ON AXIOM OF BETWEENNESS

By

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Introduction

The system A, in which a binary relation (xy) (or ρ) satisfying $P(\equiv(P1, P2, P3))$ is defined, is called a partially ordered set and is written by $A[\rho, P]$. And the system A, in which a ternary relation (xyz) (or τ) satisfying $Q\equiv(Q_1Q_2\ldots\ldots)$ is defined in A, is written by $A[\tau, Q]$. There is a problem concerning that $A[\rho, P]$ may be characterized by $A[\tau, Q]$, that is, the condition for Q in order that a binary relation ρ may be characterized by a ternary relation τ.

In this paper we shall investigate the case where the ternary relation τ becomes a betweenness. So our problem is to consider about the condition for Q, in order that a binary relation ρ (or order) satisfying P in A may be introduced from a subset $R[A, Q]$ satisfying Q in triple product space $A\times A\times A$, such that

$$\{xyz\}\in R[A, Q] \rightleftarrows x\rho y\rho z \quad \text{or} \quad z\rho y\rho x\,.$$

In other words, we purpose to investigate a complete system of axioms which characterizes "*betweenness*" in a partially ordered set A. G. Birkhoff[1], E. Pitcher and M. F. Smiley[2] have discussed about the necessary conditions for betweenness.

In chapter I, we consider about the closed betweenness[3] of a partially ordered set A. In §1, we have necessary conditions B1, B2, B3, B4, B5, B6 and B7, where B3$\equiv$(1), B4$\equiv$(2), B5$\equiv$(4) in the necessary conditions (1), (2), (3), (4) and (5) proposed by G. Birkhoff, and in §2 we inquire into the independency of them. And in §3, 4, 5, 6, 7 and 8, by considering the decomposition of the set A, by means of a collection of subsets of A: S and by the distance introduced from S, we discuss their sufficiency for closed betweenness of set A and investigate the uniqueness concerning their ordering.

1), 3) G. Birkhoff: Lattice Theory (1948).

2) E. Pitcher and M. F. Smiley: Transitivities of betweenness: Trans. Am. Math. Soc. 52 (1942).

In § 9 other complete systems of conditions for betweenness of set A are discussed and we deal with the betweenness on some set in stead of the betweenness on given set A. Further, in § 10, we treat the relation of G. Birkhoff's conditions and conditions B1, B2, B3, B4, B5, B6 and B7.

In chapter II, we treat the closed betweenness of the following five special partially ordered sets: [1] the case having radius r[1] [2] the case where $A[\rho]$ has one element a_0 such that $a_0 \rho x$ or $x \rho a_0$ for all $x \in A$ [3] the case having one extreme element [4] the case where $A[\rho]$ has two extreme elements 0 and 1; [5] the case where $A[\rho]$ satisfies the condition: $x \rho y$ or $y \rho x$ for all x and y, i.e. simply ordered set.

In chapter III we deal with the open-betweenness by reducing it to closed-betweenness, and in chapter IV the betweenness for Quasi-partially ordered set is investigated from the view of factorization.

Before we proceed with the discussion, we write the notations which are use in this paper.

A	:	a set of elements.
a, b, c etc.	:	distinct elements of A.
x, y, z etc.	:	elements which may be equal.
$\{xyz\}$	:	an element of triple product space $A \times A \times A$.
R	:	a set of $\{xyz\}$.
$R[\mathfrak{B}]$	:	R which satisfies a system $\mathfrak{B}$ of conditions.
$\|xyz\|$	:	an element of $R[\mathfrak{B}]$.
$\|\|x, y, z\|\|$	:	an element $\|xyz\|$ or $\|yzx\|$ or $\|zxy\|$.
(xyz)	:	betweenness in a partially ordered set.
$((xyz))$	:	an element (xyz) or (yzx) or (zxy).
$\|x_1 \cdots\cdots x_l\|$	:	all elements $\|x_i x_j x_k\|$ $(\|1 \leqq i \leqq j \leqq k \leqq l)$.
$\overline{ab}$	:	a distance for a and b (defined in § 5).
inner point of path of a and b	:	element $a^i (i=1, 2, \dots, l)$ in a path $a, a^1, \dots, a^l, b$.
$\|abc\|, ((abc))$ etc.	:	$\{abc\} \in R$, $((abc)) \in R$ etc.
$\|A^i A^j A^k\|$	:	There exists at least a set of elements $a^i \in A^i$, $a^j \in A^j$ $a^k \in A^k$ such that $\{a^i a^j a^k\} \in R$.
C (or) D	:	one and only one of C and D.
$C \longrightarrow D$	:	if condition C is satisfied, then condition D is satisfied; or if $C \in R$, then $D \in R$.

1) Cf. Chap. II, p. 212.

$C \Longrightarrow D$: since condition C is satisfied, so condition D is satisfied; or since $C \in R$, so $D \in R$.

$C \overset{(1)}{\Longrightarrow} D$: since condition C is satisfied, so by (1) condition D is satisfied.

$C \cdot D$: C and D.

C and $D \rightarrow E$ and F: the whole of E and F follows from the whole of C and D.

proposition ($\pm$ dual): the proposition, which is obtained by exchanging $^{+}A^{s}$ and $^{-}A^{s}$ in this proposition, is also satisfied.

partially ordered set: a system A in which a binary relation $x \geqq y$ is defined, which satisfies

P1: For all x, $x \geqq x$.

P2: If $x \geqq y$ and $y \geqq x$, then $x = y$.

P3: If $x \geqq y$ and $y \geqq z$, then $x \geqq z$.

We shall here prove that the above conditions P1, P2 and P3 may be written in the following form:

P*1: $x = x$

P*2: $x > y$ and $y > x$ are not compatible.

P*3: If $x > y$ and $y > z$, then $x > z$.

In a system $A[\rho]$ which satisfies P1, P2 and P3, we write $x > y$ when and only when $x \geqq y$ and $x \neq y$. Then in $A[\rho]$ a relation $>$ is defined. This relation $x > y$ satisfies the following conditions.

P*2: $x > y$ and $y > x$ are not compatible.

P*3: If $x > y$ and $y > z$, then $x > z$.

Proof of P*2: If $x > y$ and $y > x$, then by the above definition $x \geqq y$ and $y \geqq x$, hence by P2 $x = y$. And this contradicts the definition $x > y$. Therefore $x > y$ and $y > x$ are not compatible.

Proof of P*3: Suppose $x > y$ and $y > z$.

$$x > y \text{ and } y > z \overset{\text{def.}}{\Longrightarrow} x \geqq y \text{ and } y \geqq z \overset{\text{P3}}{\Longrightarrow} x \geqq z \qquad \text{.................(i)}$$

And if $x = z$:

$$x > y \text{ and } y > z \Longrightarrow x > y \text{ and } y > x. \qquad \text{.................(ii)}$$

This contradicts P*2, hence $x \neq z$.

Therefore, by (i) and (ii), we have $x > z$.

In a system $A[\rho^*]$ which satisfies P*1, P*2 and P*3, we define $x \geqq y$ when and only when two elements x and y satisfy $x > y$ or $x = y$.

Then, this relation $x \geqq y$ satisfies P1, P2 and P3.

Proof of P1: By P*1, $x=x \Rightarrow x \geqq x$.

Proof of P2: Suppose $x \geqq y$ and $y \geqq x$.

Case 1. $x=y$: Clearly P2 is satisfied.

Case 2. $x \neq y$: $x \geqq y$ $x \neq y \Rightarrow x > y$; $y \geqq x$ $x \neq y \Rightarrow y > x$.

And, by P*2 $x > y$ and $y > x$ are not compatible, so this case may not occur.

Therefore, P2 is satisfied.

Proof of P3: Suppose $x \geqq y$ and $y \geqq z$.

Case 1. $x=y$: $y \geqq z \Rightarrow x \geqq z$.

Hence P3 is satisfied in this case.

Case 2. $x \neq y \cdot y=z$: $x \geqq y \cdot y=z \Rightarrow x \geqq z$

Hence P3 is satisfied in this case.

Case 3. $x \neq y \cdot y \neq z$: $x \geqq y \cdot x \neq y \Rightarrow x > y$; $y \geqq z \cdot y \neq z \Rightarrow y > z$

$$x > y \cdot y > z \overset{\text{P*3}}{\Longrightarrow} x > z \rightarrow x \geqq z$$

Hence P3 is satisfied in this case.

Therefore, in all cases P3 is satisfied.

According to the above facts, we may use a set of conditions P*1, P*2 and P*3 instead of a set of conditions P1, P2 and P3.

Chapter I

On betweenness in a partially ordered set

§ 1. Necessary conditions for betweenness.

In a partially ordered set A, if either $x \geqq y \geqq z$ or $x \leqq y \leqq z$ for x, y and z, y is said to be between x and z and this relation "betweenness"[1] is represented by (xyz). In this section, we shall investigate the necessary conditions for betweenness of a partially ordered set A.

Theorem 1. *Following seven conditions must be satisfied for betweenness of set* A[2].

B1. (aaa) *for all* a.

B2. (abx) *implies* (aab).

B3. (axb) *implies* (bxa).

B4. (axb) *and* (abx) *imply* $x=b$.

B5. (axb), (xby) *and* $x \neq b$ *imply* (aby).

1) In chapter I and II, we use "betweenness" for closed betweenness.

2) In the conditions B1, B2, B3 B4, B5, B6 and B7 in theorem 1, different letters a, b ...... represent different elements, but x, y...... are not so.

B6. $(a_1a_2a_2), (a_2a_3a_3)$ ......, *and* $(a_{2n+1}a_1a_1)$ *imply* $(a_ia_{i+1}a_{i+2})$, *for at least one* i $(1 \leq i \leq 2n+1)$, $n=1, 2, \dots$ *and being* $a_{2n+1+i}=a_i$, $a_i \neq a_{i+1}$ $(i=1, 2, \dots, 2n+1)$.

B7. (abc) *and* (bdd) *imply* (abd) Ⓞⓡ (dbc)[1].

Proof.

Proof of B1: By P1, $a \geq a$ for all a $(a \in A) \Rightarrow a \geq a \geq a \Rightarrow (aaa)$. Therefore (aaa) for all a.

Proof of B2: $(abx) \overset{\text{def.}}{\Longrightarrow} a > b \geq x$ or $a < b \leq x$.
By P1 $a \geq a \Rightarrow a \geq a > b$ or $a \leq a < b \Rightarrow (aab)$.

Therefore (abx) implies (aab).

B3 is clear by the definition of (axb).

Proof of B4: In a partially ordered set, the following conditions are clearly satisfied.

P**2 $x \geq y$ and $y > x$ are not compatible.

P**3 If $x \geq y$ and $y > z$, then $x > z$.

P***3 If $x > y$ and $y \geq z$, then $x > z$.

If (axb) and (abx), then only one case among the following four may occur.

(1) $a \leq x \leq b$ and $a < b \leq x$; (2) $a \leq x \leq b$ and $a > b \geq x$
(3) $a \geq x \geq b$ and $a < b \leq x$; (4) $a \geq x \geq b$ and $a > b \geq x$.

By means of P**2 and P3, the cases (2) and (3) may not occur, and we can easily see $x=b$ for the cases (1) and (4), so we have (axb) and (abx) imply $x=b$.

Proof of B5: If (axb), (xby) and $x \neq b$, then only one case among the following four may occur.

(1) $a \leq x < b$ and $x < b \leq y$; (2) $a \leq x < b$ and $x > b \geq y$
(3) $a \geq x > b$ and $x < b \leq y$; (4) $a \geq x > b$ and $x > b \geq y$

By means of P*2, the cases (2) and (3) may not occur. In the case (1), by P***3 we have $a < b \leq y$. $a < b \leq y \Rightarrow a \leq b \leq y \Rightarrow (aby)$ and in the case (4), by P**3 we have $a > b \geq y$. $a > b \geq y \Rightarrow a \geq b \geq y \Rightarrow (aby)$.

Therefore (axb), (xby) and $x \neq b$ imply (aby).

Proof of B6: Since the number of betweenness in assumption of B6 is odd, it is impossible that the direction of the order for x_j and x_{j+1} is different from the direction of the order for x_{j+1} and x_{j+2} for all $j (j=1, 2, \cdots 2n+1)$.

