak{o}_Z$, we have $D(\theta) \equiv 0 \pmod{\mathfrak{p}_Z^r}$.

Thus we bave:

$$\mathfrak{D}(\mathfrak{o}_Z, \mathfrak{o};\ \mathfrak{o}_Z/\mathfrak{p}_Z^r) \cong \mathfrak{o}_Z/\mathfrak{o}_Z , \qquad \text{q.e d.}$$

ii) In this case, k_T/k_Z is an unramified extension and $\mathfrak{k}_T/\mathfrak{k}_Z$ is a separable extension. Let $(\mathfrak{k}_T : \mathfrak{k}_Z) = f$. Then we can choose an integral primitive element ρ of k_T/k_Z such that ρ is one of the primitive root of $\mathfrak{p}_T$ and Theorem 6 holds. Let $g(x) = x^f + b\, x^{f-1} + \cdots + b_f \equiv 0 \pmod{\mathfrak{p}_T}$ be the irreducible congruence of ρ in $\mathfrak{o}[x]$ with $\mathfrak{p}_T \| (g(\rho))$. Then we can obtain $g'(\rho) \not\equiv 0 \pmod{\mathfrak{p}_T}$ i.e. $(g'(\rho))_{\mathfrak{p}_T} = \mathfrak{o}_T$ where $g'(x)$ denotes the derivative of $g(x)$ by x and $(g'(\rho))_{\mathfrak{p}_T}$ denotes the $\mathfrak{p}_T$-component of $(g'(\rho))$. Let $f(x) = x^f + a_1 x^{f-1} + \cdots + a_f = 0$ be the irreducible defining equation of ρ in $\mathfrak{o}_Z[x]$. Then $f(x) \equiv g(x) \pmod{\mathfrak{p}_T}$. Therefore $f'(x) \equiv g'(x) \pmod{\mathfrak{p}_T}$. Since $g'(\rho) \not\equiv 0 \pmod{\mathfrak{p}_T}$, then $f'(\rho) \not\equiv 0 \pmod{\mathfrak{p}_T}$. Thus, by Lemma 7 and Theorem 7 we have:

$$\mathfrak{D}_0(k_T/k_Z)_{\mathfrak{p}_T} = \mathfrak{D}(k_T/k_Z)_{\mathfrak{p}_T} = (f'(\rho))_{\mathfrak{p}_T} = \mathfrak{o}_T .$$

iii) In this case, N_0/k_T is a completely ramified extension. Let $\mathfrak{p}_T = \mathfrak{P}_0^{e_0}$, then $(e_0, p) = 1$, p being the characteristic of the residue class field $\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$. Thus we can take a prime element Π_0 of $\mathfrak{P}_0$ as the integral primitive element of K_0/k_T which satisfies Theorem 6. Now let $\varphi(x) = x^{e_0} + a_1 x^{e_0 - 1} + \cdots + a_{e_0}$ be the canonical defining polynomial of Π_0 in $\mathfrak{o}_T[x]$, such that $\varphi(x)$ is irreducible and $\varphi(\Pi_0) = 0$. The equation $\varphi(x) = 0$ is well-known as an Eisenstein equation and $(\varphi'(\Pi_0))_{\mathfrak{P}_0}$-the $\mathfrak{P}_0$-component of the different of an element Π_0-is $\mathfrak{P}_0^{e_0 - 1}$, because $p \nmid e_0$. Then we have:

$$\mathfrak{D}_0(K_0/k_T)_{\mathfrak{P}_0} = \mathfrak{D}(K_0/k_T)_{\mathfrak{P}_0} = (\varphi'(\Pi_0))_{\mathfrak{P}_0} = \mathfrak{P}_0^{e_0 - 1} .$$

iv) The following two cases are possible to occur.

(1) The first case is the one in which N_{i+1}/N_i is a completely ramified extension of p-th degree. Thus, by the same reason as in iii), we have

the following relation for an integral primitive element Π_{i+1} of N_{i+1}/N_i satisfying $\mathfrak{P}_{i+1} \| \Pi_{i+1}$

$$\mathfrak{D}_0(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} = \mathfrak{D}(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} = (\varphi'(\Pi_{i+1}))_{\mathfrak{P}_{i+1}}$$

where $(\varphi'(\Pi_{i+1}))_{\mathfrak{P}_{i+1}}$ denotes the $\mathfrak{P}_{i+1}$-component of the different of an element Π_{i+1}. And, if $\mathfrak{P}_{i+1}^h \| (p)$, then $(\varphi'(\Pi_{i+1}))_{\mathfrak{P}_{i+1}} = \mathfrak{P}_{i+1}^s$, s being $p \leqq s < h+p$.

