THEORY OF THE SPHERICALLY SYMMETRIC SPACE-TIMES. IV. CONFORMAL TRANSFORMATIONS[1)]

By

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§1. Spherically symmetric conformal transformation.

In the previous papers we have obtained some properties of s. s. space-times.[1)] In the present paper, further, we shall study conformal transformations which transform a given s. s. space-time into s. s. one again.

A conformal transformation of a Riemannian space is given by

$$\mathrm{C}(v): \qquad g^*_{ij} = e^{2v} g_{ij}, \quad (v = v(x^i)). \tag{1.1}$$

If g_{ij} gives a s. s. space-time S_0, the space-time defined by g^*_{ij} is not necessarily s. s., but it is evident that when v is s. s. with respect to a c. s. (K) of S_0 i. e. when

$$v_i = -(\alpha^s v_s)\alpha_i + (\beta^s v_s)\beta_i, \quad (v_i \equiv \nabla_i v), \tag{1.2}$$

holds for (K), the resulting space-time becomes s. s. again. We shall call such a transformation a *spherically symmetric conformal transformation* (or s. s. c. t. in short) with respect to the S_0. If (1.1) is s. s. with respect to S_0 its inverse conformal transformation is also s. s. with respect to $S_0{}^*$ defined by g^*_{ij}. If a scalar v is s. s. with respect to a c. s. it is also s. s. with respect to any other c. s. to within m-transformation at most.[2)] Hence we know that any $'v$ obtained from v of a s. s. c. t. by any m-transformation defines a s. s. c. t. again. In this paper we shall obtain the transformation equations of the c. s. corresponding to a s. s. c. t. to within *ε-transformation* and then study the general form of C(v) which preserves the spherical symmetry of S_0.

§2. Preliminary formulae.

Transformation equations for Christoffel symbols and curvature tensor corresponding to (1.1) are given by[3)]

$$\{^k_{ij}\}^* = \{^k_{ij}\} + \delta^k_i v_j + \delta^k_j v_i - g_{ij} g^{km} v_m, \tag{2.1}$$

$$e^{-2v} K^*_{hijk} = K_{hijk} + 4g_{[h[k} v_{i]j]} + 2p g_{[h[k} g_{i]j]}, \tag{2.2}$$

where
$$v_{ij} = \nabla_i \nabla_j v - v_i v_j, \quad p = g^{ij} v_i v_j. \tag{2.3}$$

Let $C(v)$ be an arbitrary s. s. c. t. If we take any s. s. coordinate system of S_0, we have

$$ds^2 = -A(r, t)\,dr^2 - B(r, t)(d\theta^2 + \sin^2\theta d\phi^2) + C(r, t)\,dt^2, \tag{2.4}$$

and v becomes a function of r and t by a suitable $\dot{m}$-transformation at most. Then from (2.1), (2.2) and (2.3), we have

$$v_1 = v', \quad v_4 = \dot{v}, \quad v_2 = v_3 = 0, \tag{2.5}$$

$$v_{11} = v'' - (A'/2A)\,v' - (\dot{A}/2C)\,\dot{v} - v'^2, \quad \text{etc.}, \tag{2.6}$$

$$\{^1_{11}\}^* = A'/2A + v', \quad \{^4_{11}\}^* = \dot{A}/2C + (A/C)\,\dot{v}, \quad \text{etc.}, \tag{2.7}$$

$$\begin{cases} \alpha^* = e^{-2v}(\alpha + v_{11}/A + v_{22}/B - p), \quad \beta = e^{-2v}(\beta + v_{22}/B - v_{44}/C - p), \\ \gamma^* = e^{-2v}(\gamma + v_{14}/C), \quad \xi^* = e^{-2v}(\xi + v_{11}/A - v_{44}/C - p), \\ \eta^* = e^{-2v}(\eta + 2v_{22}/B - p); \quad p = -v'^2/A + \dot{v}^2/C. \end{cases} \tag{2.8}$$

