THEORY OF THE SPHERICALLY SYMMETRIC SPACE-TIMES. III. CLASS[1)]

By

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§ 1. Introduction.

It is well known that de Sitter type space-time [A][2)] and Einstein type one [C][2)] are expressible as a four dimensional fundamental hyperquadric and a hypercircular cylinder in five dimensional flat space respectively. On the other hand, Schwarzschild type space-time is expressible as a four dimensional subspace of six dimensional flat space.[3)] In other words their classes are one, one and two respectively.

In this paper we intend to determine the classes of all s. s. space-times to which the above three belong as special ones. It was shown, however, by Eiesland that the *class of any s. s. space-time S_0 is at most two,*[4)] hence we have only to determine the condition that an S_0 be of class one. After obtaining this condition in a s. s. coordinate system, we shall express it in invariant form using c. s. of S_0.[5)] Lastly we shall give the concrete form of transformation which transforms an S_0 of class one into a hypersurface of five dimensional flat space.

§ 2. Space-time $S_{\rm I}$.

Line element of an $S_{\rm I}$ is reducible to

$$ds^2 = -A(r, t)\,dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + C(r, t)\,dt^2, \tag{2.1}$$

by a suitable choice of coordinate system. Then using (2.1), we can prove:
Lemma 1.[6)] *A necessary and sufficient condition that an $S_{\rm I}$ be of class one is that A and C satisfy*

$$A \neq 1, \quad 2(\ddot{A} - C'') + (\dot{A}^2 - A'C')/(1-A) + (C'^2 - \dot{A}\dot{C})/C = 0, \tag{2.2}$$

in the coordinate system of (2.1).

Proof: In order that a space-time be of class one, it is necessary and sufficient that there exists a symmetric tensor b_{ij} satisfying

$$K_{ijkl} = e(b_{ik}b_{jl} - b_{il}b_{jk}), \qquad \text{(F)}, \qquad \nabla_k b_{ij} - \nabla_j b_{ik} = 0, \qquad \text{(G)}$$

where $i, j, \cdots = 1, \cdots 4$, and e is plus or minus one and is equal to $a_{\alpha\beta}\eta^\alpha\eta^\beta$. ($a_{\alpha\beta}$, ($\alpha, \beta = 1, \cdots, 5$), is the fundamental tensor of the five dimensional flat space and η^α is the unit normal vector of the four dimensional hypersurface.)[7] If we raise the indices k and l and use (2.1), (F) becomes

$$\begin{cases} e\alpha = b_1^1 b_2^2 - b_1^2 b_2^1, \quad e\beta = b_2^2 b_4^4 - b_2^4 b_4^2, \quad \text{etc. (8 eqs.)} \\ b_1^1 b_2^3 - b_1^3 b_2^1 = 0, \quad b_1^1 b_2^4 - b_1^4 b_2^1 = 0, \quad \text{etc. (13 eqs.)} \end{cases} \qquad \text{(F)}$$

where $\alpha = K_{12}^{\cdot\cdot 12} = K_{13}^{\cdot\cdot 13} = -A'/2A^2r$, $\beta = K_{24}^{\cdot\cdot 24} = K_{34}^{\cdot\cdot 34} = C'/2ACr$, etc.[5] And (G) becomes

$$\partial_2 b_{11} - \partial_1 b_{12} + A' b_{12}/2A + \dot{A} b_{24}/2C - b_{12}/r = 0, \text{ etc. (24 eqs.)} \qquad \text{(G)}$$

for (2.1). From (F) we can easily prove the following:

(i) *When* ($b_{12} \neq 0$, $b_{23} \neq 0$) *or* ($b_{12} \neq 0$, $b_{23} = 0$). S_I must be flat i. e. [B].[2]

(ii) *When* $b_{12} = 0$ *and* $b_{23} \neq 0$, we have $b_{1i} = 0$ and $\alpha = \beta = \gamma = \xi = 0$. Hence we have A=const. and $C = C(t)$. So by a transformation of t, we obtain

$$ds^2 = -A dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + dt^2, \quad (A = \text{const.}), \qquad (2.3)$$