For, let us assume that the direction of the order for x_j and x_{j+1} is

1) In the condition B7 (abd) Ⓞⓡ (dbc) means one and only one relation of (abd) and (dbc).

different from the direction of the order for x_{j+1} and x_{j+2} for all $j(j=1, 2, \cdots 2n+1)$. Then, since in the sequence of betweenness

$$(a_1a_2a_2)\ (a_2a_3a_3)\ \ldots\ldots\ (a_{2n+1}a_1a_1)\ (a_1a_2a_2)$$

the number of betweenness from the first $(a_1a_2a_2)$ to $(a_{2n+1}a_1a_1)$ is odd, the direction of the order for a_1 and a_2 is the same as the direction of the order for a_{2n+1} and a_1. On the other hand, since $(a_{2n+1}a_1a_2)$ is adjacent to $(a_1a_2a_2)$, so we have by the assumption that the direction of the order for a_{2n+1} and a_1 is different from the direction of the order for a_1 and a_2. These two facts contradict P*2.

Therefore, for at least one i, the direction of the order for a_i and a_{i+1} is the same as that of the order for a_{i+1} and a_{i+2}. Hence, $(a_ia_{i+1}a_{i+2})$ for at least one i.

Proof of B7: If (abc) and (bdd), then only one case among the following four may occur.

(1) $a<b<c$ and $b<d$; (2) $a<b<c$ and $b>d$

(3) $a>b>c$ and $b<d$; (4) $a>b>c$ and $b>d$

In the case (1), we have $a<b<d$, so (abd). In the case (2), we have $d<b<c$, so (dbc). In the cases (3) and (4), by conversing the order in the proof of the cases (2) and (1), we have (dbc) and (abd) respectively. And we easily see that one and only one of (abd) and (dbc) occurs.

Therefore, (abc) and (bdd) imply (abd) Ⓞⓡ (dbc).

§2. Independency of conditions.

Now, we consider a set A and R which is some collection of elements of triple product space $A\times A\times A$. And in this section, we shall investigate the independency of the conditions BI-B7 in the following system $\mathfrak{B}$:

$\mathfrak{B}$:

B1. $|aaa|$ for all a $(a\in A)$.

B2. $|abx|\to|aab|$.

B3. $|axb|\to|bxa|$

B4. $|axb|\,|abx|\to x=b$.

B5. $|axb|\,|xby|\,x\neq b\to|aby|$.

B6. $|a_1a_2a_2|\,|a_2a_3a_3|\ldots\ldots|a_{2n+1}a_1a_1|\to|a_ia_{i+1}a_{i+2}|$.
for at least one $i\,(1\leq i\leq 2n+1)$, $n=1, 2, \ldots$, $a_{2n+1+i}=a_i$, and $a_i\neq a_{i+1}$ $(i=1, 2, \ldots, 2n+1)$.

B7. $|abc|\cdot|bdd|\to|abd|$ Ⓞⓡ $|dbc|$.

where $|xyz|$ means $\{xyz\} \in R$, and $|xyz| \rightarrow |x'y'z'|$ means that $\{xyz\} \in R \rightarrow \{x'y'z'\} \in R$[1].

We shall have the following result:

Theorem 2. *The seven conditions in the system* $\mathfrak{B}$ *are independent of each other.*

Proof. We shall prove this theorem by the seven examples of R, each of which satisfies six of the seven conditions B1-B7, except one.

Example	set A	system R	
1.	a, b	$\{aab\}$ $\{abb\}$ $\{baa\}$ $\{bba\}$	only B1 is not satisfied.
2.	a, b	$\{aab\}$ $\{baa\}$ $\{aaa\}$ $\{bbb\}$	only B2 "
3.	a, b	$\{aab\}$ $\{aaa\}$ $\{bbb\}$	only B3 "
4.	a, b	$\{aba\}$ $\{aab\}$ $\{abb\}$ $\{baa\}$ $\{bba\}$ $\{aaa\}$ $\{bbb\}$	only B4 "
5.	a, b, c	$\{abc\}$ $\{cba\}$ $\{aab\}$ $\{abb\}$ $\{baa\}$ $\{bba\}$ $\{bcc\}$ $\{bbc\}$ $\{cbb\}$ $\{ccb\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$	only B5 "
6.	a, b, c	$\{aab\}$ $\{abb\}$ $\{baa\}$ $\{bba\}$ $\{bbc\}$ $\{bcc\}$ $\{cbb\}$ $\{ccb\}$ $\{aac\}$ $\{acc\}$ $\{caa\}$ $\{cca\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$	only B6 "
7.	a, b c, d	$\{abc\}$ $\{cba\}$ $\{aab\}$ $\{abb\}$ $\{baa\}$ $\{bba\}$ $\{bbc\}$ $\{bcc\}$ $\{cbb\}$ $\{ccb\}$ $\{aac\}$ $\{acc\}$ $\{caa\}$ $\{cca\}$ $\{bbd\}$ $\{bdd\}$ $\{dbb\}$ $\{ddb\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$ $\{ddd\}$	only B7 "

Remark:

Moreover, the condition B6: $(a_1a_2a_2)\ (a_2a_3a_3)\ldots\ldots(a_{2n+1}a_1a_1) \rightarrow (a_ia_{i+1}a_{i+2})$ is independent as for index n.

Proof. Now we decompose the condition B6 in the following two parts.

$B6_{n_1}$; $(a_1a_2a_2)\ldots\ldots(a_{2n_1+1}a_1a_1) \rightarrow (a_{i_1}a_{i_1+1}a_{i_1+2})$ (n_1; fixed) $(1 \leq i_1 \leq 2n_1+1)$

$B6_{n_2}$; $(a_1'a_2'a_2')\ldots(a'_{2n_2+1}a_1'a_1') \rightarrow (a'_{i_2}a'_{i_2+1}a'_{i_2+2})$ (n_2; variated) $(1 \leq i_2 \leq 2n_2+1)$

$$\begin{pmatrix} n_2 \neq n_1 \\ n_2 = 1, 2, \ldots n_1-1, n_1+1, \ldots \end{pmatrix}$$

And we consider the following system as R:

Example 8.

$\{a_1a_1a_2\}$ $\{a_1a_2a_2\}$ $\{a_2a_1a_1\}$ $\{a_2a_2a_1\}$, $\{a_2a_2a_3\}$ $\{a_2a_3a_3\}$ $\{a_3a_2a_2\}$ $\{a_3a_3a_2\}$, ...

...... $\{a_{2n_1+1}a_{2n_1+1}a_1\}$ $\{a_{2n_1+1}a_1a_1\}$ $\{a_1a_{2n_1+1}a_{2n_1+1}\}$ $\{a_1a_1a_{2n_1+1}\}$, $\{a_1a_1a_1\}$ $\{a_2a_2a_2\}$ ...

...... $\{a_{2n_1+1}a_{2n_1+1}a_{2n_1+1}\}$.

Then this system R satisfies the conditions B1, B2, B3, B4, B5, $B6_{n_2}$ and

1) $\{xyz\}$ means a element of triple product space $A \times A \times A$.

B7, but does not satisfy condition $B6_{n_1}$. Therefore, the condition $B6_{n_1}$ is independent of the other conditions B1, B2, B3, B4, B5, $B6_{n_2}$ and B7. So we have the assertion.

Furthermore, we see that this set of conditions B1, B2, B3, B4, B5, B6 and B7 does not decompose in two subsets.

For; there exists the following example which satisfies all conditions simultaneously and in which all assumptions in B1, B2, B3, B4, B5, B6 and B7 occur really[1)].

Example 9.

$|a_1a_2^1a_2^1|\ |a_2^1a_3^1a_3^1|\ |a_3^1a_1a_1|,\ |a_1a_2^1a_3^1|,$

$|a_1a_1a_2^1|\ |a_2^1a_2^1a_1|\ |a_2^1a_1a_1|,\ |a_2^1a_2^1a_3^1|\ |a_3^1a_3^1a_2^1|\ |a_3^1a_2^1a_2^1|,\ |a_3^1a_3^1a_1|\ |a_1a_1a_3^1|$

$|a_1a_3^1a_3^1|,\ |a_3^1a_2^1a_1|$

$|a_1a_1a_1|\ |a_2^1a_2^1a_2^1|\ |a_3^1a_3^1a_3^1|,$

....

....

$|a_1a_2^na_2^n|\ |a_2^na_3^na_3^n|\ \ldots\ldots\ |a_{2n+1}^na_1a_1|,\ |a_1a_2^na_3^n|$

$|a_1a_1a_2^n|\ |a_2^na_2^na_1|\ |a_2^na_1a_1|,\ |a_2^na_2^na_3^n|\ |a_3^na_3^na_2^n|\ |a_3^na_2^na_2^n|,\ \ldots\ldots$

$|a_{2n+1}^na_{2n+1}^na_1|\ |a_1a_1a_{2n+1}^n|\ |a_1a_{2n+1}^na_{2n+1}^n|,\ |a_3^na_2^na_1^n|,$

$|a_2^na_2^na_2^n|\ |a_3^na_3^na_3^n|\ \ldots\ldots\ |a_{2n+1}^na_{2n+1}^na_{2n+1}^n|,$

....

....

$|a_2^1a_4^1a_4^1|\ |a_2^1a_2^1a_4^1|\ |a_4^1a_4^1a_2^1|\ |a_4^1a_2^1a_2^1|,\ |a_1a_2^1a_4^1|\ |a_4^1a_2^1a_1|,$

$|a_1a_4^1a_4^1|\ |a_1a_1a_4^1|\ |a_4^1a_4^1a_1|\ |a_4^1a_1a_1|,\ |a_4^1a_4^1a_4^1|.$

§ 3. Properties of $\boldsymbol{R}(\mathfrak{B})$.

In the following sections, we shall make it clear that the seven conditions, which are given in the previous section, are sufficient for betweenness of set A.

When R, which is some collection of elements of triple product space $A\times A\times A$, satisfies the system $\mathfrak{B}$, R is said to be "closed" as for the system $\mathfrak{B}$ and represented by $R=R(\mathfrak{B})$. In this section, we shall investigate the properties of $R(\mathfrak{B})$.

Lemma 3. 1. $|aya|\rightarrow y=a$.

Proof. $|aya|\xrightarrow{B2}|aay|$; $|aya|\ |aay|\overset{B4}{\Longrightarrow} y=a$.

Therefore, $|aya|\rightarrow y=a$.

1) $|xyz|$ means a element of R which satisfies the system $\mathfrak{B}$.

Lemma 3. 2. $\| x, y, z \| \rightarrow |xyy|$.

Proof. In the case where $x=y$, this Lemma is clear from B1 and Lemma 3.1, so we prove the case where $x \neq y$. $\|x,y,z\| \rightarrow |xyz|$ or $|yzx|$ or $|zxy|$.
$|xyz| \xrightarrow{B2} |xxy| \xRightarrow{B3} |yxx| \xRightarrow{B2} |yyx| \xRightarrow{B3} |xyy|$.
For $x=z$, $|yzx| \longrightarrow |yyx| \xRightarrow{B2} |yyx| \xRightarrow{B3} |xyy|$, and for $x \neq z$
$|yzx| \xrightarrow{B3} |xzy| \xRightarrow{B2} |xxz| \xRightarrow{B3} |zxx|$; $|yzx|\ |zxx| \xRightarrow{B5} |yxx| \xRightarrow{B2} |yyx| \xRightarrow{B3} |xyy|$.
$|zxy| \xrightarrow{B3} |yxz| \xRightarrow{B2} |yyx| \xRightarrow{B3} |xyy|$.
Therefore, $\| x,y,z \| \rightarrow |xyy|$.

From Lemma 3.1 and 3.2, we have:

Lemma 3. 3. $\| x,y,y \| \equiv |xyy|$. $\| x,y,z \| \rightarrow \| x,y,y \|$.

Lemma 3. 4. $|abc| \cdot |bcd| \rightarrow |abd|$.

Proof. $|abc| \xrightarrow{B3} |cba|$; $|bcd| \xrightarrow{B3} |dcb|$.
$|dcb| \cdot |cba| \xRightarrow{B5} |dba| \xRightarrow{B3} |abd|$.

Therefore, $|abc|$ and $|bcd| \rightarrow |abd|$.

Lemma 3. 5. $|abc| \cdot |bcd| \rightarrow |abcd|$.

Proof. $|abc| \cdot |bcd| \xrightarrow{B5} |acd|$; $|abc| \cdot |bcd| \xrightarrow{\text{lemma 3.4}} |abd|$.