Using the formulae concerning c. s. given in (I) and the relation

$$\begin{gathered} (g^{st} - h^{st})\,v_{st} = -2v_{22}/B, \quad g^{st} v_{st} = \Box v - p, \\ (h_{ij} \equiv -\alpha_i \alpha_j + \beta_i \beta_j, \quad \Box \equiv g^{st} \nabla_s \nabla_t), \end{gathered} \tag{2.9}$$

we can prove:

Theorem [2.1] *Let v be s. s. with respect to a (K) of an S_0, then for the s. s. c. t. $C(v)$ we have*

$$\overset{1}{\rho}{}^* = e^{-2v}\,\overset{1}{\rho}, \quad F^* = F - v, \tag{2.10}$$

$$\overset{4}{\rho}{}^* = e^{-2v}\{\overset{4}{\rho} - 2(g^{st} - h^{st})\,v_{st} - 2p\} = e^{-2v}(\overset{4}{\rho} - 2\Box v + 2h^{st} v_{st}), \tag{2.11}$$

as the transformation formulae for $\overset{1}{\rho}$, F and $\overset{4}{\rho}$. When S_0 admits ω-transformation (2.11) holds for all c. s. obtained from (K) by any ω-transformation. Furthermore the formulae holds for all c. s. provided that v undergoes the same m-transformation as (K).

We shall call the property stated at the end of the theorem the *invariancy under m-transformation* for brevity's sake. It is to be noted that the tensor h_{ij} is invariant under ω-transformation.

§3. $C(v)$ which transforms S_a into S_a^*.

Theorem [3.1] *Let v be s. s. with respect to a c. s. (K) of an S_a, then a necessary and sufficient condition that a s. s. c. t. $C(v)$ transform S_a into an S_a^* again is given by*

$$P \equiv \alpha^s \beta^t v_{st} = 0, \quad Q \equiv (\alpha^s \alpha^t + \beta^s \beta^t) v_{st} = 0. \tag{3.1}$$

And this condition is invariant under m- and ω-transformations.

Proof: Taking the standard coordinate system for (K) and using the theorems concerning c. s. obtained in (I) we can prove that the condition that C(v) transform S_a into S_a^* is given by ($v_{14}=0$, $v_{11}/A+v_{44}/C=0$) which is equivalent to (3.1) by virtue of (2.6). Further if (α_i', β_i') is any pair of c. v. obtained from (α_i, β_i) by an ω-transformation, we have

$$\begin{cases} P' \equiv \alpha'^s \beta'^t v_{st} = P \cosh 2\omega + \frac{1}{2} Q \sinh 2\omega, \\ Q' \equiv (\alpha'^s \alpha'^t + \beta'^s \beta'^t) v_{st} = 2P \sinh 2\omega + Q \cosh 2\omega, \end{cases} \tag{3.2}$$

from which the latter part of the theorem is obvious.

We shall denote such a s. s. c. t. as stated above by $C_{aa}(v)$. Then we easily obtain:

Theorem [3.2] *In $C_{aa}(v)$, the transformation equation for $\overset{2}{\rho}(=\overset{3}{\rho})$ is given by*

$$\overset{2}{\rho}{}^* = \overset{3}{\rho}{}^* = e^{-2v} \{ \overset{2}{\rho} + 2(g^{st} - 2h^{st}) v_{st} \}, \tag{3.3}$$

where (α_i, β_i) is any pair of c. v. of the S_a.

(3.3) coincides with the transformation equation for M given in §5.

Theorem [3.3] *In $C_{aa}(v)$, we can establish one to one correspondence between $(\alpha_i, \beta_i; \sigma, \bar{\sigma}; \kappa, \bar{\kappa})$ of S_a and $(\alpha_i^*, \beta_i^*; \sigma^*, \cdots)$ of S_a^* by the following relations:*

$$\begin{cases} \alpha_i^* = e^v \alpha_i, \quad \beta_i^* = e^v \beta_i, \quad i.e. \quad \alpha^{*i} = e^{-v} \alpha^i, \quad \beta^{*i} = e^{-v} \beta^i; \\ \sigma^* = e^{-v} (\sigma - \beta^s v_s), \quad \bar{\sigma}^* = e^{-v} (\bar{\sigma} + \alpha v^s{}_s); \\ \kappa^* = e^{-v} (\kappa + \alpha^s v_s), \quad \bar{\kappa}^* = e^{-v} (\bar{\kappa} + \beta^s v_s), \end{cases} \tag{3.4}$$

to within an m-transformation.

