SEMI-MODULARITY IN RELATIVELY ATOMIC, UPPER CONTINUOUS LATTICES

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K. Menger [1][1] has introduced the following conditions to characterize the lattice of all subspaces of a finite dimensional affine space:

(η'') *If p is a point, then either $p \leqq a$ or $a \cup p$ covers a, for any element a.*

($\bar{\eta}$) *If h is covered by 1, then either $a \leqq h$ or a covers $a \cap h$, for any element a with $a \cap h \neq 0$.*

L. R. Wilcox [1] has shown that the lattice of all subspaces of an affine space is semi-modular in the sense that

(A) *$(b, c)M$, $b \cap c = 0$ imply $(c, b)M$,* and

(B) *$b \cap c \neq 0$ implies $(b, c)M$.*

The semi-modularity in this sense was used in my previous paper [1] to characterize the lattice of all subspaces of an affine space of arbitrary dimensions, noting that it might be replaced by the following conditions:

(ξ') *If a, b cover c and $a \neq b$, then $a \cup b$ covers a and b.*

(P) *If $p \leqq q \cup a$, $r \leqq a$, where p, q, r are points and a is any element, then there exists a point s with $p \leqq q \cup r \cup s$, $s \leqq a$.*

While L. R. Wilcox [2] has shown that in a lattice of finite dimensions, (ξ') is equivalent to the condition:

(α) *$(b, c)M$ implies $(c, b)M$,*

which follows immediately from (A) and (B).

The purpose of this paper is to show that in any relatively atomic, upper continuous lattice, (ξ') is equivalent to (α), and also the combined conditions "(η'') *and* ($\bar{\eta}$)", "(A) *and* (B)", and "(ξ') *and* (P)" are equivalent to each other.

1. We begin by listing the definitions and several known lemmas we shall employ.

DEFINITION 1. A lattice with 0 is called *relatively atomic* if $a < b$ implies $a < a \cup p \leqq b$ for some point p.

1) The numbers in square brackets refer to the list of references at the end of the paper.

LEMMA 1. *A lattice L with 0 is relatively atomic if and only if every element of L is the join of points.*

PROOF. Cf. F. Maeda [1] 88 Lemma 1.1.

DEFINITION 2. Let $\{a_\delta; \delta \in D\}$ be a directed set of elements in a complete lattice L. If $a_\delta \uparrow a$ implies $a_\delta \frown b \uparrow a \frown b$ for any element b, then L is called *upper continuous.*

LEMMA 2. *A relatively atomic, complete lattice L is upper continuous if and only if $p \leqq \bigvee(P)$ implies $p \leqq q_1 \smile q_2 \smile \cdots \smile q_n$, each q_i being in P, where P is a set of points in L.*

PROOF. Cf. F. Maeda [1] 90 Lemma 1.3.

DEFINITION 3. Let S be a set of points of a complete lattice. If $\bigvee(S_1) \frown \bigvee(S_2) = 0$ for any two disjoint subsets S_1 and S_2 of S, then S is called an *independent system* and is denoted by $(p; p \in S)\perp$ or $(S)\perp$. If in particular $S = \{p_1, p_2, \dots, p_n\}$, then we denote it by $(p_1, p_2, \dots, p_n)\perp$.

DEFINITION 4. By a *semi-modular* lattice, it is meant a lattice satisfying:

(ξ') *If a and b cover c, and $a \neq b$, then $a \smile b$ covers a and b.*

A relatively atomic, upper continuous, and semi-modular lattice is called a *matroid* lattice.

LEMMA 3. *Let $p_1, p_2, \dots, p_n$ be points of a semi-modular lattice with 0. Then $(p_1, \dots, p_n)\perp$ if and only if*

$$(p_1 \smile \cdots \smile p_k) \frown p_{k+1} = 0 \quad \textit{for } k = 1, 2, \dots, n\text{-}1.$$

PROOF. Cf. U. Sasaki and S. Fujiwara [1] 184 Lemma 2.

LEMMA 4. *Let $p_1, \dots, p_n, q_1, \dots, q_n$ be points of a semi-modular lattice with 0. If $(q_1, \dots, q_n)\perp$ and $q_j \leqq \bigvee_{i=1}^{n} p_i$ $(j = 1, 2, \dots, n)$, then $\bigvee_{i=1}^{n} p_i = \bigvee_{j=1}^{n} q_j$.*

PROOF. Cf. U. Sasaki and S. Fujiwara [1] 184 Lemma 2.

LEMMA 5. *If P is an independent system of points in a matroid lattice L, and if q is a point with $q \frown \bigvee(P) = 0$, then the set obtained by adjoining q to P is an independent system.*

PROOF. Cf. F. Maeda [2] 179 Lemma 6.