When $A=1$, this gives [B]. When $A \neq 1$ we have (2.2). This S_I is not flat and we can easily prove that (F) and (G) have a solution b_{ij} of the type b_{22}, b_{23}, $b_{33} \neq 0$ and other $b_{ij} = 0$. Hence the class of this S_I is one. In fact, by putting

$$z^1 = r\sin\theta\cos\phi, \; z^2 = r\sin\theta\sin\phi, \; z^3 = r\cos\theta, \; z^4 = \lambda r, \; z^5 = t, \quad (2.4)$$

we have

$$ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 + e_4(dz^4)^2 + (dz^5)^2, \qquad (2.5)$$

where ($\lambda = \sqrt{1-A}$, $e_4 = 1$) or ($\lambda = \sqrt{A-1}$, $e_4 = -1$) according as $1 - A > 0$ or < 0 and the S_I becomes a hypercylinder $-(z^1)^2 - (z^2)^2 - (z^3)^2 + e_4(z^4)^2/(1-A) = 0$ in the flat space (2.5).[8]

(iii) *When* $b_{12} = b_{23} = 0$. Assuming that S_I is not [B], we have $b_{12} = b_{23} = b_{13} = b_{24} = b_{34} = 0$. (iii$_a$) *When* $\beta \neq 0$, *i. e.* $C' \neq 0$: From (F), we can obtain

$$\begin{cases} b_{11} = A'/2\lambda\sqrt{A}, \quad b_{22} = -r\lambda/\sqrt{A}, \quad b_{33} = -r\lambda\sin^2\theta/\sqrt{A}, \\ b_{44} = C'/2\lambda\sqrt{A}, \quad b_{14} = \dot{A}/2\lambda\sqrt{A}, \quad \text{other } b_{ij} = 0, \quad A \neq 1, \end{cases} \qquad (2.6)$$

where $\lambda = \sqrt{e(1-A)} \neq 0$ and e is taken so as λ is real. (Double sign of b_{ij} being omitted.) Using these results, we have (2.2). Conversely if (2.2) holds and $C' \neq 0$, b_{ij} defined by (2.6) satisfies (F) and (G). (iii$_b$) *When* $\beta = 0$

i.e. $C'=0$: By a transformation of t we have $C=1$. Especially when $\gamma=0$ i. e. $\dot{A}=0$, we obtain (2. 6) in which $C'=\dot{A}=0$ as the solution of (F) and (G). Hence this S_{I} is of class one and (2. 2) is satisfied. (If $A'=0$ further, we have the S_{I} defined by (2. 3) and a new solution b_{ij} is obtained.) Similarly when $\gamma\neq0$, i. e. $\dot{A}\neq0$, we can easily obtain (2. 2) whose $C'=0$ as the condition to be satisfied by A in order that the S_{I} be of class one. b_{ij} in this case is also given by (2. 6) putting $C'=0$. Hence the lemma is proved.

The concrete form of b_{ij} is given by (2. 6) to within a sign with a proviso that in S_{I} defined by (2. 3) b_{ij}'s of different types exist.

§ 3. Condition (2. 2) in invariant forms.

By using the formulae concerning $\alpha, \beta, \cdots$ given in (I), (2. 2) can be rewritten as

$$\eta\neq0\,, \quad \xi\eta=\alpha\beta+C\gamma^2/A\,. \tag{3.1}$$

In this section we shall express this condition in invariant forms. For this purpose we shall use the relation

$$\begin{cases} \alpha\beta+C\gamma^2/A=\{4(\overset{4}{\rho})^2+2\overset{4}{\rho}(\overset{2}{\rho}+\overset{3}{\rho})+\overset{2}{\rho}\overset{3}{\rho}\}/16\,, \quad \alpha+\beta=(\overset{2}{\rho}+\overset{3}{\rho}+4\overset{4}{\rho})/4\,, \\ \xi=(\overset{1}{\rho}+\overset{2}{\rho}+\overset{3}{\rho}+2\overset{4}{\rho})/4\,, \quad \eta=\overset{4}{\rho}/2\,, \end{cases} \tag{3.2}$$

which is obtained from the formulae concerning c. s. given in (I). We can easily prove that this relation is invariant under any transformation of r and t.[9] If we substitute (3. 2) into (3. 1), we have

$$\overset{4}{\rho}\neq0\,, \quad 2\overset{1}{\rho}\overset{4}{\rho}=\overset{2}{\rho}\overset{3}{\rho}\,. \tag{3.3}$$

Hence we obtain:

Lemma 2. *A necessary and sufficient condition that an S_1 be of class one is given by (3. 3).*

Since a necessary and sufficient condition that an S_0 be conformally flat is given by $\overset{1}{\rho}=0$, we have:

Corollary 1. *A necessary and sufficient condition that a conformally flat S_{I} be of class one is that it hold either* $(\overset{4}{\rho}\neq0,\ \overset{2}{\rho}=0)$ *or* $(\overset{4}{\rho}\neq0,\ \overset{3}{\rho}=0)$.