Both S_a and S_a^* admit ω-transformation of c. s. and if we use the formulae concerning this transformation we can easily prove the theorem. When both S_a and S_a^* are neither [A] nor [B], their c. s. are invariant under m-transformation,[2] hence we obtain one to one correspondence between both c. s. Of course another way of correspondence may exist.

§ 4. C(v) which transforms S_a into S_b^*.

From [3.1] it follows that a necessary and sufficient condition that the S_0^* obtained by a s. s. c. t. from an S_a be S_b^* is that (3.1) be not satisfied. We shall denote such a C(v) by $C_{ab}(v)$. In this section we shall give transformation equations for c. s. under $C_{ab}(v)$. Here it is to be noticed that S_a admits ω-transformation of c. s. though it is not the case for S_b^*.

Theorem [4.1] *In $C_{ab}(v)$, it holds that*

$$\begin{cases} \overset{2}{\rho}{}^* = e^{-2v}\{\overset{2}{\rho}+2(g^{st}-2h^{st})v_{st}+2e\sqrt{(P^2-4Q^2)}\}, \\ \overset{3}{\rho}{}^* = e^{-2v}\{\overset{2}{\rho}+2(g^{st}-2h^{st})v_{st}-2e\sqrt{(P^2-4Q^2)}\}, \end{cases} \tag{4.1}$$

$$\alpha_i^* = e^v(\alpha_i \cosh\Omega+\beta_i\sinh\Omega), \quad \beta_i^* = e^v(\alpha_i\sinh\Omega+\beta_i\cosh\Omega), \tag{4.2}$$

where $\tanh 2\Omega=-2Q/P$ *and* $e=\pm 1$ *is chosen so as to satisfy* $eP\geqq 0$. *Furthermore if* (α_i', β_i') *is obtained from* (α_i, β_i) *by an* ω*-transformation, it holds that*

$$\begin{cases} \overset{2}{\rho}{}^* = e^{-2v}\{\overset{2}{\rho}+2(g^{st}-h'^{st})v_{st}+2e'\sqrt{(P'^2-4Q'^2)}\}, \\ \overset{3}{\rho}{}^* = e^{-2v}\{\overset{2}{\rho}+2(g^{st}-h'^{st})v_{st}-2e'\sqrt{(P'^2-4Q'^2)}\}, \end{cases} \tag{4.1'}$$

$$\begin{cases} \alpha_i^* = e^v\{\alpha_i'\cosh(\Omega'-\omega)+\beta_i'\sinh(\Omega'-\omega)\}, \\ \beta_i^* = e^v\{\alpha_i'\sinh(\Omega'-\omega)+\beta_i'\cosh(\Omega'-\omega)\}, \end{cases} \tag{4.2'}$$

where ω *is the parameter of the* ω*-transformation,* e' *is chosen so as to satisfy* $e'P'\geqq 0$, *and*

$$\tanh 2\Omega' = (P'\sinh 2\omega-2Q'\cosh 2\omega)/(P'\cosh 2\omega-2Q'\sinh 2\omega). \tag{4.3}$$

Of course (4.1), (4.2), (4.1') and (4.2') are invariant under m-transformation.

Proof: Using the relation[4)]

$$\alpha_i^* = e^v(\alpha_i\cosh\zeta^*+\beta_i\sinh\zeta^*), \quad \beta_i^* = e^v(\alpha_i\sinh\zeta^*+\beta_i\cosh\zeta^*), \tag{4.4}$$

$$\tanh 2\zeta^* = 2\sqrt{C/A}\,\gamma^*/(\alpha^*-\beta^*) = -2Q/P, \tag{4.5}$$

$(P=v_{11}/A+v_{44}/C,\ Q=-v_{14}/\sqrt{AC})$, and putting $\zeta=\Omega$, we can obtain (4.2) and (4.2'). Then from (2.8), we have

$$M^* = e^{-2v}\{M+2(g^{st}-2h^{st})v_{st}\}, \quad (M=\overset{2}{\rho}=\overset{3}{\rho});\ N^*=2e^{-2v}\sqrt{(P^2-4Q^2)}. \tag{4.6}$$

But h^{st} and P^2-4Q^2 are invariant under ω-transformation, so we have (4.1) and (4.1').