Furthermore, $|abc|$ and $|bcd| \in R[\mathfrak{B}]$ by the assumption of this lemma. Therefore, by the definition of $|xyzu|$, we haye $|abcd|$.

Lemma 3. 6. $|abc| \cdot |acd| \rightarrow |bcd|$.

Proof. $|abc| \xrightarrow{\text{lemma 3.2}} |cbb|$.
$|acd| \cdot |cbb| \xRightarrow{B7} |bcd|$ (or) $|acb|$.

If $|acb|$, then $|abc| \cdot |acb| \xRightarrow{B4} b=c$. This contradicts the assumption: $b \neq c$. Hence, $|acb|$ may not occur, and so $|bcd|$.
Therefore, $|abc| \cdot |acd| \rightarrow |bcd|$.
Note: This lemma follows from the conditions B2, B3, B4 and B7.

Lemma 3. 7. $|abc| \cdot |acd| \rightarrow |abcd|$.

PROOF. $|abc| \cdot |acd| \xrightarrow{\text{lemma 3.6}} |bcd|$; $|abc| \cdot |bcd| \xrightarrow{\text{lemma 3.4}} |abd|$.

Furthermore, $|abc|$ and $|acd| \in R[\mathfrak{B}]$ by the assumption of this lemma. Therefore, by the definition of $|xyzu|$, we have $|abcd|$.

Lemma 3. 8. $|abd| \cdot |bcd| \rightarrow |abc|$.

Proof. $|abd| \xrightarrow{B3} |dba|$; $|bcd| \xrightarrow{B3} |dcb|$.
$|dcb| \cdot |dba| \xRightarrow{\text{lemma 3.6}} |cba| \xRightarrow{B3} |abc|$.

Therefore, $|abd| \cdot |bcd| \rightarrow |abc|$.

Lemma 3. 9. $|abd| \cdot |bcd| \rightarrow |abcd|$.

Proof. $|abd| \cdot |bcd| \xrightarrow{\text{lemma 3.8}} |abc|$; $|abc| \cdot |bcd| \xRightarrow{B5} |acd|$.

Furthermore, $|abd|$ and $|bcd| \in R[\mathfrak{B}]$ the assumption of this lemma. Therefore, by the definition of $|xyzu|$, we have $|abcd|$.

Lemma 3. 10. $|x_1ax_2|\cdot|x_2ax_3|\cdots\cdots|x_{2n+1}ax_1|\rightarrow x_i=a.$

(for all $n=1,2,\dots,$ and at least one i $(1\leqq i\leqq 2n+1)$).

Proof. If $|x_1ax_2|\cdot|x_2ax_3|\cdots\cdots|x_{2n+1}ax_1|$, then by lemma 3.2 we have $|x_1x_2x_2|,\ |x_2x_3x_3|,\ \cdots\cdots,\ |x_{2n+1}x_1x_1|.$

$|x_1x_2x_2|\cdot|x_2x_3x_3|\cdots\cdots|x_{2n+1}x_1x_1|\overset{\text{B6}}{\Longrightarrow}|x_{i-1}x_ix_{i+1}|$ for at least one i:

$|x_{i-1}x_ix_{i+1}|\cdot|x_iax_{i+1}|\overset{\text{lemma 3.8}}{\Longrightarrow}|x_{i-1}x_ia|.$

$|x_{i-1}x_ia|\cdot|x_{i-1}ax_i|\overset{\text{B4}}{\Longrightarrow}x_i=a.$

Therefore, $|x_1ax_2|\cdot|x_2ax_3|\cdots\cdots|x_{2n+1}ax_1|\rightarrow x_i=a$ for at least one i.

Lemma 3. 11. *If* $R=R[\mathfrak{B}]$, $A'\subset A$ *and* $R'=\{|a'b'c'|\ |\ |a'b'c'|\in R[\mathfrak{B}]$ *and* $a',b',c'\in A'\}$, *then* $R'=R'[\mathfrak{B}]$.

Proof. The elements of A, which appear in each conclusion of these conditions B1, B2, B3, B4, B5, B6 and B7, are all contained in the assumption of these conditions.

Hence, if R' does not satisfy the conditions B1, B2, B3, B4, B5, B6 and B7, then it contradicts $R=R[\mathfrak{B}]$. Therefore, R' satisfies the conditions, so $R'=R'[\mathfrak{B}]$.

§ 4. Decomposition of a set A by a family of subsets and introduction of distance.

In a set A, we shall consider a collection $S(\alpha,\beta,\dots\dots)(\neq 0)$ of subsets $\alpha,\beta,\dots\dots$ of A. In this section we shall decompose A by means of $S(\alpha,\beta,\dots\dots)$ and introduce a distance in A.

First, we shall decompose S[1].

α and $\beta(\alpha,\beta\in S)$ are said to be *connected* when for α and β there exists a sequence $\alpha_i(\alpha_i\in S,\ i=1,2,\dots,l$ and l being finite), such that $\alpha\frown\alpha_1\neq 0,\ \alpha_1\frown\alpha_2\neq 0,\ \dots\dots,\ \alpha_l\frown\beta\neq 0.$

And the ordered collection $\alpha,\alpha_1,\dots\dots,\alpha_l,\beta$ is said to be a *chain* of α and β. A subset T of S is said to be *connected* when any two elements of T are connected.

If we write $\alpha\sim\beta$ when α and β are connected, then the relation $\alpha\sim\beta$ is clearly an equivalent relation. By means of this relation we decompose S into the sum of connected S_μ:

$$S=\sum S_\mu \qquad \dots\dots\dots\dots\dots\dots\dots\dots(4.1)$$

It is clear that each S_μ is a maximal connected subset of S.

1) The decomposition of S: (4.1) is the same as that in S. Lefschetz: Algebraic Topology (1941) p. 15.

We use the following notation:

$$S^* \equiv \{x \mid x \in \alpha,\ \alpha \in S\}\ .$$

Lemma 4. 1. $S_\mu^* \frown S_\nu^* = 0\ (\mu \neq \nu)$.

Proof. If $S_\mu^* \frown S_\nu^* \ni x\ (\neq 0)$, then there exists α and β such that $x \in \alpha$, $\alpha \in S_\mu$, $x \in \beta$ and $\beta \in S_\nu$. Hence $\alpha \sim \beta$ and so $S_\mu = S_\nu$, this contradicts $\mu \neq \nu$.

From Lemma 4.1 we have:

Lemma 4. 2. *The element α of S such that $x \in S_\mu^*$ and $x \in \alpha$, belongs to S_μ. And for $x \in S_\mu^*$ and $y \in S_\nu^*\ (\mu \neq \nu)$, there exists no element α of S such that $x, y \in \alpha\ (\alpha \in S)$.*

Next, we shall decompose A by S. Since $S^* = \sum S^*_\mu$, if we write $A - S^* = A_0$, then we have

$$A = A_0 + \sum S_\mu^*\ .$$

From Lemma 4.1 the above is a direct sum: $A_0 \frown S_\mu^* = 0$ and $S_\mu^* \frown S_\nu^* = 0$.

Definition 1. When for x and $y\ (x, y \in A)$ there exists α and β such that $x \in \alpha$; $y \in \beta\ (\alpha, \beta \in S)$ and $\alpha \sim \beta$ (i.e. $\alpha, \beta \in S_\mu$), x and y are said to be *connected* in A by S, and it is denoted by $x \sim y$.

It is clear that the relation $x \sim y$ is an equivalent relation.

Theorem 3. *The decomposition of S^* by this equivalent relation is just $S^* = \sum S_\mu^*$.*

Proof. 1°: From Lemma 4.1 we have $S_\mu^* \frown S_\nu^* = 0$. 2°: If $x, y \in S_\mu^*$, then there exists α and β such that $x \in \alpha$; $y \in \beta\ (\alpha, \beta \in S_\mu)$, and therefore $\alpha \sim \beta$. Hence from the definition 1, we have $x \sim y$. 3°: $x \in S_\mu^*$ and $y \in S_\nu^*$ $\mu \neq \nu \Rightarrow x \nsim y$. For; if $x \in S_\mu^*$, $y \in S_\nu^*$ and $x \sim y$, then there exist α and β such that $x \in \alpha, y \in \beta$ and $\alpha \sim \beta$. But, from 1° α such that $x \in S_\mu^*$ and $x \in \alpha$, belong to S_μ. So $\alpha \in S_\mu$, $\beta \in S_\nu$, and since $\alpha \sim \beta$, we have $S_\mu = S_\nu$. This contradicts $\mu \neq \nu$. So we have assertion.

If we write $A_\mu \equiv S_\nu^*$, then from the above theorem we have the following relations:

$$A = A_0 + \sum A_\mu \quad \text{(decomposition by } x \sim y\text{)},$$
$$S = \sum S_\mu \quad \text{(decomposition by } \alpha \sim \beta\text{)},$$

and

$$A_\mu = S_\mu^*$$

And any element of A_0 can not be connected to any element in A by S.

When, for two elements x and y of A, there exists an element α of S which contains x and y simultaneously, we say that the *distance* of x

and y is one, and it is denoted by $\overline{xy}=1\,(x\neq y)$[1].

When $x\sim y$, there exists a sequence of elements $x_1, \dots, x_m$ of A such that

$$\overline{xx_1}=1,\quad \overline{x_1x_2}=1,\quad \dots\dots,\quad \overline{x_{m-1}x_m}=1,\quad \overline{x_my}=1. \qquad \dots\dots\dots(4.2)$$

For; Since $x\sim y$, then for some α and $\beta\,(x\in\alpha, y\in\beta)$ there is a chain $\alpha, \alpha_1, \dots\dots, \alpha_{m-1}, \beta$ such that

$$\alpha\wedge\alpha_1\neq 0,\quad \alpha_1\wedge\alpha_2\neq 0,\ \dots\dots,\ \alpha_{m-1}\wedge\beta\neq 0.$$

So, let $\alpha\wedge\alpha_1\ni x_1$, $\alpha_1\wedge\alpha_2\ni x_2$, ..., $\alpha_{m-1}\wedge\beta\ni x_m$, then $x, x_1\in\alpha$: $x_1, x_2\in\alpha_1$; ......; $x_m, y\in\beta$, hence by the definition we have

$$\overline{xx_1}=1,\quad \overline{x_1x_2}=1,\quad \dots\dots,\quad \overline{x_my}=1.$$

For two elements x and y of A, a sequence of elements $x, x_1, \dots x_m, y$ which satisfies (4.2) is said to be a *chain* of x and y. Generally, for a pair of elements x and y of A, there may exist many chains of x and y. We select from them a chain $x, x_1, \dots\dots, x_l, y$ whose index l is minimum among them, and any one of these is said to be a *path* for x and y, and $l+1$ is said to be a *distance* of x and y, $x\neq y$. We can easily see that this denomination "distance" is justified since it satisfies the usual distance axioms.

And, from the minimumity of length of path for x and y, we have

Lemma 4.3. *In a path* $\widehat{xy}\,(\equiv x, x_1, \dots\dots x_l, y)$ *a connected part* $x_i, x_{i+1}, \dots\dots, x_{i+h}$ $(0\leqq i,\ i+h\leqq l+1$ *being* $x_0=x,\ x_{l+1}=y)$ *is a path for* x_i *and* x_{i+h}, *and* $\|x_j; x_{j+k}, z\|\notin S\ (0\leqq j\leqq l+1;\ 2\leqq k$ *and being* $x_{l+2}=x,\ x_{l+3}=x_1)$.

Similarly, when $\alpha\sim\beta$, among the chains of α and β the one whose index is minimum is said to be a *path* of α and β.

From the above discussion, we have the following:

Lemma 4.4. *A and S are decomposed in a direct sum:*

$$A=A_0+\sum_{\mu}A_\mu,\quad S=\sum_{\mu}S_\mu,$$

where A_μ *and* S_μ *have the following properties,*

1. $A_\mu=S_\mu{}^*$.
2. *For* x *and* y *of* A_μ, *there exists a path* $x, x_1, \dots, x_l, y$.
3. *For* α *and* β *of* S_μ, *there exists a path* $\alpha, \alpha_1, \dots, \alpha_l, \beta$.
4. *For* $x\in A_\mu$ *and* $y\in A_\nu\ (\mu\neq\nu)$, *there exists no path.*

4'. *For* $x\in A_\mu$ *and* $y\in A_\nu\ (\mu\neq\nu)$, *there exists no element* α *of* S *such that* $x, y\in\alpha\ (\alpha\in S)$.