(2) The second case is the one in which N_{i+1}/N_i is an unramified extension and $\mathfrak{K}_{i+1}/\mathfrak{K}_i$ is a purely inseparable extension of p-th degree. Let us consider $\mathfrak{K}_{i+1} = \mathfrak{K}_i(\bar{\alpha})$. Then we can choose an primitive element α of N_{i+1}/N_i such that α is a representative of $\bar{\alpha}$ in $\mathfrak{O}_{i+1}$. On the other hand, one of the prime elements of $\mathfrak{P}_{i+1}$ belongs to $\mathfrak{O}_i$. Therefore, by Lemma 6 we can take the above α for which Theorem 6 holds.

Suppose that α satisfies the following congruence:

$$\alpha^p + a \equiv 0 \qquad (\text{mod } \mathfrak{P}_{i+1}) \quad (\mathfrak{P}_i \not| a)$$

where $a (\in \mathfrak{O}_i)$ denotes a representative of $\bar{a} \in \mathfrak{K}_i$. Then the canonical defining equation of α is written as follows:

$$\varphi(x) = x^p + a_1 x^{p-1} + \cdots + a_{p-1} x + a_p = 0 \qquad (\in \mathfrak{O}_i[x])$$

where $a \equiv a_p \ (\text{mod } \mathfrak{P}_i)$ and a_i $(i=1, 2, \cdots, p-1)$ is divisible by $\mathfrak{P}_i$. Then we have:

$$\mathfrak{D}_0(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} = \mathfrak{D}(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} = (\varphi'(\alpha))_{\mathfrak{P}_{i+1}}.$$

And, $(\varphi'(\alpha))_{\mathfrak{P}_{i+1}}$ is always divisible by $\mathfrak{P}_{i+1}$.

v) Of couse, the chain-theorem is held also in the definition of the relative different which was introduced by R. Dedekind. Summarising our results obtained in i), ii), iii) and iv) we have:

$$\mathfrak{D}_0(N/k)_{\bar{\mathfrak{P}}} = \mathfrak{D}(N/k)_{\bar{\mathfrak{P}}}.$$

Lemma 8. $\qquad \mathfrak{D}_0(K/k)_{\mathfrak{P}} = \mathfrak{D}(K/k)_{\mathfrak{P}}.$

Proof. Let N be a separable and normal extension of finite degree over k containing K, $\bar{\mathfrak{O}}$ the ring of all integral elements in N with respect to $\mathfrak{o}$, and $\mathfrak{P}$, $\bar{\mathfrak{P}}$ the prime ideal in $\mathfrak{O}$, $\bar{\mathfrak{O}}$ respectively, with $\mathfrak{P} \subset \bar{\mathfrak{P}}$. Then, since N is also a normal extension over K, we can obtain the following equation by Theorem 9 v)

$$\mathfrak{D}_0(N/k)_{\bar{\mathfrak{P}}} = \mathfrak{D}(N/k)_{\bar{\mathfrak{P}}} \text{ and } \mathfrak{D}_0(N/K)_{\bar{\mathfrak{P}}} = \mathfrak{D}(N/K)_{\bar{\mathfrak{P}}}.$$

Now, put $\mathfrak{D}_0(N/k)_{\bar{\mathfrak{P}}} = \bar{\mathfrak{P}}^H$, $\mathfrak{D}_0(K/k)_{\mathfrak{P}} = \mathfrak{P}^g$, $\mathfrak{D}_0(N/K)_{\bar{\mathfrak{P}}} = \bar{\mathfrak{P}}^h$ and $\bar{\mathfrak{P}}^e \| \mathfrak{P}$, then we can prove the relation $H = h + eg$ by the analogous method with the proof of Theorem 8. That is, for a sufficiently large natural number r we have only to see the isomorphism

$$\frac{\mathfrak{D}(\bar{\mathfrak{O}}, \mathfrak{o}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})}{\mathfrak{D}(\bar{\mathfrak{O}}, \mathfrak{O}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})} \cong \mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er}).$$