Theorem [4.2] *In* $C_{ab}(v)$, *it holds that*

$$\begin{cases} \sigma^* = e^{-v}\{(\sigma-\alpha^s\Omega_s-\beta^s v_s)\cosh\Omega-(\bar\sigma+\beta^s\Omega_s+\alpha^s v_s)\sinh\Omega\}\,, \\ \bar\sigma^* = e^{-v}\{-(\quad ,, \quad)\sinh\Omega+(\quad ,, \quad)\cosh\Omega\}\,; \end{cases} \tag{4.7}$$

$$\begin{cases} \kappa^* = e^{-v}\{(\kappa+\alpha^s v_s)\cosh\Omega+(\bar\kappa+\beta^s v_s)\sinh\Omega\}\,, \\ \bar\kappa^* = e^{-v}\{(\quad ,, \quad)\sinh\Omega+(\quad ,, \quad)\cosh\Omega\}\,, \end{cases} \tag{4.8}$$

$$\begin{cases} \sigma^* = e^{-v}[\{\sigma'-\alpha'^s(\Omega_s'-\omega_s)-\beta'^s v_s\}\cosh(\Omega'-\omega)-\{\bar\sigma+\beta'^s(\Omega_s'-\omega_s)+\alpha'^s v_s\} \\ \qquad\qquad \sinh(\Omega'-\omega)]\,, \\ \bar\sigma^* = e^{-v}[-\{\quad ,, \quad\}\sinh(\Omega'-\omega)+\{\quad ,, \quad\}\cosh(\Omega'-\omega)]\,; \end{cases} \tag{4.7'}$$

$$\begin{cases} \kappa^* = e^{-v}\{(\kappa'+\alpha'^s v_s)\cosh(\Omega'-\omega)+(\bar\kappa'+\beta'^s v_s)\sinh(\Omega'-\omega)\}\,, \\ \bar\kappa^* = e^{-v}\{(\quad ,, \quad)\sinh(\Omega'-\omega)+(\quad ,, \quad)\cosh(\Omega'-\omega)\}\,, \end{cases} \tag{4.8'}$$

where $\Omega_s=\partial_s\Omega$ *and* $\omega_s=\partial_s\omega$, *corresponding to* (α_i, β_i) *and* (α_i', β_i') *respectively as in* [4.1].

Proof: Using the relations[4)]

$$\sigma^* = \bar M^*\cosh\zeta - \bar N^*\sinh\zeta^*\,,\quad \bar\sigma^* = -\bar M^*\sinh\zeta^*+\bar N^*\cosh\zeta^*\,, \tag{4.9}$$

$$\begin{aligned} \bar M^* &= e^{-v}(\sigma-\alpha^s\zeta_s^*-\beta^s v_s) \\ &= e^{-v}\{(\sigma'+\alpha'^s\omega_s-\alpha'^s\zeta_s^*-\beta'^s v_s)\cosh\omega+\cdots\}\,,\quad \text{etc.}, \end{aligned} \tag{4.10}$$

and putting $\zeta^*=\Omega$, we can prove (4.7) and (4.7′). Similarly we have (4.8) and (4.8′). Of course the right hand sides of (4.7′) and (4.8′) are independent of ω, and (4.7), ⋯ (4.8′) are invariant under m-transformation.

§ 5. C(v) which transforms S_b into S_b^*.

As in the preceding sections we can easily prove:

Theorem [5.1] *A necessary and sufficient condition that a s. s. c. t. be* $C_{ba}(v)$, *(i. e. a s. s. c. t. which transforms* S_b *into* S_a^**), is given by*

$$(\overset{2}{\rho}-\overset{3}{\rho})+4P=0\,,\quad (i.\,e.\ N+2P=0)\,,\quad and\quad Q=0\,. \tag{5.1}$$

When this condition is satisfied for a c. s. (K) *of* S_b *it holds for all c. s. of the* S_b *with a proviso that* v *undergoes the same m-transformation as* (K).