LEMMA 6. *If P is any independent system of points with $\bigvee(P) \leqq a$ in a matroid lattice, then there is a set $Q \supseteqq P$ which is a basis of a.* By a *basis* of an element a, we mean an independent system Q of points with $a = \bigvee(Q)$.

PROOF. Cf. F. Maeda [2] 179 Lemma 7.

LEMMA 7. *Let P be an independent system of points in a matroid lattice. Then for any subsets P_1, P_2 of P,*

$$\bigvee(P_1) \frown \bigvee(P_2) = \bigvee(P_1 \frown P_2).$$

PROOF. Cf. F. Maeda [2] 180 Theorem 1.

LEMMA 8. *In a relatively atomic, upper continuous lattice, the following conditions are equivalent.*

(ξ') *If a and b cover c, and $a \neq b$, then $a \cup b$ covers a and b.*

(η'') *If p is a point, then either $p \leqq a$ or $a \cup p$ covers a.*

(η') *If p, q are points, and if $q \leqq p \cup a$, and $q \cap a = 0$, then $p \leqq q \cup a$.*

PROOF. Cf. F. Maeda [2] 180 Theorem 2.

DEFINITION 5. By a *strongly plane* lattice, we mean a lattice satisfying the condition:

(P) *If $p \leqq q \cup a$, $r \leqq a$, where p, q, r are points and a is any element, then there exists a point s with $p \leqq q \cup r \cup s$, $s \leqq a$.*

DEFINITION 6. By $(b, c)M$, we mean that

$$a \leqq c \text{ implies } (a \cup b) \cap c = a \cup (b \cap c).$$

A lattice with 0 is called *semi-modular in the sense of Wilcox* if

(A) $(b, c)M$, $b \cap c = 0$ *imply* $(c, b)M$, and

(B) $b \cap c \neq 0$ *implies* $(b, c)M$.

Obviousely (A) and (B) imply the following condition:

(α) $(b, c)M$ *implies* $(c, b)M$.

2. In this section, we shall show the equivalence of (ξ') and (α).

LEMMA 9. *In a matroid lattice of arbitrary dimensions, if $(b, c)M$ then $(c, b)M$.*

PROOF. First suppose $b \cap c \neq 0$. Let d be any element with $0 < d \leqq b$ and let p be any point such that $p \leqq (d \cup c) \cap b$. It follows from Lemma 6 that there exist point sets P, Q, and R such that P and Q are bases of d and $b \cap c$ respectively, and $Q \cup R$ is a basis of c, since $d > 0$, $b \cap c > 0$, and $b \cap c \leqq c$.[2)]

Since $p \leqq d \cup c = \bigvee(P) \cup \bigvee(Q) \cup \bigvee(R)$, we have by Lemma 2:

(1) $p \leqq p_1 \cup p_2 \cup \cdots \cup p_l \cup q_1 \cup q_2 \cup \cdots \cup q_m \cup r_1 \cup r_2 \cup \cdots \cup r_n$,

for some $p_i \in P$ $(i=1, 2, \ldots, l)$, $q_j \in Q$ $(j=1, 2, \ldots, m)$, and $r_k \in R$ $(k=1, 2, \ldots, n)$.

We can assert that by deleting the redundant points in (1), we obtain:

(2) $p \leqq p_{i_1} \cup \cdots \cup p_{i_{l'}} \cup q_{j_1} \cup \cdots \cup q_{j_{m'}}$.

For, let us assume the contrary and suppose that no point in (1) is irredundant. If $n=1$, then we have by (η'):

2) If $b \cap c = c$, then R is the void set and $c \leqq b$, whence the result is obvious.

$$r_1 \leqq p_1 \cup \cdots \cup p_l \cup q_1 \cup \cdots \cup q_m \cup p \leqq b .$$[3]

Hence it holds $r_1 \leqq b \frown c = \bigvee(Q)$, contradicting the fact that $Q \cup R$ is an independent system. If $n>1$, then by a similar way we have:

$$r_n \leqq p_1 \cup \cdots \cup p_l \cup q_1 \cup \cdots \cup q_m \cup r_1 \cup \cdots \cup r_{n-1} \cup p .$$

Put $r_1 \cup \cdots \cup r_{n-1} = a$, then we have $r_n \leqq (a \cup b) \frown c$, since $p_i \leqq d \leqq b$, $q_j \leqq b$, and $p \leqq b$. While it holds by the hypothesis $(b, c)M$, whence $r_n \leqq a \cup (b \frown c) = r_1 \cup \cdots \cup r_{n-1} \cup \bigvee(Q)$, contrary to $(Q, R)\perp$. Thus (2) has been proved, whence $p \leqq \bigvee(P) \cup \bigvee(Q) = d \cup (b \frown c)$.