Now, by taking $\mathfrak{O}$ as its domain, a derivation $\bar{D} \in \mathfrak{D}(\bar{\mathfrak{O}}, \mathfrak{o}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})$ induces a derivation $\varphi(\bar{D})=D' \in \mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})$ and the kernel of the mapping φ is $\mathfrak{D}(\bar{\mathfrak{O}}, \mathfrak{O}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})$. On the other hand, from the structure of $\bar{\mathfrak{O}}$ with respect to $\mathfrak{O}$, it is easily shown by the mathematical induction that for every derivation $D \in \mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})$ there exists an extension $D^* \in \mathfrak{D}(\bar{\mathfrak{O}}, \mathfrak{o} : \bar{\mathfrak{O}}/\bar{\mathfrak{P}}^{er})$ of D. Thus we have:

$$\mathfrak{D}_0(N/k)_{\bar{\mathfrak{P}}} = \mathfrak{D}_0(K/k)_{\mathfrak{P}}\, \mathfrak{D}_0(N/K)_{\bar{\mathfrak{P}}}.$$

From this result and $\mathfrak{D}(N/k)_{\bar{\mathfrak{P}}}=\mathfrak{D}(K/k)_{\mathfrak{P}}\, \mathfrak{D}(N/K)_{\bar{\mathfrak{P}}}$ for the relative differents introduced by R. Dedekind. it follows:

$$\mathfrak{D}_0(K/k)_{\mathfrak{P}} = \mathfrak{D}(K/k)_{\mathfrak{P}}.$$

Theorem 10. $$\mathfrak{D}_0(K/k) = \mathfrak{D}(K/k).$$

Proof. Let $\mathfrak{D}_0(K/k)=\mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_t^{e_t}$ be the prime ideal decomposition of $\mathfrak{D}_0(K/k)$ in $\mathfrak{O}$. Then, from the definition of the relative different by derivations there exists an ideal $\mathfrak{A}$ in $\mathfrak{O}$ such that $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$ has the maximal dimension $\sum_{i=1}^{t} e_i$ in the set of dimensions of all derivation modules in $\mathfrak{O}$.

Next let $\mathfrak{A}=\mathfrak{P}_1^{r_1} \cdots \mathfrak{P}_t^{r_t}$ be the prime ideal decomposition of $\mathfrak{A}$ in $\mathfrak{O}$. Then, by Theorem 1, it holds that:

$$\mathfrak{D}(\mathfrak{O}, \mathfrak{o} : \mathfrak{O}/\mathfrak{A}) \cong \sum_{i=1}^{t} \oplus\, \mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}_i^{r_i}).$$

Since the dimension of $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$ is maximal, then for each $i\,(1 \leqq i \leqq t)$, $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}_i^{r_i})$ must have the maximal dimesion e_i. On the other hand, $\mathfrak{D}(K/k)_{\mathfrak{P}_i}$ is the ideal $\mathfrak{P}_i^{e_i}$ such that the dimension of $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}_i^{r_i})$'s is equal to e_i. By definition, it must be:

$$\mathfrak{D}_0(K/k) = \prod_{i=1}^{t} \mathfrak{D}(K/k)_{\mathfrak{P}_i}.$$

Let $\mathfrak{P}$ be a prime ideal divisor of $\mathfrak{D}(K/k)$ which is different from $\mathfrak{P}_i\,(i=1, 2\, \cdots, t)$. Then, by Lemma 8, for a positive integer r the dimension of $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{P}^r)$ is greater than zero and $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A}\mathfrak{P}^r)$ has a greater dimension than that of $\mathfrak{D}(\mathfrak{O}, \mathfrak{o}; \mathfrak{O}/\mathfrak{A})$, but this gives a contradiction. Hence it must be:

$$\mathfrak{D}_0(K/k)=\prod_{i=1}^{t}\mathfrak{D}(K/k)_{\mathfrak{P}_i}=\mathfrak{D}(K/k), \qquad \text{q. e. d.}$$

5. In regard to the relative different $\mathfrak{D}_0(K/k)$ by derivations, we shall show that the chain-theorem and the different-theorem hold independently of the usual one which was introduced by R. Dedekind.