By this theorem we know that *an* S_b *is transformed into* S_b^* *by a s. s. c. t. when and only when (5.1) is not satisfied.* In the following we shall consider the transformation law of the c. s. under such a $C_{bb}(v)$.

Using the formulae concerning c. s. of S_b given in (I) we can easily obtain:

Lemma [5.2] *In $C_{bb}(v)$, the following relation concerning the transformation law for ζ, the quantity which appears in the formulae for c. s. of S_b, holds*

$$\zeta^* = \zeta + \Omega , \tag{5.2}$$

where

$$\tanh 2\Omega = -2Q/\{(\overset{2}{\rho}-\overset{3}{\rho})/4+P\} . \tag{5.3}$$

By using this lemma, we have

Theorem [5.3] *In $C_{bb}(v)$, we have the transformation equations (4.2), (4.7) and (4.8) for (α_i, β_i), $(\sigma, \bar{\sigma})$ and $(\kappa, \bar{\kappa})$ respectively, with a proviso that Ω is given by (5.3), and for $\overset{2}{\rho}$ and $\overset{3}{\rho}$, it holds that*

$$\left\{\begin{aligned} \overset{2}{\rho}{}^* &= e^{-2v}\{2(g^{st}-h^{st})v_{st}+(\overset{2}{\rho}+4\alpha^s\alpha^t v_{st})\cosh^2\Omega-(\overset{3}{\rho}-4\beta^s\beta^t v_{st}) \\ &\qquad \sinh^2\Omega+4Q\sinh 2\Omega\} , \\ \overset{3}{\rho}{}^* &= e^{-2v}\{2(\quad ,, \quad)v_{st}-(\quad ,, \quad)\sinh^2\Omega+(\quad ,, \quad) \\ &\qquad \cosh^2\Omega-4Q\sinh 2\Omega\} , \end{aligned}\right. \tag{5.4}$$

where Ω is given by (5.3) again. These equations are invariant under m-transformation of S_b.

Proof: We can prove the former part by using the formulae concerning c. s. of an S_b given in (I), and then by substituting this result into the identities (3.9) of (I) we can obtain (5.4). From this theorem we have:

Corollary [5.4] *In $C_{bb}(v)$, it holds that*

$$M^* = e^{-2v}\{M+2(g^{st}-2h^{st})v_{st}\} , \quad N^* = e^{-2v}\{(N+2P)\cosh 2\Omega+4Q\sinh 2\Omega\} .$$

$$(N^*)^2 = e^{-4v}\{(N+2P)^2-16Q^2\} . \tag{5.5}$$

§ 6. C(v) which transforms S_b into $S_a{}^*$.

The condition to be satisfied by $C_{ba}(v)$ is given by [5.1]. If this condition is satisfied Ω defined by (5.3) is indeterminate and we can not determine α_i^*, β_i^*, σ^*, $\cdots$ from the formulae given in the last section. This fact is natural since the c. s. of an S_a^* admits ω-transformation.

Theorem [6.1] *In $C_{ba}(v)$, it holds that*

$$\overset{2}{\rho}{}^* = \overset{3}{\rho}{}^* = e^{-2v}\{(\overset{2}{\rho}+\overset{3}{\rho})/2+2(g^{st}-2h^{st})v_{st}\} ,$$

and this holds for all c s. of S_b. $(\alpha_i^*, \beta_i^*; \sigma^*, \cdots)$ is given by

$$\left\{\begin{aligned} &\alpha_i^* = e^v\alpha_i , \quad \beta_i^* = e^v\beta_i ; \quad \sigma^* = e^{-v}(\sigma-\beta^s v_s) , \quad \bar{\sigma}^* = e^{-v}(\bar{\sigma}+\alpha^s v_s) , \\ &\kappa^* = e^{-v}(\kappa+\alpha^s v_s) , \quad \bar{\kappa}^* = e^{-v}(\bar{\kappa}+\beta^s v_s) , \end{aligned}\right. \tag{6.2}$$

and those obtained from this by m- and ω-transformations.