Examples: In [A] and [C] it holds that $(\overset{1}{\rho}=\overset{2}{\rho}=\overset{3}{\rho}=0.\ \overset{4}{\rho}=\text{const.}\neq0)$ and $(\overset{1}{\rho}=\overset{2}{\rho}=0,\ \overset{4}{\rho}\neq0)$ respectively, so (3. 3) is satisfied. In general, we can easily prove that in $S(L)$ except [B], (i. e. the space-time used in relativistic cosmology), it holds that $\overset{1}{\rho}=\overset{2}{\rho}=0$ and $\overset{4}{\rho}\neq0$,[2] hence this $S(L)$ is also of class one.[10] For Schwarzschild type $S_{\rm I}$ it holds that $\overset{1}{\rho}=-2\overset{2}{\rho}=-2\overset{3}{\rho}=6\overset{4}{\rho}\neq 0$, hence (3.3) is not satisfied and it is of class two.

There is another way of expressing (2.2) in invariant form. The characteristic equation of the matrix $[K_{A}{}^{\cdot B}]$ in the coordinate system of (2.1) is given by

$$(\lambda-\xi)(\lambda-\eta)\{\lambda^2-(\alpha+\beta)\lambda+\alpha\beta+C\gamma^2/A\}^2=0, \qquad (3.4)$$

where $K_{A}{}^{\cdot B}=K_{ij}{}^{\cdot\cdot lm}$, $(A=(i,j),\ B=(l,m);\ A,B=1,\cdots,6;\ (1,2)=1,\cdots,(3,4)=6)$.[5),11)] The six roots of this equation which are invariant under any coordinate transformation are $\xi,\eta;\ \lambda_1,\lambda_1;\ \lambda_2,\lambda_2$ where λ_1 and λ_2 are roots of $\lambda^2-(\alpha+\beta)\lambda+\alpha\beta+C\gamma^2/A=0$. When $A=1$, (3.4) becomes $\lambda^3(\lambda-\xi)(\lambda-\beta)^2=0$. The second equation of (3.1) is equivalent to $\xi\eta=\lambda_1\lambda_2$. Therefore, *from the standpoint of the characteristic roots i. e. principal invariants of* $[K_{A}{}^{\cdot B}]$, *we can determine the class of given* $S_{\rm I}$. For example, when six roots are $p, q;\ r, r;\ s, s,\ (p, q, r, s\neq)$, the class is one or two according as $pq=rs$ or $pq\neq rs$.

§ 4. Space-time $S_{\rm II}$.

The line element of an $S_{\rm II}$ is reducible to the form

$$ds^2=-A(r,t)\,dr^2-B(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)\,dt^2,\quad (B=\text{const.}>0), \qquad (4.1)$$

by a suitable transformation of coordinates. Since $\eta\equiv K_{23}{}^{\cdot\cdot 23}=-1/B\neq 0$ in this coordinate system, an $S_{\rm II}$ can not be flat and its class must be one or two by virtue of the theorem stated in §1.

Solving (F) and (G) as in the case of $S_{\rm I}$, we have $b_{11}=b_{14}=b_{44}=0$ and $\xi=K_{14}{}^{\cdot\cdot 14}=0$. So by a transformation of (r,t)-space, we have

$$ds^2=-dr^2-B(d\theta^2+\sin^2\theta d\phi^2)+dt^2,\quad (B=\text{const.}>0). \qquad (4.2)$$

Evidently this $S_{\rm II}$ is of class one and becomes a hypersuface $(z^1)^2+(z^2)^2+(z^3)^2=B$ in a flat space $ds^2=-(ds^1)^2-(dz^2)^2-(dz^3)^2-dr^2+dt^2$.[12)] Hence we have