Theorem 11. (*chain-theorem*) *Let K be a separable extension of finite degree over k containing K', we have*:

$$\mathfrak{D}_0(K/k)=\mathfrak{D}_0(K'/k)\mathfrak{D}_0(K/K').$$

Proof. Let N be a separable and normal extension of finite degree over k containing K; $\mathfrak{O}'$, $\mathfrak{O}$ and $\bar{\mathfrak{O}}$ the rings of all integral elements in K', K and N with respect to $\mathfrak{o}$ respectively; $\mathfrak{P}'$, $\mathfrak{P}$ and $\bar{\mathfrak{P}}(\mathfrak{p}\subset\mathfrak{P}'\subset\mathfrak{P}\subset\bar{\mathfrak{P}})$ some prime ideals in $\mathfrak{O}'$, $\mathfrak{O}$ and $\bar{\mathfrak{O}}$ respectively. Then since N/k is a normal extension, N/K' and N/K are also the normal extensions. Hence, from the results that we obtained on the way of the proof in Lemma 8, it follows:

(A) $\mathfrak{D}_0(N/k)_{\bar{\mathfrak{P}}}=\mathfrak{D}_0(K/k)_{\mathfrak{P}}\mathfrak{D}_0(N/K)_{\bar{\mathfrak{P}}}=\mathfrak{D}_0(K'/k)_{\mathfrak{P}'}\mathfrak{D}_0(N/K')_{\bar{\mathfrak{P}}}$;

(B) $\mathfrak{D}_0(N/K')_{\bar{\mathfrak{P}}}=\mathfrak{D}_0(K/K')_{\mathfrak{P}}\mathfrak{D}_0(N/K)_{\bar{\mathfrak{P}}}$.

Using (B) in place of $\mathfrak{D}_0(N/K')_{\bar{\mathfrak{P}}}$ of (A), we have:

$$\mathfrak{D}_0(K/k)_{\mathfrak{P}}=\mathfrak{D}_0(K'/k)_{\mathfrak{P}'}\mathfrak{D}_0(K/K')_{\mathfrak{P}}.$$

Thus, by the definition of $\mathfrak{D}_0(K/k)$ we can easily prove our assertion.

Next, in order to prove the different-theorem we shall prepare some notations. Let N be a separable and normal extension of finite degree over k containing K, and let G be its Galois group; let H be the subgroup of G corresponding to K. Let $k\subset k_T\subset N_0\subset N$, $\mathfrak{o}\subset\mathfrak{o}_T\subset\mathfrak{O}_0\subset\bar{\mathfrak{O}}$, $\mathfrak{p}\subset\mathfrak{p}_T\subset\mathfrak{P}_0\subset\bar{\mathfrak{P}}$ and $\mathfrak{k}\subset\mathfrak{k}_T\subset\mathfrak{K}_0\subset\bar{\mathfrak{K}}$ be the same notations as in Theorem 9. Next let T and V_0 be the subgroup of G corresponding to k_T and N_0 respectively. Then there are two cases to consider with respect to $H\cap T$ and T:

(1) $$H\cap T=T;$$

(2) $$H\cap T\subsetneqq T.$$

Now, we shall prove the following

Lemma 9. *The following three conditions are equivalent*:

(A) $H\cap T=T$;

(B) $K\subseteqq k_T$;

(C) $\mathfrak{P}\,\|\,\mathfrak{p}$ *and* $\mathfrak{K}$ *is a separable extension over* $\mathfrak{k}$.

Proof. (A)→(B) and (B)→(A) are obvious. Since $\mathfrak{p}$ is unramified in $\mathfrak{o}_T$ and $\mathfrak{k}_T/\mathfrak{k}$ is a separable extension, (B)→(C) is also obvious.

Next, in order to prove (C)→(B), we shall show that, when Kk_T[1] $\supsetneqq k_T$, either $e>1$ for $\mathfrak{P}^e \| \mathfrak{p}$ or $\mathfrak{K}/\mathfrak{k}$ is an inseparable extension. Let $Kk_T \supsetneqq k_T$, Kk_T being an inertia field of N/K with respect to $\overline{\mathfrak{P}}$. Let $\mathfrak{O}_T$ be the ring of all integral elements in Kk_T with respect to $\mathfrak{o}$, $\mathfrak{P}_T$ the prime ideal in $\mathfrak{O}_T$ with $\mathfrak{P}_T \subset \overline{\mathfrak{P}}$, and $\mathfrak{K}_T = \mathfrak{O}_T/\mathfrak{P}_T$. Then, in $k \subset k_T \subset Kk_T$, either Kk_T/k_T is a completely ramified extension or $\mathfrak{K}_T/\mathfrak{k}_T$ is an inseparable extension. Hence K/k must satisfy that either $e>1$ for $\mathfrak{P}^e \| \mathfrak{p}$ or $\mathfrak{K}/\mathfrak{k}$ is an inseparable extension.