In the above lemma we can replace "path" by "distance".

1) We define to be $\overline{xx}=0$.

§ 5. Decomposion of A by distance.

We take $R[\mathfrak{B}]$ in § 3 as $S(\alpha, \beta, \ldots\ldots)$ in § 4 and we take the subset consisting of elements x, y, z in A which is contained in $\{xyz\}$ $(\in R)$ as subset α in A (in § 4). In this section, we shall decompose the connected set A_μ by the distance which we defined in the previous section.

First, we take one element $a_\omega(\mu)$ from each A_μ respectively and fix it. And we represent by $A_\mu^s(a_\omega(\mu))$ the set of all elements which have the distance s from $a_\omega(\mu)$, then the set A is decomposed in the form:

$$A = \sum_\mu A_\mu = \sum_\mu (\sum_s A_\mu^s(a_\omega(\mu)) + a_\omega(\mu)), \qquad \ldots\ldots\ldots\ldots\ldots(5.1)$$

where s represents the distance of $a_\omega(\mu)$ and element of A_μ^s. $(s=1, 2, 3 \ldots$ from the definition of distance.) $A_0 \equiv 0$ follows from B1.

Since A is the direct sum of connected sets such that A_μ and A_ν have no connection to each other as for $R[\mathfrak{B}]$: that is,

There is no element in $R[\mathfrak{B}]$ which contains both of the two elements x and y such that $x \in A_\mu$, $y \in A_\nu (\mu \neq \nu)$. $\ldots\ldots\ldots\ldots\ldots(5.2)$

So, first we deal with the connected set A_μ and next with $A: A = \sum_\mu A_\mu$ (direct sum).

When A is connected, decomposition (5.1) becomes the form:

$$A = a_\omega + \sum_{s=1} A^s(a_\omega).$$

Furthermore, we shall consider the decomposition of A^1 and $A^s (s \geqq 2)$.

[1] Decomposition of $A^1(a_\omega)$.

We take any element $a_0 (\neq a_\omega)$ of A^1 and fix it[1]. We define ${}^+A^1$ and ${}^-A^1$ as follows:

${}^+A^1 \equiv \{y \mid \{a_0 a_\omega y\} \in R \text{ and } y \neq a_\omega\}$ and ${}^-A^1 \equiv A^1 - {}^+A^1$,

(if there is no y, then ${}^+A^1 \equiv 0$, that is $A^1 \equiv {}^-A^1$). Then

$$A^1 = {}^-A^1 + {}^+A^1 \quad \text{(direct sum).} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

We have the following lemmas concerning ${}^-A^1$ and ${}^+A^1$.

Lemma 5. 1. *The decomposition* (5. 3) *is independent of* a_0 *disregarding for* $\pm$*dual.*

Proof. When ${}^+A^1(a_0) \equiv 0$, this lemma is clear, so we have only to consider the case where ${}^+A^1(a_0) \not\equiv 0$.

Let $a \in {}^+A^1(a_0)$ and $b \in {}^-A^1(a_0)$, then

$$|a_0 a_\omega a| \cdot |a_\omega b b| \xrightarrow{B7} |b a_\omega a| \text{ (or) } |a_0 a_\omega b|.$$

1) When $A \equiv \{a\}$, it is trivial.

And by the definition of $^{-}A^1$, $|a_0 a_\omega b|$ may not occur.
Therefore $|b a_\omega a|$ for $a \in {}^{+}A^1(a_0)$ and $b \in {}^{-}A^1(a_0)$.
Then $a \in {}^{+}A^1(a_0)$ and $b \in {}^{-}A^1(a_0) \Rightarrow a \in {}^{+}A^1(b)$.
Hence $\quad {}^{+}A^1(a_0) \subset {}^{+}A^1(b)$.
Since we may consider b instead of a_0 in the above discussion, we have

$$ {}^{+}A^1(a_0) \supset {}^{+}A^1(b). $$

So, we have ${}^{+}A^1(a_0) \equiv {}^{+}A^1(b)$.

And by the similar way, ${}^{+}A^1(a_0) \equiv {}^{-}\dot{A}^1(a)$ $(a \in {}^{+}A^1(a_0))$.

Therefore, the decomposition $A^1 = {}^{-}A^1 + {}^{+}A^1$ is independent of a_0 disregarding for dual.

From the above lemma, it follows that:
If $a \in {}^{+}A^1$ and $b \in {}^{-}A^1$, then $\{a a_\omega b\}$ is always contained in $R[\mathfrak{B}]$. ...(5.4)

Lemma 5.2. *$b_1, b_2 \in {}^{-}A^1$ or $b_1, b_2 \in {}^{+}A^1 \rightarrow \{b_1 a_\omega b_2\}$ is not contained in $R[\mathfrak{B}]$.*

Proof. Now, let $\{b_1 a_\omega b_2\}$ be contained in $R[\mathfrak{B}]$.

Case 1. $b_1, b_2 \in {}^{-}A^1$:

$$ |b_1 a_\omega b_2| \cdot b_1 \in {}^{-}A^1 \xrightarrow{\text{lemma 5.1}} b_2 \in {}^{+}A^1. $$

This contradicts the assumption $b_2 \in {}^{-}A^1$.

Therefore $\{b_1 a_\omega b_2\}$ is not contained in $R[\mathfrak{B}]$.

Case 2. $b_1, b_2 \in {}^{+}A^1$:

$$ |b_1 a_\omega b_2| \cdot |a_\omega a_0 a_0| \xrightarrow{B^7} |a_0 a_\omega b_2| \text{ (or) } |b_1 a_\omega a_0| \quad (a_0 \neq b_1, b_2) $$

This is a contradiction, because both $|a_0 a_\omega b_2|$ and $|b_1 a_\omega a_0|$ are contained in $R[\mathfrak{B}]$ by the definition of ${}^{+}A^1$. Therefore $\{b_1 a_\omega b_2\}$ is not contained in $R[\mathfrak{B}]$. And when $a_0 = b_1$ or b_2, the assertion is evident.

So, we have the lemma 5.2.

Lemma 5.3. *$|{}^{-}a_1^1\, {}^{-}a_2^1\, {}^{-}a_2^1|$ and ${}^{-}a_1^1, {}^{-}a_2^1 \in {}^{-}A^1 \rightarrow |{}^{-}a_1^1\, {}^{-}a_2^1 a_\omega|$ (or) $|{}^{-}a_2^1\, {}^{-}a_1^1 a_\omega|$.*

Proof. ${}^{-}a_1^1 \in {}^{-}A^1 \rightarrow |a_\omega\, {}^{-}a_1^1\, {}^{-}a_1^1|$.; ${}^{-}a_2^1 \in {}^{-}A^1 \rightarrow |{}^{-}a_2^1\, {}^{-}a_2^1\, a_\omega|$.

$|a_\omega\, {}^{-}a_1^1\, {}^{-}a_1^1| \cdot |{}^{-}a_1^1\, {}^{-}a_2^1\, {}^{-}a_2^1| \cdot |{}^{-}a_2^1 a_\omega a_\omega| \xrightarrow{B^6} |a_\omega\, {}^{-}a_1^1\, {}^{-}a_2^1|$ or $|{}^{-}a_1^1\, {}^{-}a_2^1 a_\omega|$ or $|{}^{-}a_2^1 a_\omega\, {}^{-}a_1^1|$.

And $|a_\omega\, {}^{-}a_1^1\, {}^{-}a_2^1| \xrightarrow{B^3} |{}^{-}a_2^1\, {}^{-}a_1^1 a_\omega|$, and also by lemma 5.2 $|{}^{-}a_2^1 a_\omega\, {}^{-}a_1^1|$ is not contained in $R[\mathfrak{B}]$.

Therefore $|{}^{-}a_1^1\, {}^{-}a_2^1\, {}^{-}a_2^1|$ and ${}^{-}a_1^1, {}^{-}a_2^1 \in {}^{-}A^1 \rightarrow |{}^{-}a_1^1\, {}^{-}a_2^1 a_\omega|$ (or) $|{}^{-}a_2^1\, {}^{-}a_1^1 a_\omega|$.

[2] Decomposition of $A^s (s \geqq 2)$

We shall consider the path $\widehat{a_\omega a^s} \equiv a_\omega, a_1^1, a_1^2, \dots, a_1^{s-1} a^s$ for a_ω and an element a^s of A^s. According to that a_1^1 in this path belongs to either ${}^{+}A^1$ or ${}^{-}A^1$, this path is said to be (+) path or (−) path respectively, and a (+) path and a (−) path are said to be of different sort.

In A^s, let ${}^+A^s$ be the set of all elements a^s which have (+) path and ${}^-A^s$ be the set of all elements which have (−) path.

Inner points a^t of (−) path clearly have (−) path so that from Lemma 4.3 a^t belongs to ${}^-A^t$.

Then $A^s = {}^-A^s + {}^+A^s$. (this is not necessarily direct sum)

Let us define as follows:

${}^-A^s \wedge {}^+A^s = {}^\frown A^s$; ${}^-A^s - {}^\frown A^s = {}^\ominus A^s$; ${}^+A^s - {}^\frown A^s = {}^\oplus A^s$,

then $A^s = {}^\ominus A^s + {}^\frown A^s + {}^\oplus A^s$ (direct sum).

So, we have the decomposition of A:

$$A = a_\omega + \sum_{s=1} ({}^-A^s + {}^+A^s) = a_\omega + \sum_{s=1} ({}^\ominus A^s + {}^\frown A^s + {}^\oplus A^s). \quad \text{.........(5. 5)}$$

§ 6. Lemma concerning $R \equiv R[\mathfrak{B}]$.

In this section we shall investigate the properties of $R \equiv R[\mathfrak{B}]$. We consider the decomposition (5. 5) of A.

Definition 2. When an element $a^s \in A^s$ has two paths $a_\omega, a^1, \dots, a^{s-1}, a^s$ and $a_\omega, 'a^1, \dots, 'a^{s-1}, a^s$, then a set of elements

$a_\omega, a^1, \dots, a^{s-1}, a^s, 'a^{s-1}, \dots, 'a^1, a_\omega$: for two paths of the same sort,

$a^1, \dots, a^{s-1}, a^s, 'a^{s-1}, \dots, 'a^1, a^1$: for two paths of the different sorts,

is said to be a *cyclic path* constructed by $\widehat{a_\omega a^s} : \hat{a}^t$ and $\widehat{a^s a_\omega} : '\hat{a}^t$.

Since by (5. 4), for ${}^-a^1 \in {}^-A^1$, ${}^+a^1 \in {}^+A^1$, $\{{}^-a^{1+}a^{1+}a^1\}$ is always contained in R, so we can omit a_ω in the latter case.

Lemma 6. 1. *For a cyclic path of different sort constructed by* $\widehat{a_\omega a^s} : \hat{a}^t$ *and* $\widehat{a^s a_\omega} : '\hat{a}^t$ *we have* $\{a^{s-1}a^s\, 'a^{s-1}\} \in R[\mathfrak{B}]$.

Proof. From the definition of path we can easily see that a cyclic path of different sort $a^1, a^2, \dots, a^{s-1}, a^s, 'a^{s-1}, \dots, 'a^1, a^1$ is a sequence constituting odd elements and satisfying the condition $\{xyy\} \in R[\mathfrak{B}]$ for all pairs of the adjacent elements x and y. So, by B6 and Lemma 4. 3 we have that at least one of the following three

$$\{a^{s-1}a^s\, 'a^{s-1}\}, \quad \{'a^1a^1a^2\}, \quad \{'a^2\, 'a^1a^1\} \quad \text{..............(6.1)}$$

must be contained in $R[\mathfrak{B}]$.

Suppose $\{'a^1a^1a^2\} \in R[\mathfrak{B}]$. From (5. 4) we have $|a^1a_\omega\, 'a^1|$.

$$|a^2a^1\, 'a^1| \cdot |a^1a_\omega\, 'a^1| \xrightarrow{\text{lemma 3.9}} |a^2a^1a_\omega\, 'a^1|.$$

This contradicts Lemma 4. 3. Hence $|'a^1a^1a^2|$ may not occur, and in similar way $|'a^2 'a^1a^1|$ may not occur. So, we have from (6. 1) $\{a^{s-1}a^s\, 'a^{s-1}\} \in R[\mathfrak{B}]$.