Hence we have by Lemma 1, $(d \cup c) \frown b \leqq d \cup (b \frown c)$, which secures $(c, b)M$, since the converse inequarity is true in any lattice.

If $b \frown c = 0$, the set Q is a void set and the proof is similar to the above.

However it is well known that if a lattice satisfies the condition (α), then it is semi-modular. Cf. G. Birkhoff [1] 101, Ex. 1, and L. R. Wilcox [2] Theorem 1.

Thus we have the following

THEOREM 1.[4] *A relatively atomic, upper continuous lattice is semi-modular if and only if it satisfies the condition*:

(α) $(b, c)M$ *implies* $(c, b)M$.

3. Now we shall show several lemmas in order to prove the equivalence of the combined conditions "(A) and (B)", "(η'') and $(\bar{\eta})$", and "(ξ') and (P)".

LEMMA 10. *Let L be a lattice satisfying the condition (B). Then L satisfies the condition*:

$(\bar{\eta})$ *If h is covered by* 1, *and a is any element of L with* $h \frown a \neq 0$, *then either* $a \leqq h$ *or a covers* $a \frown h$.

PROOF. Assume $a \not\leqq h$, and let b be any element of L such that $h \frown a \leqq b \leqq a$. Since $h \frown a \neq 0$, we have in view of the condition (B), $(h, a)M$. It follows $(b \cup h) \frown a = b \cup (h \frown a)$, whence we have $h \frown a = b$ if $b \leqq h$, and $a = b$ if $b \not\leqq h$, since $b \not\leqq h$ yields $b \cup h = 1$. Consequently $h \frown a$ is covered by a.

LEMMA 11. *Let L be a matroid lattice satisfying the condition* $(\bar{\eta})$. *Then L is strongly plane.*

3) Since no point in (1) is redundant, $(p_1 \cup \cdots \cup p_l \cup q_1 \cup \cdots \cup q_m) \frown p = 0$, whence we have the former inequality by applying (η') to (1). The latter is obvious, since $p_i \leqq d \leqq b$, $q_j \leqq b$, and $p \leqq b$.

4) As to the case of a lattice of finite dimensions, cf. G. Birkhoff [1] 101 Theorem 1, and L. R. Wilcox [2] Theorem 2.

PROOF. Let $p \leqq q \cup a$, $r \leqq a$, where p, q, r are points and a is an element of L. We shall show that there exists a points s with

$$p \leqq q \cup r \cup s, \quad s \leqq a.$$

We can assume $p \nleqq a$, since otherwise the result is trivial. It follows also $q \nleqq a$.

We may also assume $(p, q, r) \perp$, since the contrary implies in view of Lemma 3, $p=q$ or $p \neq q$ and $r \leqq p \cup q$, whence $p \leqq q \cup r$, in either case the results being obvious.

Now since $r \leqq a$, there exists by Lemma 6, an independent system of points S such that $(r, S) \perp$ and $a = r \cup \bigvee(S)$. Since $q \cap a = 0$, it holds $(q, r, S) \perp$ and $q \cup a = q \cup r \cup \bigvee(S)$. By making use of Lemma 6 again, there exists an independent system of points T such that

(1) $(q, r, S, T) \perp$, and $1 = q \cup r \cup \bigvee(S) \cup \bigvee(T)$.

Put $h = r \cup \bigvee(S) \cup \bigvee(T)$, then h is covered by 1, and $h \cap (p \cup q \cup r) \geqq r > 0$. Furthermore $h \ngeqq (p \cup q \cup r)$, since the contrary would imply $h \geqq q$, contradicting (1). It follows from $(\bar{\eta})$:

(2) $p \cup q \cup r$ covers $h \cap (p \cup q \cup r)$.

While $p \cup q \cup r \leqq q \cup a$, since $p \leqq q \cup a$. It follows:

(3) $(p \cup q \cup r) \cap h = (p \cup q \cup r) \cap \{q \cup r \cup \bigvee(S)\} \cap \{r \cup \bigvee(S) \cup \bigvee(T)\}$
$= (p \cup q \cup r) \cap a,$

the latter equality following from Lemma 7.

From (2) and (3), it follows that $p \cup q \cup r$ covers $(p \cup q \cup r) \cap a$, whence $(p \cup q \cup r) \cap a > r$. Therefore there is a point s such that

(4) $s \leqq p \cup q \cup r$, $s \leqq a$, and $s \neq r$.