Lemma 3. *A necessary and sufficient condition that an* $S_{\rm II}$ *be of class one is given by* $\xi=0$ *in the coordinate system of (4.1).*

In this coordinate system it holds that $\alpha=\beta=\gamma=0$ and $\eta\neq 0$, so $\xi=0$ coincides with the second equation of (3.1). And further an S_0 is $S_{\rm II}$ when and only when $\overset{2}{\rho}=\overset{3}{\rho}=-2\overset{4}{\rho}=4/B$, $(\neq 0)$, and it holds that $\overset{1}{\rho}=4(\xi-1/B)$.[5)] Hence $\xi=0$ is equivalent to $\overset{1}{\rho}=-\overset{2}{\rho}$, $(\neq 0)$, and also to the second of (3.3). Summarizing the results obtained, we have:

Theorem 1. *A necessary and sufficient condition that an S_0 be of class one is given by (3.3). Especially when S_0 is S_{II} the condition becomes* $\overset{1}{\rho}=-\overset{2}{\rho}$.

Corresponding to the corollary 1 in §3, we easily obtain:

Corollary 2. *Conformally flat S_{II} can not be of class one i. e. it is of class two.*

A characterizing condition of S_{II} from the standpoint of the characteristic roots of $[K_A^{\cdot B}]$ is that four of the six roots are 0 and one of the remaining two is non-vanishing constant (i. e. $-1/B$).[11] Hence as in §3, we have:

Lemma 4. *A necessary and sufficient condition that an S_0 be S_{II} of class one is that five of the six characteristic roots of the matrix $[K_A^{\cdot B}]$ be 0 and the remaining one be non-vanishing constant.*

§5. Concrete form of z^α in general S_0 of class one.

In an S_0 of class one there exist five functions $z^\alpha=f^\alpha(x^i)$ satisfying

$$ds^2 = g_{ij}dx^i dx^j = \sum_\alpha e_\alpha(dz^\alpha)^2\,,\quad (\alpha=1,\cdots,5)\,, \tag{5.1}$$

where the e's are plus or minus one according to the character of the S_0. In this section we shall obtain the concrete forms of these z's by solving

$$\nabla_j\nabla_i z^\alpha = e b_{ij}\eta^\alpha\,, \qquad \text{(L)}, \qquad \nabla_j\eta^\alpha = -b_{ij}g^{im}\nabla_m z^\alpha\,, \qquad \text{(M)}$$

where b_{ij} is the solution of (F) and (G).[13]

In the first place, we shall deal with an S_{I} satisfying (2.2) taking the coordinate system of (2.1). When S_{I} is of the type (2.3), z^α is given by (2.4), so we assume that the S_{I} is not of this type and take b_{ij} given by (2.6). Then (L) and (M) become

$$\partial_{11}z^\alpha-(A'/2A)\,\partial_1 z^\alpha-(\dot{A}/2C)\,\partial_4 z^\alpha=(A'/2\sqrt{A}\,\lambda)\,\eta^\alpha\,,\ \text{etc.}\quad \text{(10 eqs.)}\quad \text{(L)}$$

and $\partial_1\eta^\alpha=(A'/2A\sqrt{A}\,\lambda)\,\partial_1 z^\alpha-(\dot{A}/2C\sqrt{A}\,\lambda)\,\partial_4 z^\alpha$, etc. (4 eqs.) (M)

respectively, where $\lambda=\sqrt{e(1-A)}$ and ∂_{ij} means $\partial^2/\partial x^i\partial x^j$. By solving this we know that such z^α are given by

$$\begin{aligned} &z^1=r\sin\theta\cos\phi\,,\quad z^2=r\sin\theta\sin\phi\,,\quad z^3=r\cos\theta\,,\\ &z^4=\tau(r,t)\,,\quad z^5=\mu(r,t)\,, \end{aligned} \tag{5.2}$$

where τ and μ are determined in the following way:

(i) *When $\dot{A}$ or C' is not* 0. Let $\omega(r,t)$ be any non-constant solution of $C'\partial_1 f-\dot{A}\partial_4 f=0$, then the general solution of this equation is given by