Definition 3. The points a^i $(i=1, \dots, l)$ of path $\widehat{ab} : a, a^1, a^2, \dots, a^l, b$ are said to be *inner points* of this path.

Lemma 6. 2. $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^s| \to |{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$ (or) $|{}^-a_2^{s-1}\,{}^-a_2^s\,{}^-a_1^s|$ *for some element* ${}^-a_1^{s-1}$ *or* ${}^-a_2^{s-1} \in {}^-A^{s-1}$, *(± dual,* $s \geqq 2$*)*[1].

Proof. We consider the paths $\widehat{a_\omega {}^-a_1^s}$ bnd $\widehat{a_\omega {}^-a_2^s}$, then we have an odd cyclic path $a_\omega, {}^-a_1^1, \dots, {}^-a_1^{s-1}, {}^-a_1^s, {}^-a_2^s, {}^-a_2^{s-1}, \dots, {}^-a_2^1, a_\omega$. For this seqence constituting odd elements, we have, by the similar way as the proof of Lemma 6.1, that at least one of $\{{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\}$ and $\{{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}\}$ is contained in $R[\mathfrak{B}]$

And if $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$ and $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}|$, then

$$|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s| \cdot |{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}| \xrightarrow{\text{lemma 3.5}} |{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}|. \qquad \dots\dots(6.2)$$

Then a sequence of elements $a_\omega, {}^-a_1^1, \dots, {}^-a_1^{s-1}, {}^-a_2^{s-1}, \dots, {}^-a_2^1, a_\omega$ is an odd cyclic path. By the same way as the above, at least one of $\{{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}\}$ and $\{{}^-a_1^{s-1}\,{}^-a_2^{s-1}\,{}^-a_2^{s-2}\}$ is contbined in $R[\mathfrak{B}]$. ......(6. 3)
If $\{{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}\}$ is contained in $R[\mathfrak{B}]$, then by $|{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}|$ and (6. 2), we have $|{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}|$. This contracicts Lemma 4. 3. Therefore $\{{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}\}$ is not contained. Similarly, $\{{}^-a_1^{s-1}\,{}^-a_2^{s-1}\,{}^-a_2^{s-2}\}$ is not contained. These contradict (6. 3).

Therefore the case when $\{{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\}$ and $\{{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}\} \in R[\mathfrak{B}]$ may not occur, $\{{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\}$ (or) $\{{}^-a_2^{s-1}\,{}^-a_2^s\,{}^-a_1^s\}$ is contained in $R[\mathfrak{B}]$ for some element ${}^-a_1^{s-1}$ or ${}^-a_2^{s-1} \in {}^-A^{s-1}$.

Lemma 6. 3. $\{{}^-a_1^{s-1}\,{}^-a^s\,{}^-a_2^{s-1}\}$ *is not contained in* $R[\mathfrak{B}]$. *(± dual,* $s \geqq 2$*)*

Proof. $|{}^-a_1^{s-1}\,{}^-a^s\,{}^-a_2^{s-1}| \xrightarrow{\text{lemma 3.2}} |{}^-a_1^{s-1}\,{}^-a_2^{s-1}\,{}^-a_2^{s-1}|$

Let us consider the path $\widehat{a_\omega {}^-a_1^{s-1}}$ and $\widehat{a_\omega {}^-a_2^{s-1}}$, then by the same process as that used in Lemma 6. 2, we see that at least one of $\{{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}\}$ and $\{{}^-a_1^{s-1}\,{}^-a_2^{s-1}\,{}^-a_2^{s-2}\}$ is contained in $R[\mathfrak{B}]$.
If $\{{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}\} \in R$, then

$$|{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}| \cdot |{}^-a_1^{s-1}\,{}^-a^s\,{}^-a_2^{s-1}| \xrightarrow{\text{lemma 3.8}} |{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a^s|.$$

This contradicts Lemma 4. 3, so $\{{}^-a_1^{s-2}\,{}^-a_1^{s-1}\,{}^-a_2^{s-1}\}$ is not contained in $R[\mathfrak{B}]$. Similarly, $\{{}^-a_1^{s-1}\,{}^-a_2^{s-1}\,{}^-a_2^{s-2}\}$ is not contained in $R[\mathfrak{B}]$.

Therefore $\{{}^-a_1^{s-1}\,{}^-a^s\,{}^-a_2^{s-1}\}$ is not contained in $R[\mathfrak{B}]$.

Lemma 6. 4. $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s| \cdot |{}^-a_2^{s-1}\,{}^-a_1^s\,{}^-a_1^s| \to |{}^-a_2^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$ *(± dual* $s \geqq 2$*)*

Proof. $|{}^-a_2^{s-1}\,{}^-a_1^s\,{}^-a_1^s| \xrightarrow{\text{lemma 3.2}} |{}^-a_1^s\,{}^-a_2^{s-1}\,{}^-a_2^{s-1}|$

1) The notation (±dual) means that the propositfon exchanging ${}^-a^s$ and ${}^+a^s$ in this proposition is also satisfied.

So, $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|\cdot|{}^-a_1^s\,{}^-a_2^{s-1}\,{}^-a_2^{s-1}|\xrightarrow{B7}|{}^-a_2^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$ Ⓞⓡ $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^{s-1}|$
And, by the Lemma 6.3 $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^{s-1}|$ is not contained in $R[\mathfrak{B}]$.
Therefore $\{{}^-a_2^{s-1}\,{}^-a_1^s\,{}^-a_2^s\}$ is contained in $R[\mathfrak{B}]$.

From Lemma 6.2 and Lemma 6.4 we have

Lemma 6. 5. $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^s|\to|{}^-a^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$ Ⓞⓡ $|{}^-a^{s-1}\,{}^-a_2^s\,{}^-a_1^s|$.

Lemma 6. 6. $|{}^-a^{s-1}\,{}^-a_1^s\,{}^-a_2^s|\cdot|{}^-a_1^s\,{}^-a^{s+1}\,{}^-a^{s+1}|\to|{}^-a^{s+1}\,{}^-a_1^s\,{}^-a_2^s|$ ($\pm$*dual*) $(s\geqq 2)$

Proof. $|{}^-a^{s-1}\,{}^-a_1^s\,{}^-a_2^s|\cdot|{}^-a_1^s\,{}^-a^{s+1}\,{}^-a^{s+1}|\xrightarrow{B7}|{}^-a^{s+1}\,{}^-a_1^s\,{}^-a_2^s|$ Ⓞⓡ $|{}^-a^{s-1}\,{}^-a_1^s\,{}^-a^{s+1}|$.
And from Lemma 4.3 $|{}^-a^{s-1}\,{}^-a_1^s\,{}^-a^{s+1}|$ may not occur, so $|{}^-a^{s+1}\,{}^-a_1^s\,{}^-a_2^s|$ is contained in $R[\mathfrak{B}]$.

Lemma 6. 7. $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^s|\cdot|{}^-a_2^s\,{}^+a^{s-1}\,{}^+a^{s-1}|\to|{}^-a_1^{s-1}\,a_1^s\,{}^-a_2^s\,{}^+a^{s-1}|$ Ⓞⓡ
$|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}|$ *and* ${}^-a_2^s\in{}^\frown A^s$.

Proof. $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^s|\xrightarrow{\text{lemma 6.2}}|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$ Ⓞⓡ $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}|$ ......(6. 4)

If $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|$, then a cyclic path ${}^+a^1,\dots,{}^+a^{s-1}\,{}^-a_2^s,\dots,{}^-a^1$ is obtained from the path $\widehat{a_\omega\,{}^+a^{s-1}}$, path $\widehat{{}^-a_2^s a_\omega}$ and $|{}^-a_2^s\,{}^+a^{s-1}\,{}^+a^{s-1}|$.
Hence, by B6 $\{{}^-a_1^{s-1}\,{}^-a_2^s\,{}^+a^{s-1}\}$ is contained in $R[\mathfrak{B}]$.
So, $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|\cdot|{}^-a_1^{s-1}\,{}^-a_2^s\,{}^+a^{s-1}|\xrightarrow{\text{lemma 3.7}}|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\,{}^+a^{s-1}|$ ....(6. 5)
From (6. 4) and (6. 5), we have $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\,{}^+a^{s-1}|$ Ⓞⓡ $|{}^-a_1^s\,{}^-a_2^s\,{}^-a_2^{s-1}|$.
And $|{}^-a_2^s\,{}^+a^{s-1}\,{}^+a^{s-1}|\to{}^-a_2^s\in{}^\frown A^s$.

Lemma 6. 8. $|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s|\cdot|{}^-a_2^s\,{}^+a^{s-1}\,{}^+a^{s-1}|\to|{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\,{}^+a^{s-1}|$. $(s\geqq 2)$

Proof. By the same process as that deriving (6. 5) in Lemma 6.7, we may prove that $\{{}^-a_1^{s-1}\,{}^-a_1^s\,{}^-a_2^s\,{}^+a^{s-1}\}$ is contained in $R[\mathfrak{B}]$.

Lemma 6. 9. *$\{{}^\frown a^{s-1}a^s a^s\}$ is not contained in* $R[\mathfrak{B}]$. $(s\geqq 3)$

Proof. For ${}^\frown a^{s-1}\in{}^\frown A^{s-1}$, there exist two paths which are of different sort. So, since for ${}^+a^1\in{}^+A^1$ and ${}^-a^1\in{}^-A^1$, $\{{}^+a^1\,{}^-a^1\,{}^-a^1\}$ is contained in $R[\mathfrak{B}]$, a cyclic path is obtained from the two paths ${}^\frown a^{s-1}, a_1^{s-2},\dots,{}^+a^1\,a_\omega$ and ${}^\frown a^{s-1}\,a_2^{s-2},\dots,{}^-a^1\,a_\omega$. So, by means of B6, $\{a_1^{s-2}\,{}^\frown a^{s-1}\,a_2^{s-2}\}$ is contained in $R[\mathfrak{B}]$. Then

$$|a_1^{s-2}\,{}^\frown a^{s-1}\,a_2^{s-2}|\cdot|{}^\frown a^{s-1}\,a^s\,a^s|\xrightarrow{B7}|a_1^{s-2}\,{}^\frown a^{s-1}\,a^s|\ \text{Ⓞⓡ}\ |a^s\,{}^\frown a^{s-1}\,a_2^{s-2}|$$

And $|a_1^{s-2}\,{}^\frown a^{s-1}\,a^s|$ and $|a^s\,{}^\frown a^{s-1}\,a_2^{s-2}|$ both contradicts Lemma 4. 3.
Therefore $\{{}^\frown a^{s-1}\,a^s\,a^s\}$ is not contained in $R[\mathfrak{B}]$.

Lemma 6. 10. *All inner points of the path of a_ω and $x(x\in{}^+A^t)$: $a^1,\dots,a^{t-1}$ consist of the elements which are contained in ${}^\oplus A^s$ $(s=1,\dots,t-1;\ t\geqq 1)$.*

Proof. For $x\in{}^+A^s$, there exists always y such that $\{xyy\}$ is contained in $R[\mathfrak{B}]$ and $y\in{}^+A^{s-1}$ by the definition of ${}^+A^s$. Then, for $x\in{}^+A^t$, there is a path $\widehat{a_\omega x}$ which consists of the elements of ${}^+A^s$ $(s=1,\dots,t-1)$. And by

Lemma 6.9 $\{{}^{\frown}a^{s-1}a^{s}a^{s}\}$ is not contained in $R[\mathfrak{B}]$. Then all the inner points of this path $\widehat{a_{\omega}x}$ belong to ${}^{\oplus}A^{s}(s=1, \dots, t-1)$.

By this Lemma, we can define as follows:

The path $(\widehat{a_{\omega}{}^{\frown}a^{t}})^{+}$ means the path whose inner points belong to ${}^{\oplus}A^{s}$, and the path $(\widehat{a_{\omega}{}^{\frown}a^{t}})^{-}$ means the path whose inner points belong to ${}^{\ominus}A^{s}$ $(s=1, \dots, t-1)$.