Corollary. *The following three conditions are equivalent*:

(A) $H \cap T \subsetneqq T$;

(B) $Kk_T \supsetneqq k_T$;

(C) *either* $e>1$ *for* $\mathfrak{P}^e \| \mathfrak{p}$ *or* $\mathfrak{K}$ *is an inseparable extension over* $\mathfrak{k}$.

Next we shall prove the different-theorem.

Theorem 12. (*different-theorem*) *A necessary and sufficient condition that a prime ideal* $\mathfrak{P}$ *satisfies* $\mathfrak{D}_0(K/k) \subset \mathfrak{P}$, *is given by the following relation*: *either*

(A) *If* $\mathfrak{P}^e \| \mathfrak{p}$, *then* $e>1$

or

(B) *The residue class field* $\mathfrak{K} = \mathfrak{O}/\mathfrak{P}$ *is an inseparable extension over* $\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$.

Proof. (Necessity) By supposing that $\mathfrak{P} \| \mathfrak{p}$ and $\mathfrak{K}/\mathfrak{k}$ is a separable extension, we shall show $\mathfrak{D}_0(K/k) \not\subset \mathfrak{P}$. From Lemma 9 this assumption is equivalent to $K \subseteqq k_T$. Then, from Theorem 9 and 11 we have:

$$\mathfrak{D}_0(K/k)_{\mathfrak{P}} \supseteqq \mathfrak{D}_0(k_T/k)_{\mathfrak{p}_T} = \mathfrak{o}_T .$$

Hence, we obtain $\mathfrak{D}_0(K/k) \not\subset \mathfrak{P}$, q. e. d.

(Sufficiency) By supposing that either $e>1$ for $\mathfrak{P}^e \| \mathfrak{p}$ or $\mathfrak{K}/\mathfrak{k}$ is an inseparable extension, we shall show $\mathfrak{D}_0(K/k) \subset \mathfrak{P}$.

From Corollary of Lemma 9 this assumption is equivalent to $Kk_T \supsetneqq k_T$. There are two cases to consider:

(1) $K \supset k_T$ $(K \neq k_T)$;

(2) $Kk_T \supset k_T$ $(K \not\supset k_T)$.

Case (1). $K \supset k_T$ $(K \neq k_T)$.

1) Kk_T denotes the composite extension of K and k_T.

If $N_0 \supseteqq K$, then our assertion is obvious. Therefore we assume $N_0 \not\supseteqq K$. Let KN_0 be the first ramification field of N/K with respect to $\mathfrak{P}$, $\mathfrak{O}_0^*$ the ring of all integral elements in KN_0 with respect to $\mathfrak{o}$, and $\mathfrak{P}_0^*$ be the prime ideal in $\mathfrak{O}_0^*$ with $\mathfrak{P}_0^* \subset \mathfrak{P}$.

Since $\mathfrak{p}_T$ is completely ramified in $\mathfrak{O}_0$, we can choose an integral primitive element Π_0 of N_0/k_T which satisfies $\mathfrak{P}_0 \parallel \Pi_0$. Let $\mathfrak{p}_T = \mathfrak{P}_0^{e_0}$, then the canonical defining equation $f(x) = x^{e_0} + a_1 x^{e_0-1} + \cdots + a_{e_0} = 0$ $(\mathfrak{o}_T[x])$ of an element Π_0 is an Eisenstein equation. By Theorem 9, it follows:

$$\mathfrak{D}_0(N_0/k_T)_{\mathfrak{P}_0} = (f'(\Pi_0))_{\mathfrak{P}_0} = \mathfrak{P}_0^{e_0-1}.$$

Since $N_0 = k_T(\Pi_0)$, we have $KN_0 = K(\Pi_0)$. Let $f(x) = f_1(x)\, g(x)$ be the decomposion of $f(x)$ in $K[x]$, where $f_1(\Pi_0) = 0$ and $f_1(x)$ is irreducible in $K[x]$. Then we have:

$$(f'(\Pi_0))_{\mathfrak{P}_0} = (f_1'(\Pi_0))_{\mathfrak{P}_0^*}(g(\Pi_0))_{\mathfrak{P}_0^*} \qquad (g(\Pi_0) \neq 0)$$

and

$$\mathfrak{D}_0(KN_0/K)_{\mathfrak{P}_0^*} \supseteqq (f_1'(\Pi_0))_{\mathfrak{P}_0^*} \supseteqq (f'(\Pi_0))_{\mathfrak{P}_0} = \mathfrak{D}_0(N_0/k_T)_{\mathfrak{P}_0}.$$