$F(\omega)$ where F is an arbitrary function. If we put $\nu=\dot{A}/(2\lambda\omega'\sqrt{C})$ $=C'/(2\lambda\dot{\omega}\sqrt{C})$ and use (2.2) we obtain $C'\partial_1\nu-\dot{A}\partial_4\nu=0$ so that $\nu=\nu(\omega)$. Then we consider the differential equation

$$\frac{d^2F}{d\omega^2}-\frac{\nu'}{\nu}\frac{dF}{d\omega}+e\nu^2F=0, \quad \left(\nu'=\frac{d\nu}{d\omega}\right). \tag{5.3}$$

The solution $F(\omega)$ contains two arbitrary constants and we can take two solutions F and $\bar{F}$ of (5.3) satisfying

$$F^2+e\bar{F}^2=1, \quad eFF'+\bar{F}\bar{F}'=0, \quad eF'^2+\bar{F}'^2=\nu^2, \quad (F'=dF/d\omega), \tag{5.4}$$

initially. If we put $F'=\nu G$ and $\bar{F}'=\nu\bar{G}$, we have $G'=-e\nu F$ and $\bar{G}'=-e\nu\bar{F}$. Hence (5.4) becomes

$$F^2+e\bar{F}^2=1, \quad eFG+\bar{F}\bar{G}=0, \quad eG^2+\bar{G}^2=1. \tag{5.5}$$

By repeated differentiations of the left hand sides of (5.5), we know that there exist τ and μ satisfying

$$\partial_1\tau=\lambda F, \quad \partial_4\tau=\sqrt{C}\,G; \quad \partial_1\mu=\lambda\bar{F}, \quad \partial_4\mu=\sqrt{C}\,\bar{G}. \tag{5.6}$$

We have only to take these τ and μ.

Then from (5.6) we have

$$-(dz^1)^2-(dz^2)^2-(dz^3)^2+e(dz^4)^2+(dz^5)^2=ds^2 \text{ of } (2.1), \tag{5.7}$$

and corresponding η^α is given by

$$\eta^\alpha=[-(\lambda/\sqrt{A})\sin\theta\cos\phi, \quad -(\,,,\,)\sin\theta\sin\phi, \quad -(\,,,\,)\cos\theta, \quad -F/\sqrt{A}, \; -\bar{F}/\sqrt{A}\,], \tag{5.8}$$

and so

$$-(\eta^1)^2-(\eta^2)^2-(\eta^3)^2+e(\eta^4)^2+(\eta^5)^2=e. \tag{5.9}$$

(ii) *When* $\dot{A}=C'=0$. By a transformation of t we take the coordinate system in which $C=1$. From (L) and (M), we have

$$\tau=\int\sqrt{e(1-A(r))}\,dr, \quad \mu=t, \tag{5.10}$$

$$\eta^\alpha=[-(\lambda/\sqrt{A})\sin\theta\cos\phi, \quad -(\,,,\,)\sin\theta\sin\phi, \quad -(\,,,\,)\cos\theta, \quad -1/\sqrt{A}, \; 0], \tag{5.11}$$

and (5.7) and (5.9) also hold.

In both cases if we express r^2 as a function of z^4 and z^5 i.e. $r^2=\psi(z^4, z^5)$, then the S_{I} is expressible as a hypersurface $(z^1)^2+(z^2)^2+(z^3)^2=\psi(z^4, z^5)$ in flat space whose ds^2 is given by the left hand side of (5.7).

Next we shall deal with an S_{II} satisfying $\xi=0$ in the coordinate system of (4.2). Then, evidently, the solution z^α of (L) and (M) is given by

$$z^1=\sqrt{B}\sin\theta\cos\phi\,,\quad z^2=\sqrt{B}\sin\theta\sin\phi\,,\quad z^3=\sqrt{B}\cos\theta\,,$$
$$z^4=r\,,\quad z^5=t\,, \tag{5.12}$$

b_{ij} being omitted, and similar results as in the case of S_I are obtained.