Lemma 6.11. *If there are two paths $(\widehat{a_{\omega}x})^{+}$ and $(\widehat{a_{\omega}x})^{-}$ for $x \in {}^{-}A^{s}$, then $\{{}^{-}a^{s-1}\, x\, {}^{+}a^{s-1}\}$ is contained in $R[\mathfrak{B}]$.*

Proof. From the two paths $(\widehat{a_{\omega}x})^{+}$ and $(\widehat{a_{\omega}x})^{-}$, an odd cyclic path ${}^{-}a^{1}, \dots, {}^{-}a^{s-1}, x, {}^{+}a^{s-1}, \cdots, {}^{+}a^{1}$ is obtained. Hence, by means of Lemma 6.1 and 6.10, $\{{}^{-}a^{s-1}\, x\, {}^{+}a^{s-1}\}$ is always contained in $R[\mathfrak{B}]$.

Lemma 6.12. *When $a^{s+1} \in {}^{\frown}A^{s+1}$; $a_{1}^{s} \in {}^{\oplus}A^{s}$ and $a_{2}^{s} \in {}^{\ominus}A^{s}$ (or $a_{1}^{s} \in {}^{\ominus}A^{s}$ and $a_{2}^{s} \in {}^{\oplus}A^{s}$), the necessary and sufficient condition for $\| a^{s+1}, a_{1}^{s}, a_{2}^{s} \| \in R[\mathfrak{B}]$ is that there are $|a_{1}^{s}\, {}^{\frown}a^{s+1}\, {}^{\frown}a^{s+1}|$ and $|a_{2}^{s}\, {}^{\frown}a^{s+1}\, {}^{\frown}a^{s+1}|$.*

Proof. Since it is clear that the above condition is necessary, we shall only prove its sufficiency.

As $\{a_{1}^{s}\, {}^{\frown}a^{s+1}\, {}^{\frown}a^{s+1}\} \in R[\mathfrak{B}]$, path $(\widehat{a_{\omega}a_{1}^{s}})^{+}$ and ${}^{\frown}a^{s+1}$ is a path of a_{ω} and ${}^{\frown}a^{s+1}$, and as $\{a_{2}^{s}\, {}^{\frown}a^{s+1}\, {}^{\frown}a^{s+1}\} \in R[\mathfrak{B}]$, path $(\widehat{a_{2}^{s}a_{\omega}})^{-}$ and ${}^{\frown}a^{s+1}$ is a path of a^{s+1} and a_{ω}. From these paths and Lemma 6.11, we have that $\{a_{1}^{s}\, {}^{\frown}a^{s+1}\, a_{2}^{s}\}$ is contained in $R[\mathfrak{B}]$.

Therefore $|a_{1}^{s}\, {}^{\frown}a^{s+1}\, {}^{\frown}a^{s+1}|$ and $|a_{2}^{s}\, {}^{\frown}a^{s+1}\, {}^{\frown}a^{s+1}|$ are necessary and sufficient for $\| a^{s+1}, a_{1}^{s}, a_{2}^{s} \|$.

Lemma 6.13. *When ${}^{+}a_{1}^{s} \in {}^{\oplus}A^{s}$, then*

$$|{}^{-}a^{s-1}\, {}^{-}a_{1}^{s}\, {}^{-}a_{2}^{s}| \cdot |{}^{-}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}| \rightarrow |{}^{\oplus}a_{1}^{s}\, {}^{-}a_{1}^{s}\, {}^{-}a_{2}^{s}|. \quad (s \geq 2)$$

Proof. $|{}^{-}a^{s-1}\, {}^{-}a_{1}^{s}\, {}^{-}a_{2}^{s}| \cdot |{}^{-}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}| \xrightarrow{B7} |{}^{\oplus}a_{1}^{s}\, {}^{-}a_{1}^{s}\, {}^{-}a_{2}^{s}|$ (or) $|{}^{-}a^{s-1}\, {}^{-}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}|$.
But, $|{}^{-}a^{s-1}\, {}^{-}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}| \xrightarrow{\text{lemma 3.2}} |{}^{-}a^{s-1}\, {}^{\oplus}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}|$. So ${}^{\oplus}a_{1}^{s} \in {}^{\frown}A^{s}$, this is contradiction, hence $\{{}^{-}a^{s-1}\, {}^{-}a_{1}^{s}\, {}^{\oplus}a_{1}^{s}\}$ is not contained in $R[\mathfrak{B}]$.

Therefore $\{{}^{\oplus}a_{1}^{s}\, {}^{-}a_{1}^{s}\, {}^{-}a_{2}^{s}\}$ is contained in $R[\mathfrak{B}]$.

§ 7. Sufficient condition for betweenness.

In this section, we shall prove that the system of seven conditions: $\mathfrak{B} \equiv (B1, \dots, B7)$ is sufficient for betweenness of set A by the following process:

First, we introduce by means of $R[\mathfrak{B}]$ the binary relation ρ in A which satisfies P1, P2, and P3 and prove that if $x\rho y\rho z$ or $z\rho y\rho x$ in $A[\rho]$ concerning the above binary relation, then $\{xyz\}$ is contained in $R[\mathfrak{B}]$ and conversely if $\{xyz\}$ is contained in $R[\mathfrak{B}]$, then $x\rho y\rho z$ or $z\rho y\rho x$ in $A[\rho]$.

We consider at first the case when $A[\tau]$ is connected, and then the general case $A[\tau]$ as the direct sum of connected systems.

[1] Definition of binary relation ρ in A when $R[Q]$ is connected.

We docompose the set A as follows:

$$A = a_\omega + \sum_{s=1} ({}^-A^s + {}^+A^s) = a_\omega + \sum_{s=1} ({}^\ominus A^s + {}^\frown A^s + {}^\oplus A^s),$$

and define an binary relation for the elements of A as follows:

(I) ${}^-A^1 < a_\omega < {}^+A^1$.

where ${}^-A < a_\omega < {}^+A^1$ means that ${}^-a^1 < a_\omega$, $a_\omega < {}^+a^1$ and ${}^-a^1 < {}^+a^1$ for ${}^-a^1 \in {}^-A^1$ and ${}^+a^1 \in {}^+A^1$.

(II) ${}^-A^1 \otimes {}^-A^2 \otimes \cdots\cdots \otimes {}^-A^{2m-2} \otimes {}^-A^{2m-1} \otimes {}^-A^{2m} \otimes \cdots\cdots,$

where $A^i \otimes A^{i+1}$ means that for $a^i \in A^i$ and $a^{i+1} \in A^{i+1}$, if $\{a^i a^{i+1} a^{i+1}\}$ is contained in $R[\mathfrak{B}]$, then $a^i < a^{i+1}$.[1] And the notation $\otimes$ represents the binary relation of elements of each adjacent A^j's alone.

(III) ${}^+A^1 \otimes {}^+A^2 \otimes \cdots\cdots \otimes {}^+A^{2l-2} \otimes {}^+A^{2l-1} \otimes A^{2l} \otimes \cdots\cdots.$

(IV) When $a^s \in A^s$ and $a^{s+k} \in A^{s+k} (k \geqq 2)$, we do not define the binary relation of a^s and a^{s+k}. (where $A^\circ \equiv a_\omega$).

$(\mathrm{V})_1$ ${}^\oplus A^s \otimes {}^-A^s$; ${}^+A^s \otimes {}^\ominus A^s$ (s=even, $s \geqq 2$),

$(\mathrm{V})_2$ ${}^\oplus A^s \otimes {}^-A^s$; ${}^+A^s \otimes {}^\ominus A^s$ (s=odd, $s \geqq 3$).

$(\mathrm{VI})_1$ In ${}^-A^s$ (s=even), for ${}^-a_1^s, {}^-a_2^s \in {}^-A^s$,

$|{}^-x^{s-1}\, {}^-a_1^s\, {}^-a_2^s| \rightarrow {}^-a_1^s < {}^-a_2^s$.

$(\mathrm{VI})_2$ In ${}^-A^s$ (s=odd) for ${}^-a_1^s, {}^-a_2^s \in {}^-A^s$,

$|{}^-x^{s-1}\, {}^-a_1^s\, {}^-a_2^s| \rightarrow {}^-a_1^s > {}^-a_2^s$ (${}^-x^{s-1} \equiv a_\omega$ for $s=1$).

(VII) The binary relation $>$ in ${}^+A^s$ is defined by the converse of direction in $(\mathrm{VI})_1$ and $(\mathrm{VI})_2$.

The process, by which the expression $u, v, w, \ldots$ and $u > v, w < z \ldots$ in (a) are replaced by $u^1, v^1, w^1 \ldots$ and $u^1 < v^1$, $w^1 > z^1 \ldots$ respectively, is called the conversion of direction in (a).

(VIII) The relation $x=y$ means that x and y are the same elements of A.

From the above definition of order ρ, we may prove the following lemmas:

Lemma 7. 1. ${}^-A^s \otimes {}^\oplus A^{s-1}$ $(\equiv {}^\frown A^s \otimes {}^\oplus A^{s-1})$ (s=*even*, $s \geqq 2$).

Proof. The binary relation ρ of A^s and A^{s-1} ($s \geqq 2$) is defined by (II) and (III) only. Among (II) and (III), the relation containing ${}^\oplus A^{s-1}$ is (III) only. And ${}^-a^s \in {}^-A^s$, ${}^\oplus a^{s-1} \in {}^\oplus A^{s-1}$ and $\| {}^-a^s, {}^\oplus a^{s-1}, x \| \rightarrow {}^-a^s \in {}^\frown A^s$. Hence the

1) If $A^{i+1} \not\equiv 0$ and $a^{i+1} \in A^{i+1}$, there exists at least one element a^i of A^i such that $\{a^i a^{i+1} a^{i+1}\}$ is cotained in $R[\mathfrak{B}]$.

element of $^{-}A^{s}$ which has binary-relation with an element of $^{\oplus}A^{s-1}$ belongs to $^{\frown}A^{s}$. So, from (III) we have

$$^{-}A^{s} \oslash {}^{\oplus}A^{s-1} \; (\equiv {}^{\frown}A^{s} \oslash {}^{\oplus}A^{s-1}) \; (s=even,\ s\geq 2)$$

When we define the binary reiation in A be (I), (II), (III), (IV), (V), (VI), (VII) and (VIII), A^{t} which has binary-relation with A^{s} (s=even) is A^{s-1}, A^{s} and A^{s+1} by (IV). From the meaning of $\oslash$ and lemma 6.9, A^{t}'s which has binary-relation with $^{-}A^{s}$ (s=even) by the above definition of binary-relation are $^{\ominus}A^{s-1}$, $^{\oplus}A^{s-1}$, A^{s} and $^{-}A^{s+1}$, and their relations are as follows:

$^{-}A^{s} \oslash {}^{\ominus}A^{s-1}$ (by (II)); $^{-}A^{s} \oslash {}^{\oplus}A^{s-1} \equiv {}^{\frown}A^{s} \oslash {}^{\oplus}A^{s-1}$ (by Lemma 7.1)

$^{-}A^{s} \oslash {}^{\oplus}A^{s}$ (by (V)); $^{-}A^{s} \oslash {}^{-}A^{s+1}$ (by (II))

The binary relation in $^{-}A^{s}$ is decided by (VI).

So, we have the following lemma.

Lemma 7.2. *When* $y \in {}^{-}A^{s}$ ($s=even$),

$$\left.\begin{array}{l} x>y \rightarrow x \in {}^{\oplus}A^{s-1} \text{ (or) } {}^{-}A^{s} \; (x \in {}^{\oplus}A^{s-1} \textit{ may occur only in the case} \\ \quad \textit{where } y \textit{ belongs to } {}^{\frown}A^{s}), \\ y>x \rightarrow x \in {}^{\ominus}A^{s-1} \text{ (or) } {}^{-}A^{s} \text{ (or) } {}^{\oplus}A^{s} \text{ (or) } {}^{-}A^{s+1}. \end{array}\right\} \quad \text{.........(7.1)}$$

when $y \in {}^{-}A^{s}(s=odd, \neq 1)$, *we have the proposition (conversing the direction in* (7.1)).
.........(7.2)

When $y \in {}^{+}A^{s}$, *we have the proposition (conversing the direction and exchanging the sign in* (7.1) *and* (7.2)).

For $s=\pm 1$, we have:

Lemma 7.3. *When* $y \in {}^{-}A^{1}$,

$$\left.\begin{array}{l} x>y \rightarrow x \in {}^{-}A^{1} \text{ (or) } a_{\omega} \text{ (or) } {}^{+}A^{1} \text{ (or) } {}^{-}A^{2}, \\ y>x \rightarrow x \in {}^{-}A^{1}. \end{array}\right\} \quad \text{.........(7.3)}$$

When $y \in {}^{+}A^{1}$, *we have the proposition (conversing the direction (see p. 195) and exchanging the sign in* (7.3)).