While it holds $s \nleqq q \cup r$, since otherwise $q \leqq s \cup r$, contrary to $q \nleqq a$. It follows at once from (4), $p \leqq q \cup r \cup s$, completing the proof.

LEMMA 12. *Let L be a strongly plane matroid lattice, and p be a point such that $p \leqq a \cup b$; a, b being elements $(\neq 0)$ of L. Then there exist points q, r $(\leqq a)$; s, t $(\leqq b)$ such as $p \leqq q \cup r \cup s \cup t$.*

PROOF. In view of Lemma 2, there exist points $p_1, p_2, \dots, p_n$; $q_1, q_2, \dots, q_m$ such that $p_i \leqq a$ $(i=1, 2, \dots, n)$; $q_j \leqq b$ $(j=1, 2, \dots, m)$ and

(1) $p \leqq p_1 \cup p_2 \cup \cdots \cup p_n \cup q_1 \cup q_2 \cup \cdots \cup q_m$.

First we shall show by induction that there exists a point $t (\leqq b)$ with

(2) $p \leqq p_1 \cup p_2 \cup \cdots \cup p_n \cup q_1 \cup t$.

When $n=1$, (2) is trivial, since L is strongly plane. Let us assume that (2) is true for $n=k-1$. It follows from (1) that there exists a point p' with

$$p \leqq p_1 \cup p_2 \cup p', \quad p' \leqq p_2 \cup p_3 \cup \cdots \cup p_k \cup q_1 \cup q_2 \cup \cdots \cup q_m .$$

By the induction hypothesis, it holds:

$$p' \leqq p_2 \cup p_3 \cup \cdots \cup p_k \cup q_1 \cup t, \text{ for some point } t (\leqq b).$$

Hence (2) holds for the point t.

Since L is strongly plane, there exists, from (2), a point s with

$$p \leqq t \cup q_1 \cup s, \quad s \leqq q_1 \cup p_n \cup p_{n-1} \cup \cdots \cup p_1 ,$$

whence again there exists a point r with

$$s \leqq q_1 \cup p_n \cup r, \quad r \leqq p_n \cup p_{n-1} \cup \cdots \cup p_1 \leqq a$$

Consequently $p \leqq t \cup q_1 \cup p_n \cup r$, where $t, q_1 \leqq b$; $p_n, r \leqq a$.

LEMMA 13. *In a strongly plane matroid lattice L,*

$$b \cap c \neq 0 \textit{ implies } (b, c)M .$$

PROOF. Let $b \cap c \neq 0$. It is sufficient to prove that $a \leqq c$ implies $(a \cup b) \cap c \leqq a \cup (b \cap c)$, since the converse inequality is true in general. In view of Lemma 1, we need only to show that if $a \leqq c$ and if p is a point with $p \leqq (a \cup b) \cap c$, then $p \leqq a \cup (b \cap c)$.

From Lemma 12, there are points p_1, p_2, q_1, q_2 such that

(1) $p \leqq p_1 \cup p_2 \cup q_1 \cup q_2$, where $p_1, p_2 \leqq a$; $q_1, q_2 \leqq b$.

It may be assumed that $p \not\leqq p_1 \cup p_2$, and $p \not\leqq q_1 \cup q_2$, since otherwise the result is trivial, because $p \leqq c$. Hence we can suppose that we obtain by deleting the redundant points from (1), $p \leqq p_1 \cup q_1$, or $p \leqq p_1 \cup p_2 \cup q_1$, or $p \leqq p_1 \cup q_1 \cup q_2$, or $p \leqq p_1 \cup p_2 \cup q_1 \cup q_2$.

In the case $p \leqq p_1 \cup q_1$, it holds $q_1 \leqq p_1 \cup p \leqq c$, whence $q_1 \leqq b \cap c$. It follows that $p \leqq a \cup (b \cap c)$, which is to be proved.

In the case $p \leqq p_1 \cup p_2 \cup q_1$, the proof is similar to the above.

In the case $p \leqq p_1 \cup q_1 \cup q_2$, let r be a point with $r \leqq b \cap c$, then it follows $p \leqq p_1 \cup r \cup q_1 \cup q_2$. Since L is strongly plane, there exists a point s with

(2) $p \leqq p_1 \cup r \cup s$, and (3) $s \leqq r \cup q_1 \cup q_2$.

We may assume $p \not\leqq p_1 \cup r$, since otherwise the result is obvious. It follows from (2), $s \leqq p_1 \cup r \cup p \leqq c$, while $s \leqq b$ by (3), whence $s \leqq b \cap c$. Consequently the result follows immediately from (2).