Thus, putting $(f'(\Pi_0))_{\mathfrak{P}_0}/\mathfrak{D}_0(KN_0/K)_{\mathfrak{P}_0^*} = \mathfrak{A}(\subset \mathfrak{O}_0^*)$, we have:

(I) $$\mathfrak{D}\,(K/k)_{\mathfrak{P}} = \mathfrak{D}_0(K/k_T)_{\mathfrak{P}} = \mathfrak{A}\mathfrak{D}_0(KN_0/N_0)_{\mathfrak{P}_0^*}.$$

Let $N_0 \subset N_1 \subset \cdots \subset N_m = N$ and $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_m = \mathfrak{P}$ be the same notations as in 4. Let $\mathfrak{O}_0^* \subseteqq \mathfrak{O}_1^* \subseteqq \cdots \subseteqq \mathfrak{O}_m^* = \bar{\mathfrak{O}}$ be the series of the rings of all integral elements in $KN_0 \subseteqq KN_1 \subseteqq \cdots \subseteqq KN_m = N$ with respect to $\mathfrak{o}$, and $\mathfrak{P}_0^* \subseteqq \mathfrak{P}_1^* \cdots \subseteqq \mathfrak{P}_m^* = \mathfrak{P}$ be the series of the prime ideals in $\mathfrak{O}_0^* \subseteqq \mathfrak{O}_1^* \subseteqq \cdots \subseteqq \mathfrak{O}_m^* = \bar{\mathfrak{O}}$. Then, by the same way as the above, we have:

(II) $$\mathfrak{D}_0(KN_{i+1}/KN_i)_{\mathfrak{P}_{i+1}^*} \supseteqq \mathfrak{D}_0(N_{i+1}/N_i)_{\mathfrak{P}_{i+1}} \quad (i = 0, 1, \dots, m-1).$$

However, in the relation

$$[KN_0 : N_0] \prod_{i=0}^{m-1} [KN_{i+1} : KN_i] = \prod_{i=0}^{m-1} [N_{i+1} : N_i] \qquad [N_{i+1} : N_i] = p,$$

since $[KN_0 : N_0] > 1$ and $[KN_{i+1} : KN_i] \leqq [N_{i+1} : N_i]$, there exists at least one sufix i of $0 \leqq i \leqq m-1$ such that $KN_{i+1} = KN_i$ i.e. $\mathfrak{D}_0(KN_{i+1}/KN_i) = \mathfrak{O}_{i+1}^*$. Then, by applying this fact, (II) and Theorem 9 iv) in the relation

$$\mathfrak{D}_0(KN_0/N_0) = \prod_{i=0}^{m-1} \mathfrak{D}_0(N_{i+1}/N_i) / \prod_{i=0}^{m-1} \mathfrak{D}_0(KN_{i+1} : KN_i),$$

we have:

(III) $$\mathfrak{D}_0(KN_0/N_0) \subset \mathfrak{P}_0^*$$

Thus from (I) and (III), we obtain $\mathfrak{D}_0(K/k) \subset \mathfrak{P}$.

Case (2). $Kk_T \supsetneqq k_T$ $(K \not\supset k_T)$.

Using the relations

$$\mathfrak{D}_0(K/k)_{\mathfrak{P}} \mathfrak{D}_0(Kk_T/K)_{\mathfrak{P}_T} = \mathfrak{D}_0(k_T/k)_{\mathfrak{p}_T} \mathfrak{D}_0(Kk_T/k_T)_{\mathfrak{P}_T},$$
$$\mathfrak{D}_0(Kk_T/K)_{\mathfrak{P}_T} = \mathfrak{O}_T \quad \text{and} \quad \mathfrak{D}_0(k_T/k)_{\mathfrak{p}_T} = \mathfrak{o}_T,$$

we have:

$$\mathfrak{D}_0(K/k)_{\mathfrak{P}} = \mathfrak{D}_0(Kk_T/k_T)_{\mathfrak{P}_T}.$$

Hence, by applying Case (1) i.e. using Kk_T in place of K, we have $\mathfrak{D}(Kk_T/k_T) \subset \mathfrak{P}_T$. Thus we obtain $\mathfrak{D}_0(K/k) \subset \mathfrak{P}$. q. e. d.

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