[2] Proof of P1, P2 and P3.

We shall prove that the binary relation which is defined in [1] satisfies P1, P2 and P3. As we have proved in Introduction that system P1, P2, P3 is equivalent to system P*1, P*2, P*3, we have only to prove here that the binary relation defined in A satisfies P*1, P*2, P*3.

P*1 is clear by (VIII) in the definition.

Proof of P*2:

According to A^{t} to which y belongs, we consider about six cases:

Case 1 $y=a_{\omega}$; Case 2 $y \in {}^{-}A^{s}$ ($s\geq 2$, even)
Case 3 $y \in {}^{-}A^{1}$; Case 4 $y \in {}^{-}A^{s}$ ($s\geq 3$, odd)
Case 5 $y \in {}^{+}A^{s}$ ($s\geq 2$, even); Case 6 $y \in {}^{+}A^{s}$ (s=odd)

Case 1. $y=a_\omega$:

$$x>y \xrightarrow{(1)(IV)} x\in {}^+A^1\ ;\ y>x \xrightarrow{(1)(IV)} x\in {}^-A^1.$$

This contradicts ${}^-A^1 \frown {}^+A^1 \equiv 0$.

Therefore, $x>y$ and $y>x$ are not compatible.

Case 2. $y\in {}^-A^s$ ($s\geqq 2$, even):

$$x>y \xrightarrow{\text{lemma 7.2}} x\in {}^{\oplus}A^{s-1} \text{ (or) } {}^-A^s; \qquad \cdots\cdots\cdots(7.4)$$

$$y>x \xrightarrow{\text{lemma 7.2}} x\in {}^{\ominus}A^{s-1} \text{ (or) } {}^-A^s \text{ (or) } {}^{\oplus}A^s \text{ (or) } {}^-A^{s+1} \qquad \cdots\cdots\cdots(7.5)$$

If $x>y$ and $y>x$ are compatible, then x must be contained in ${}^-A^s$ which is the intersection of (7.4) and (7.5), and then it follows from $x, y\in {}^-A^s$ that

$$x>y \xrightarrow{(VI)} |{}^-a_1^{s-1}\, yx|\ ;\ y>x \xrightarrow{(VI)} |{}^-a_2^{s-1}\, xy|.$$

$$|{}^-a_1^{s-1}\, yx| \xRightarrow{\text{lemma 3.2}} |x\ {}^-a_1^{s-1}\ {}^-a_1^{s-1}|,$$

So,
$$|{}^-a_2^{s-1}\, xy|\cdot|x\ {}^-a_1^{s-1}\ {}^-a_1^{s-1}| \xRightarrow{\text{lemma 6.4}} |{}^-a_1^{s-1}\, xy|,$$

Hence,
$$|{}^-a_1^{s-1}\, yx|\,|{}^-a_1^{s-1}\, xy| \xRightarrow{B4} x=y.$$

This contradicts $x\neq y$, so $x>y$ and $y>x$ are not compatible.

Case 3. $y\in {}^-A^1$:

$$x>y \xrightarrow{\text{lemma 7.3}} x\in {}^-A^1 \text{ (or) } a_\omega \text{ (or) } {}^+A^1 \text{ (or) } {}^-A^2. \qquad \cdots\cdots\cdots(7.6)$$

$$y>x \xrightarrow{\text{lemma 7.3}} x\in {}^-A^1. \qquad \cdots\cdots\cdots(7.7)$$

If $x>y$ and $y>x$ are compatible, then x must be contained in A^{-1} which is the intersection of (7.6) and (7.7), and then it follows from $x, y\in {}^-A^1$ that

$$x>y \xRightarrow{(VI)} |a_\omega xy|\ ;\ y>x \xRightarrow{(VI)} |a_\omega yx|.$$

So,
$$|a_\omega xy|\cdot|a_\omega yx| \xRightarrow{B4} x=y.$$

This contradicts $x\neq y$, so $x>y$ and $y>x$ are not compatible.

Case 4. $y\in {}^-A^s$ ($s\geqq 3$, *odd*);

By coversing the direction in the proof of Case 2, we can prove this case in the similar way.

Case 5. $y\in {}^+A^s$ ($s\geqq 2$, even)

By conversing the direction and exchanging the sign in the proof of Case 2, we can prove this case in the similar way.

Case 6. $y\in {}^+A^s$ (s is odd):

By exchanging the sign in the proof of Case 2, we can prove this case in the similar way. And when $y\in {}^+A^1$, by conversing the direction and by exchanging the sign in the proof of Case 3, we can prove in the similar way.

Therefore, in all cases $x>y$ and $y>x$ are not compatible.

Before proceeding the proof of P*3, we prove the lemma:

Lemma 7. 4. *When and only when $\{xyy\}$ is contained in $R[\mathfrak{B}]$, one and only one binary relation of x and y is defined.*

Proof. If there exists the binary relation of x and y, then from (5. 4) and the definition of binary relation, it follows that $\{xyy\}$ is contained in $R[\mathfrak{B}]$. And conversely, we shall prove that if $\{xyy\}$ is contained in $R[\mathfrak{B}]$, then there is defined the binary relation for x and y.

Suppose $y \in {}^{-}A^{s}$.

$$|xyy| \cdot y \in {}^{-}A^{s} \xrightarrow[\text{and } 6.9]{\text{lemma } 4.3} x \in {}^{\ominus}A^{s-1} \text{ (or) } {}^{\oplus}A^{s-1} \text{ (or) } A^{s} \text{ (or) } {}^{-}A^{s+1}.$$

We consider separately about two cases:

Case 1. $x \in {}^{\ominus}A^{s-1}$ (or) ${}^{\oplus}A^{s-1}$ (or) ${}^{\oplus}A^{s}$ (or) ${}^{-}A^{s+1}$

There exists an binary relation of x and y from (II), (III) and (V) in the definition of the binary relation. And by P*2 there is only one binary relation of x and y.

Case 2. $x \in {}^{-}A^{s}$.

By Lemma 6. 5,

$$|xyy| \to |{}^{-}a^{s-1}\, xy| \text{ (or) } |{}^{-}a^{s-1}\, yx| \qquad \text{.........(7. 8)}$$

So, by P*2 and (7. 8), there is defined one and only one binary relation for x and y.

Therefore, if $\{xyy\}$ is contained in $R[\mathfrak{B}]$, then there exists one and only one binary relation for x and y.

Proof of P*3:[1)]

We consider about the six cases, similarly in the proof of P*2.

Case 1. $y = a_{\omega}$:

$$x>y \xrightarrow{(1)(IV)} x \in {}^{+}A^{1} \;;\; y>z \xrightarrow{(1)(IV)} z \in {}^{-}A^{1}$$

Then, from (I) in the definition of binary relation, we have $x>z$.

And, by means of (5. 4), $\{xa_{\omega}z\} \equiv \{xyz\}$ is contained in $R[\mathfrak{B}]$. ....(7. 9)

Case 2. $y \in {}^{-}A^{s}$ ($s \geq 2$, even):

$x>y \cdot y \in {}^{-}A^{s} \xrightarrow{\text{lemma } 7.2} x \in {}^{-}A^{s}$ (or) ${}^{\oplus}A^{s-1}$ (${}^{\oplus}A^{s-1}$ may occur only in the case where $y \in {}^{\frown}A^{s}$).

According to A^{t} to which x belongs, we consider about the two cases.

1) In the proof of this assertion, at the same time we shall prove that if $x>y$ and $y>z$ then $\{xyz\}$ is contained in $R[\mathfrak{B}]$ at the end of each case.

(a) $x \in {}^{-}A^s$, $y \in {}^{-}A^s$; (b) $x \in {}^{\oplus}A^{s-1}$, $y \in {}^{\frown}A^s$.

(a) $x \in {}^{-}A^s$, $y \in {}^{-}A^s$:

$$x > y \text{ and } x, y \in {}^{-}A^s \xrightarrow{\text{def. (VI)}} |{}^{-}a_1^{s-1}yx| \quad \cdots\cdots\cdots(7.10)$$

$$y > z \xrightarrow{\text{lemma 7.4}} |yzz| \quad \cdots\cdots\cdots(7.11)$$

On the other hand,

$$y > z \xrightarrow{\text{lemma 7.2}} z \in {}^{\ominus}A^{s-1} \text{ ⓞⓡ } {}^{-}A^s \text{ ⓞⓡ } {}^{\oplus}A^s \text{ ⓞⓡ } {}^{-}A^{s+1}$$

According to A^t to which z belongs, we consider three cases:

(a_1) $x, y \in {}^{-}A^s$, $z \in {}^{\ominus}A^{s-1}$ ⓞⓡ ${}^{-}A^{s+1}$,

(a_2) $x, y \in {}^{-}A^s$, $z \in {}^{-}A^s$,

(a_3) $x, y \in {}^{-}A^s$, $z \in {}^{\oplus}A^s$.

(a_1) $x, y \in {}^{-}A^s$, $z \in {}^{\ominus}A^{s-1}$ ⓞⓡ ${}^{-}A^{s+1}$

From (7.10) and (7.11) we have $|{}^{-}a^{s-1}yx|$ and $|yzz|$.

$$|{}^{-}a_1^{s-1}yx| \cdot |yzz| \xrightarrow{\text{lemma 6.4 (6.6)}} |zyx| \xRightarrow{\text{lemma 3.2}} |zzx|. \quad \cdots\cdots\cdots(7.12)$$

Hence, from (7.12), (II) in the definition of binary relation and $x \in {}^{-}A^s$, $z \in {}^{\ominus}A^{s-1}$ ⓞⓡ ${}^{-}A^{s+1}$, we have $x > z$.

(a_2) $x, y \in {}^{-}A^s$, $z \in {}^{-}A^s$:

$$y > z \xrightarrow{\text{(VI)}} |{}^{-}a_2^{s-1}zy| \xRightarrow{\text{lemma 3.2}} |{}^{-}a_2^{s-1}yy|.$$

From (7.6), we have $|{}^{-}a_1^{s-1}yx|$ and $|{}^{-}a_2^{s-1}yy|$.

So,
$$|{}^{-}a_1^{s-1}yx| \cdot |{}^{-}a_2^{s-1}yy| \xRightarrow{\text{lemma 6.4}} |{}^{-}a_2^{s-1}yx|.$$

$$|{}^{-}a_2^{s-1}zy| \cdot |{}^{-}a_2^{s-1}yx| \xRightarrow{\text{lemma 3.7}} |{}^{-}a_2^{s-1}zyx|. \quad \cdots\cdots\cdots(7.13)$$

Hence from (VI) in the definition of binary relation and $x, z \in {}^{-}A^s$, we have $x > z$.

(a_3) $x, y \in {}^{-}A^s$, $z \in {}^{\oplus}A^s$:

From (7.10) and (7.11) we have $|{}^{-}a_1^{s-1}yx|$ and $|yzz|$.

$$|{}^{-}a_1^{s-1}yx| \cdot |yzz| \xRightarrow{\text{lemma 6.13}} |zyx| \Rightarrow |zzx|. \quad \cdots\cdots\cdots(7.14)$$

Hence, from (V) in the definition of binary relation and $x \in {}^{-}A^s$, $z \in {}^{\oplus}A^s$, we have $x > z$.

(b) $x \in {}^{\oplus}A^{s-1}$, $y \in {}^{\frown}A^s$:

$$y > z \xrightarrow{\text{lemma 7.2}} z \in {}^{\ominus}A^{s-1} \text{ ⓞⓡ } {}^{+}A^s.$$

According to A^t to which z belongs, we consider two cases:

(b_1) $z \in {}^{\ominus}A^{s-1}$:

$$x > y \rightarrow |xyy| \; ; \; y > z \rightarrow |yzz|.$$

$$x \in {}^{\oplus}A^{s-1}, \; y \in {}^{\frown}A^s, \; z \in {}^{\ominus}A^{s-1},$$

Hence, by Lemma 6.12, $\{xyz\}$ is contained in $R[\mathfrak{B}]$. ....(7.15)

So, $$|xyz| \xRightarrow{\text{lemma 3.2}} |xzz|$$

Hence from (V) in the definition of binary relation and $x \in {}^{\oplus}A^{s-1}$, $z \in {}^{\ominus}A^{s-1}$, we have $x > z$,

(b_2) $z \in {}^{+}A^{s}$:

$$y > z \ (s=\text{even}) \xrightarrow{\text{def. (VI)}} |{}^{+}a^{s-1}yz| \ ; \ x > y \to |xyy|.$$

$$|{}^{+}a^{s-1}yz| \cdot |xyy| \xRightarrow{\text{lemma 6.4 dual}} |xyz| \xRightarrow{\text{lemma 3.2}} |xzz|. \qquad ....(7.16)$$

Hence from (III) in the definition of binary relation and $x \in {}^{\oplus}A^{s-1}$, $z \in {}^{+}A^{s}$, we have $x > z$.