Proof: To prove the latter part of the theorem we have only to show that $(\alpha_i^*, \cdots)$ given by (6.2) gives a c.s. of S_a^*. Since the proof is easy but tedious, we shall omit it here.

By the discussions given in §§ 3, 4, 5 and 6, we have succeeded in expressing the transformation formulae for c. s. of an S_0 under any s. s. c. t. in invariant form in terms of the c. s. of given S_0 and the conformal factor v.

§ 7. Conformal transformation which preserves spherical symmetry of an S_0.

Let S^* be the space-time obtained from an S_0 by (1.1) whose v is any function of x^i. Obviously S^* is not necessarily s. s. but in some special cases both S_0 and S^* can be s. s. notwithstanding that (1.1) is not s. s. In this section we shall give some theorems concerning this problem.

We shall denote by S_{20} the s. s. space-time whose line element can be brought into the form

$$ds^2 = -A(r,\ t)\,dr^2 + C(r,\ t)\,dt^2 - B(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.1}$$

where B=const. and ξ=const. Hence S_{20} is an $S_{\rm II}$ whose $(r,\ t)$-space is of constant curvature (i. e. $\overset{1}{\rho}$=const.). Specially when S_{20} is conformally flat we have $\xi=-\eta=1/B$. Therefore in the coordinate system in which $A=1$, we have $\xi=(\sqrt{C})''/\sqrt{C}=-1/B$ and the general form of C is given by $(ae^{mr}+be^{-mr})^2$ where a and b are arbitrary functions of t and $m=1/\sqrt{B}$.

Theorem [7.1] *A necessary and sufficient condition that an S_0 be conformal to an S_{20} is given by $\overset{1}{\rho}e^{-2F}$=const.* (7.2)

Proof: (i) (7.2) is necessary: The condition for S_0 to be conformally flat is $\overset{1}{\rho}=0$ i. e. a special form of (7.2), hence we have only to deal with S_0 whose $\overset{1}{\rho}\neq 0$. Then we have

$$ds^2 = e^{2v}ds^{*2}, \quad ds^{*2} = -A^*dr^2 + C^*dt^2 - B^*(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.3}$$

where ds^{*2} gives an S_{20} and $v=v(r,\ \theta,\ \phi,\ t)$. By the same method as the one used in the next theorem we know that a c. s. of S_0 must be either of the following two forms: (a) $P_{14}\neq 0$ and other $P_{ij}=0$, (b) $P_{23}\neq 0$ and other $P_{ij}=0$ where $P_{ij}=\alpha_{[i}\beta_{j]}$. Then we can easily obtain (7.2). (ii) By taking standard coordinate system for c. s. of S_0 and using (2.10) we can prove that $ds^{*2}=ds^2/B(r,\ t)$ gives an S_{20}.

Theorem [7.2] *Let S_0 be not conformal to any S_{20}. A necessary and sufficient condition that the S^* obtained by $g_{ij}^*=e^{2v}g_{ij}$ from the S_0 be s. s. again is that v be s. s. with respect to a c. s. of the S_0. (i. e. the $C(v)$ be s. s. c. t.).*

Proof: The sufficiency is evident. Next, let S^* be s. s. and take any standard coordinate system for g_{ij}. Using the first fundamental equation for c. s. (F_1) and (2.2), we can prove that only either of the following two cases is possible[5]: (i) When $P_{14}\neq 0$ and other $P_{ij}=0$. From this we have $\alpha_2=\alpha_3=\beta_2=\beta_3=0$ and further by using (F_2), (F_3)[6] and (2.1) we can prove $v_2=v_3=0$ i. e. the C(v) is s. s. (ii) When $P_{23}\neq 0$ and other $P_{ij}=0$. In the same way we have $\alpha_1=\alpha_4=\beta_1=\beta_4=0$ and from (F_2) we have $e^{2v}=e^{2\bar{v}}/B(r, t)$ where $\bar{v}=\bar{v}(\theta, \phi)$. Then using ($F_1$) and (2.2) we can prove that $d\bar{s}^2$ defined by $\bar{g}_{ij}=e^{-2\bar{v}}g_{ij}^*$ gives an S_{20} contrary to our assumption. Thus the theorem is proved.