As for the case of S_0 of class two we can easily show that the concrete form of z^α, ($\alpha=1,\cdots,6$), in the case of S_I given by (2.1), is given by

$$z^1=r\sin\theta\cos\phi\,,\quad z^2=r\sin\theta\sin\phi\,,\quad z^3=r\cos\theta\,,$$
$$z^4=z^4(r,t)\,,\quad z^5=z^5(r,t)\,,\quad z^6=z^6(r,t)\,, \tag{5.13}$$

where z^4, z^5 and z^6 are solutions of

$$\sum_\rho e_\rho\left(\frac{\partial z^\rho}{\partial r}\right)^2=1-A\,,\qquad \sum_\rho e_\rho\frac{\partial z^\rho}{\partial r}\frac{\partial z^\rho}{\partial t}=0\,,$$
$$\sum_\rho e_\rho\left(\frac{\partial z^\rho}{\partial t}\right)^2=C\,,\quad (\rho=4,5,6)\,, \tag{5.14}$$

and $e_\rho=\pm1$ are taken so as z's become real. Similarly we can easily obtain z's for S_{II} of class two.

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Notes and References

1) This paper is a continuation of Journ. Math. Soc. Japan **3** (1952), 317; Journ. Sci. Hiroshima Univ., **16** (1952), 67. They are cited as (I) and (II) respectively and the same notations as in those papers are used throughout the present paper.

2) See (II).

3) L. P. Eisenhart, ***Riemannian Geometry***, Princeton (1926), 189. See also next 4).

4) J. Eiesland, Trans. Amer. Math. Soc., **27** (1925), 213. Although this theorem is true certainly as is seen from his proof, his argument concerning S_{II} seems to be somewhat insufficient. See also §4 of the present paper.

5) See (I).

6) This theorem is also proved by Eiesland, loc. cit., 240. But his conclusion is somewhat insufficient i. e. the first condition $A\neq1$(i e. $D=(23,23)/\sin^2\theta\neq0$ by his notations) is missed.

7) L. P. Eisenhart, loc. cit., 149.

8) In this space-time rank of the matrix $[b_{ij}]$ is two and only two of its principal radii of normal curvature in five dimensional flat space are finite. And the solution b_{ij} of (F) and (G) is not determined uniquely. The transformation (2.4) corresponds to the b_{ij} whose $b_{22}=b_{33}=\sqrt{e(1-A)}/r\sqrt{A}$ and other $b_{ij}=0$ where e is taken so as $e(1-A)>0$. Transformations in case of $b_{23}\neq0$ are somewhat complicated.

9) Further, if we express α, β and γ by ρ's, we have

$$\begin{cases} \alpha = \{2\overset{4}{\rho} + (\overset{2}{\rho}\sin^2\omega + \overset{3}{\rho}\cos^2\omega)\}/4, \quad \beta = \{2\overset{4}{\rho} + (\overset{2}{\rho}{}^2\cos^2\omega + \overset{3}{\rho}\sin^2\omega)\}/4, \\ \gamma = \frac{1}{4} i\sqrt{A/C}\,(\overset{2}{\rho} - \overset{3}{\rho})\sin\omega\cos\omega, \end{cases} \tag{N. 1}$$

where ω is a parameter. This relation is not invariant under transformation of r and t. If we use (3.1) and (N.1), scalars made from g_{ij} and K_{ijlm} by algebraic processes are expressible in terms of ρ's e. g.,

$$K = -4(\alpha+\beta) - 2(\xi+\eta) = -\tfrac{1}{2}\{\overset{1}{\rho} + 3(\overset{2}{\rho} + \overset{3}{\rho}) + 8\overset{4}{\rho}\},$$

$$\varepsilon^{ijlm}\varepsilon_{pqrs} K_{ij}^{\cdot\cdot pq} K_{lm}^{\cdot\cdot rs} = 32\{2(\alpha\beta + C\gamma^2/A) + \xi\eta\} = 4\{6(\overset{4}{\rho})^2 + 3\overset{4}{\rho}(\overset{2}{\rho} + \overset{3}{\rho}) + \overset{1}{\rho}\overset{4}{\rho} + \overset{2}{\rho}\overset{3}{\rho}\}, \text{ etc.}$$

10) R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, Oxford (1934), 370.

11) H. Takeno, This Journal, **12** (1942), 125.

12) Using this coordinate system we can obtain

$$b_{22} = \sqrt{B}, \quad b_{33} = \sqrt{B}\sin^2\theta, \quad \text{other } b_{ij} = 0,$$

as a solution of (F) and (G). Another solutions of different types may exist. The rank of $[b_{ij}]$ is two as in the case of (2. 3).

13) L. P. Eisenhart, loc. cit., 197.