Finally we shall assume that no point is redundant in (1), then

(4) $(p_1, p_2, q_1, q_2)\perp$, and (5) $(p_1, p_2, p, q_1)\perp$.

It follows from Lemma 4,

(6) $p_1 \cup p_2 \cup p \cup q_1 = p_1 \cup p_2 \cup q_1 \cup q_2$.

Let r be a point with $r \leqq b \cup c$, then the following cases occur:

Case I. $r \nleqq p_1 \cup p_2 \cup p \cup q_1$.

It follows from Lemma 5 and (6),

(7) $(r, p, p_1, p_2, q)\perp$, and (8) $(r, p_1, p_2, q, q_2)\perp$.

By (1) and (5), it holds $q_2 \leqq q_1 \cup p_1 \cup p_2 \cup p$, whence we have:

$$q_2 \leqq q_1 \cup (r \cup p_1 \cup p_2 \cup p).$$

Therefore there exists a point s with

(9) $q_2 \leqq q_1 \cup r \cup s$, and (10) $s \leqq r \cup p_1 \cup p_2 \cup p$.

Then $s \nleqq r \cup p_1 \cup p_2$, since otherwise (9) would yield $q_2 \leqq q_1 \cup r \cup p_1 \cup p_2$, contrary to (8). Hence we have from (10);

(11) $p \leqq p_1 \cup p_2 \cup r \cup s$.

It holds $s \leqq c$ by (10), and $s \leqq b$ by (9) and (8). And so the result is obvious from (11).

Case II. $r \leqq p_1 \cup p_2 \cup p \cup q_1$, *but* $r \nleqq p_1 \cup p_2 \cup p$.

It follows from (5) and Lemma 5 that $(r, p_1, p_2, p)\perp$, whence we have from (6) and Lemma 4:

$$r \cup p_1 \cup p_2 \cup p = p_1 \cup p_2 \cup p \cup q_1 = p_1 \cup p_2 \cup q_1 \cup q_2.$$

Therefore $q_1 \cup q_2 \leqq r \cup p_1 \cup p_2 \cup p \leqq c$, while $q_1 \cup q_2 \leqq b$, whence $q_1 \cup q_2 \leqq b \cap c$. Hence the result is immediate from (1).

Case III. $r \leqq p_1 \cup p_2 \cup p$, *but* $r \nleqq p_1 \cup p_2$.

It follows at once, $p \leqq p_1 \cup p_2 \cup r \leqq a \cup (b \cap c)$, which is to be proved.

Case IV. $r \leqq p_1 \cup p_2$.

We can assume $r \neq p_1$, without loss of generality. So we have $p_1 \cup r = p_1 \cup p_2$, and it holds by (5),

(12) $(p, p_1, r)\perp$.

It follows from (1), $p \leqq p_1 \cup r \cup q_1 \cup q_2$. Hence there is a point s with

(13) $p \leqq p_1 \cup r \cup s$, and (14) $s \leqq r \cup q_1 \cup q_2$.

We have $s \leqq p_1 \cup r \cup p \leqq c$ by (12), (13), and $s \leqq b$ by (14). Therefore the result is obvious in view of (13).

This completes the proof.

From Theorem 1, Lemma 8, 10, 11 and 13, we obtain the following

THEOREM 2. *In a relatively atomic, upper continuous lattice, the combined conditions* "(A) *and* (B)", "(η'') *and* $(\bar{\eta})$", *and* "(ξ') *and* (P)" *are equivalent to each other*:

(A) $(b, c)M$, $b \cap c = 0$ *imply* $(c, b)M$, *and*
(B) $b \cap c \neq 0$ *implies* $(b, c)M$.

$\left\{\begin{array}{ll}(\eta'') & \textit{If } p \textit{ is a point, then either } p \leqq a \textit{ or } a \cup p \textit{ covers } a, \textit{ for any element } a, \textit{ and} \\ (\bar{\eta}) & \textit{If } h \textit{ is covered by } 1, \textit{ then either } a \leqq h \textit{ or } a \textit{ covers } a \cap h, \textit{ for any element } a \textit{ with } a \cap h \neq 0.\end{array}\right.$

$\left\{\begin{array}{ll}(\xi') & \textit{If } a, b \textit{ cover } c \textit{ and } a \neq b, \textit{ then } a \cup b \textit{ covers } a \textit{ and } b, \textit{ and} \\ (\mathrm{P}) & \textit{If } p \leqq q \cup a,\ r \leqq a, \textit{ where } p, q, r \textit{ are points and } a \textit{ is any element, then there exists a point } s \textit{ with } p \leqq q \cup r \cup s,\ s \leqq a.\end{array}\right.$

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