From the proof of this theorem we know that when (7.2) holds there exist some C(v)'s which are not s. s. and yet transform S_0 into s. s. S^* again. (The constant in (7.2) is invariant under s. s. c. t. but not necessarily so under other ones). Concerning this we have the theorem:

Theorem [7.3] *The general form of the conformal transformation $g_{ij}^*=e^{2v}g_{ij}$ where both g_{ij}^* and g_{ij} define [B] (i. e. Minkowski type S_0) is given by the following two types: (a) The one whose $v_{ij}=-pg_{ij}/2\neq 0$. In this case the $C(v)$ is s. s. (b) The one whose $v_{ij}=0$. In this case the $C(v)$ is not s. s.*

Proof: Using the coordinate system in which $ds^2=-dx^2-dy^2-dz^2+dt^2$, we have $e^{-v}=-mx^ix_i+n_ix^i+s$ where $x^i=x_i$, and m, n_i, and s are constants satisfying $n_in^i=-4ms$, $(n_i=n^i)$. Then we can prove that when $m\neq 0$ or $m=0$ we have (a) or (b) respectively. (In the case of (a) we must use m-transformation).

In connection with the above we shall add the following theorem.

Theorem [7.4] *The fundamental form of any conformally flat S_0 is reducible to*

$$ds^2 = A\{-(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+\lambda dt^2\}, \tag{7.4}$$

where $A=A(r, t)$ and $\lambda=\{a(t)+r^2b(t)\}^2$. Namely any conformally flat S_0 is given by a suitable choice of $A(r, t)$, $a(t)$ and $b(t)$.

Proof: (2.4) is transformable into isotropic form

$$ds^2 = -A(r, t)(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+C(r, t)\,dt^2, \tag{7.5}$$

using which we can easily obtain (7.4).

From this theorem it must be that by a suitable $C(v)$ (7.4) is transformable into [B]. If we assume that $v=v(r, t)$, however, this is not necessarily the case. This will be seen if we look for the condition to be satisfied by λ in order that (7.4) may give [B].

When $b=0$, by taking $A=$const. or $=(t^2-r^2)^2$ taking $a=1$ by a transformation of t, we have [B][7]. Hence we assume that $b\neq 0$ and by a transformation of t we take $b=1$. Calculating $K_{ij}{}^{lm}=0$ for (7.4) we can show that (7.4) can be [B] when and only when $a=0$ or ± 1 or $(2t+q)^{-2}$ where q is any const., and we can obtain the corresponding $A(r, t)$. Examples of the line element of [B] in these cases are given by

When $a=0$: $$ds^2=-r^4dm^2+dt^2, \quad (dm^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2). \tag{7.6}$$

When $a=1$: $$ds^2=-e^{4tt}(1+r^2)^{-2}dm^2+e^{4tt}dt^2, \tag{7.7}$$

or putting $e^{2tt}=2i\bar{t}$, $$ds^2=4\bar{t}^2(1+r^2)^{-2}dm^2+d\bar{t}^2. \tag{7.8}$$

When $a=-1$: $$ds^2=4t^2(1-r^2)^{-2}dm^2+dt^2.$$

When $a=(2t+q)^{-2}$: $$ds^2=(2t+q)^{-2}(r^2-a)^{-2}\{-dm^2+((r^2+a)^2dt^2\}. \tag{7.10}$$

(7.8) was obtained by the writer from another point of view.[7]

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Notes and References

1) This paper is a continuation of Journ. Math. Soc. Japan **3** (1952), 317; this Journal **16** (1952), 67; this Journal **16** (1952), 291. These are cited as (I), (II) and (III) respectively and the same notations as in these papers are used throughout the present paper.

2) (I) and (II).

3) L. P. Eisenhart, ***Riemannian Geometry***, Princeton (1926), 89.

4) § 5 of (I).

5) Here (α_i, β_i) denotes any pair of c. v. of S^*.

6) § 1 of (I).

7) H. Takeno, This Journal **11** (1941), 224.