Case 3. $y \in {}^{-}A^{1}$:

$$y > z \xrightarrow{\text{lemma 7.3}} z \in {}^{-}A^{1}; \quad \text{Hence } y, z \in {}^{-}A^{1}.$$

So, from $y > z$ we have $|a_{\omega}yz|$. ....(7.17)

On the other hand,

$$x > y \xrightarrow{\text{lemma 7.3}} x \in {}^{-}A^{1} \text{ (or) } a_{\omega} \text{ (or) } {}^{+}A^{1} \text{ (or) } {}^{-}A^{2}.$$

According to ${}^{-}A^{1}$, a_{ω}, ${}^{+}A^{1}$, ${}^{-}A^{2}$ to which x belongs, we consider about the four cases:

(a) $x \in {}^{-}A^{1}$, $y, z \in {}^{-}A^{1}$:

$$x > y \xrightarrow{\text{def. (VI)}} |a_{\omega}xy|.$$

And from (7.17) we have $|a_{\omega}xy|$ and $|a_{\omega}yz|$.

So, $$|a_{\omega}xy| \cdot |a_{\omega}yz| \xRightarrow{\text{lemma 3.7}} |a_{\omega}xyz|. \qquad ....(7.18)$$

Hence, from (VI) in the definition of binary relation and $x, z \in {}^{-}A^{1}$, we have $x > z$.

(b) $x = a_{\omega}$, $y, z \in {}^{-}A^{1}$:

From (I) in the definition of binary relation, we have $x > z$.

Furthermore, from (7.17) $|a_{\omega}yz| \equiv |xyz| \in R$. ....(7.19)

(c) $x \in {}^{+}A^{1}$, $y, z \in {}^{-}A^{1}$:

From (I) in the definition of binary relation, we have $x > z$.

Furthermore, $x \in {}^{+}A^{1}$, $y \in {}^{-}A^{1} \xRightarrow{(5.4)} |xa_{\omega}y|$.

And from (7.17), we have $|a_{\omega}yz|$ and $|xa_{\omega}y|$.

$$|xa_{\omega}y| \cdot |a_{\omega}yz| \xRightarrow{\text{B5}} |xyz|. \qquad ....(7.20)$$

(d) $x \in {}^{-}A^{2}$, $y, z \in {}^{-}A^{1}$:

From (7.10) and (7.11) we have $|a_{\omega}yz|$ and $|xyy|$.

So, $|a_\omega yz| \cdot |xyy| \xrightarrow{\text{lemma 6.6}} |xyz| \xrightarrow{\text{lemma 3.2}} |xzz|$. ....(7.21)

Hence, from (II) in the definition of binary relation and $x \in {}^-A^2$, $z \in {}^-A^1$, we have $x > z$.

Case 4. $y \in {}^-A^s$ (s is odd and ≥ 3);

By exchanging x and z and by conversing the direction in the proof of case 2, we can prove this in the similar way.

Case 5. $y \in {}^+A^s$ (s is even and ≥ 2):

By exchanging x and z and the sign respectively, and by conversing the direction in the proof of Case 2, we can prove this case in the similar way.

Case 6. $y \in {}^+A^s$ (s is odd):

By exchanging the sign in the proof of Case 2 ($s \geq 3$), we can prove this case in the similar way. And when $s=1$, by exchanging x and z and sign respectively, and by conversing the direction in the proof of Case 3, we can prove this case in the similar way.

Therefore, in all cases it is proved that the binary relation satisfies P*3. From the above discussion, we can conclude that the binary relation which is defined in [1] satisfies P*1, P*2 and P*3, and so is a patial order.

Moreover, we stated in the proof of each case of P*3 that if $x > y$ and $y > z$, then $\{xyz\}$ is contained in $R[\mathfrak{B}]$. ....(7.22)

[3] Proof of the fact that $x \geq y \geq z$ or $x \leq y \leq z \rightarrow |xyz|$.

We consider about the case when $x \geq y \geq z$. In this case we consider about the following four cases:

Case 1. $x > y > z$; Case 2. $x = y$, $y > z$

Case 3. $x > y$, $y = z$; Case 4. $x = y = z$

Case 1. $x > y > z$:

If $x > y > z$, then from (7.22), $\{xyz\}$ is contained in $R[\mathfrak{B}]$.

Case 2. $x = y$, $y > z$:

$$y > z \xrightarrow{\text{lemma 7.4}} |yyz|, \text{ hence } |yyz| \equiv \{xyz\} \in R[\mathfrak{B}].$$

Case 3. $x > y$, $y = z$:

$$x > y \xrightarrow{\text{lemma 7.4}} |xyy|, \text{ hence } |xyy| \equiv \{xyz\} \in R[\mathfrak{B}].$$

Case 4. $x = y = z$:

From B1, $\{xxx\}$ is contained in $R[\mathfrak{B}]$, so $\{xyz\}$ is contained in $R[\mathfrak{B}]$.

Therefore, in all cases, $x \geq y \geq z \rightarrow |xyz|$.

And we can prove the case when $x \leq y \leq z$ in the similar way. Thus, we have $x \geq y \geq z$ or $x \leq y \leq z \rightarrow |xyz|$.

[4] Proof of the fact that $|xyz| \to x \geqq y \geqq z$ or $x \leqq y \leqq z$.

$$|xyz| \xrightarrow{\text{lemma 3.2}} (|xyy|,\ |yzz| \text{ and } |xzz|).$$
$$|xyy| \xRightarrow{\text{lemma 7.4}} x > y \ \text{(or)}\ x = y \ \text{(or)}\ x < y.$$
$$|yzz| \xRightarrow{\text{lemma 7.4}} y > z \ \text{(or)}\ y = z \ \text{(or)}\ y < z.$$

By combining the above cases, we see, from Lemma 7.4, that only one case among the following nine may occur:

$$\left.\begin{array}{lll} (1)\ x>y\cdot y>z\,; & (4)\ x=y\cdot y>z\,; & (7)\ x<y\cdot y>z, \\ (2)\ x>y\cdot y=z\,; & (5)\ x=y\cdot y=z\,; & (8)\ x<y\cdot y=z, \\ (3)\ x>y\cdot y<z\,; & (6)\ x=y\cdot y<z\,; & (9)\ x<y\cdot y<z. \end{array}\right\} \qquad \ldots\ldots(7.23)$$

In (7.23), (3) $x>y$ and $y<z$ and (7) $x<y$ and $y>z$ may not occur.

For; (3) $x>y\cdot y<z$:

$$|xyz| \xrightarrow{\text{lemma 3.2}} |xzz| \xRightarrow{\text{lemma 7.4}} x>z \ \text{(or)}\ x=z \ \text{(or)}\ x<z. \qquad \ldots\ldots(7.24)$$

We consider about the three cases according to the relation of x and z.

(a) $x>z$:

$$x>z \text{ and } z>y \xRightarrow{(7.22)} |xzy|.$$

So, $|xyz|\cdot|xzy| \xRightarrow{\text{P4}} y=z$. This contradicts $y<z$. Hence, $x>z$ may not occur.

(b) $x=z$:

$|xyz| \equiv |xyx| \xRightarrow{\text{lemma 3.1}} x=y$. This contradicts $x>y$. Hence $x=z$ may not occur.

(c) $x<z$:

$$z>x \text{ and } x>y \xRightarrow{(7.22)} |zxy|\,;\ |xyz| \xRightarrow{\text{P3}} |zyx|.$$

Hence, $|zyx|$ and $|zxy| \xRightarrow{\text{P4}} x=y$. This contradicts to $x>y$. So, $x<z$ may not occur.

Therefore, in this case no order between x and z exists. But this fact contradicts (7.24).

Thus, Case (3): $x>y$ and $y<z$ may not occur.

(7) $x<y$ and $y>z$:

By conversing the order in the proof of the case (3), we can prove similarly that the case (7) may not occur.

Therefore, if $\{xyz\}$ is contained in $R[\mathfrak{B}]$, then $x \geqq y$ and $y \geqq z$ or $x \leqq y$ and $y \leqq z$. And, as this order satisfies P3, then we have

$$|xyz| \to x \geqq y \geqq z \quad \text{or} \quad x \leqq y \leqq z.$$

From the above discussion, we can conclude that when A is connected, conditions B1, B2, B3, B4, B5, B6 and B7 are sufficient for betweenness of set A.

By conversing the order in a partially ordered set we have a partially ordered set (which is dual to proceed). So, in A_μ we may introduce at least two kinds of order from $R_\mu[\mathfrak{B}]$ when $R_\mu{}^*[\mathfrak{B}]$ is not the only one element.

(IX) In general case when $A=\sum_\mu A_\mu$, we do not define the order of the element of A_μ and that of A_ν $(\mu\neq\nu)$. ....(7.25)
And, we define the binary relation in each A_μ as in [1], then $A[\rho]\equiv\sum_\mu A_\mu[\rho_\mu]$ becomes a partially ordered set. For, $A[\rho]$ is a direct sum of partially ordered sets $A_\mu[\rho_\mu]$ (from § 4 and (5.1)). Moreover from the Lemma 4.4, (5.1) and (5.2) if $x\geqq y\geqq z$ or $x\leqq y\leqq z$ in $A[\rho]$, then $\{xyz\}$ is contained in $R[\mathfrak{B}]$, conversely if $\{xyz\}$ is contained in $R[\mathfrak{B}]$, then $x\geqq y\geqq z$ or $x\leqq y\leqq z$ in $A[\rho]$.

Therefore, in general case, conditions B1, B2, B3, B4, B5, B6 and B7 are also sufficient for betweenness of set A.

From the above result and Theorem 1 and 2, we have:

Theorem 4. *The system of conditions $\mathfrak{B}$ (≡B1, B2, B3, B4, B5, B6 and B7) is necessary and sufficient for betweenness of set A. And these conditions are mutually independent.*

§ 8. Uniqueness of order.

In this section, we shall prove that if the binary relation which satisfies P1, P2 and P3, i.e. partial order, may be introduced in A from $R[\mathfrak{B}]$, then this order is unique.[1]

We perform the decomposition of A into connected systems: $A=\sum_\mu A_\mu$.

First, we shall prove that:

If the partial order can be introduced in A_μ from $R_\mu[\mathfrak{B}]$, then there are only two kinds of order, and the one is dual to the other.

Proof. We assume that a partial order ρ is introduced in A_μ from $R_\mu[\mathfrak{B}]$. We take any element $a_\omega(\mu)$ of A_μ, and fix it. And similarly as in § 5, by means of distance for $R_\mu[\mathfrak{B}]$ we decompose the A_μ in the form

$$A_\mu = a_\omega(\mu)+\sum_{s=1}({}^+A^s+{}^-A^s)^{2)}.$$

1) Equivalent partially ordered sets are regarded as the same.

2) In this section, putting off the suffix μ in ${}^+A^s_\mu$ (or ${}^-A^s_\mu$) we write ${}^+A^s$ (or ${}^-A^s$) for ${}^+A^s_\mu$ (or ${}^-A^s_\mu$) in the case when any confusion may not occur by it.

From (5.4), when ${}^{+}A^{1} \not\equiv 0$,

$$ {}^{-}a^{1}_{0} \in {}^{-}A^{1} \text{ and } {}^{+}a^{1}_{0} \in {}^{+}A^{1} \rightarrow |{}^{-}a^{1}_{0}\, a_{\omega}\, {}^{+}a^{1}_{0}|. $$

When ${}^{+}A^{1} \equiv 0$, we have only to consider the proof concerning to a_{ω} and ${}^{-}A^{i}$ in the following proof.

Since a partial order ρ is introduced in A_{μ},

$$ |{}^{-}a^{1}_{0}\, a_{\omega}\, {}^{+}a^{1}_{0}| \Rightarrow {}^{-}a^{1}_{0} \leqq a_{\omega} \leqq {}^{+}a^{1}_{0} \text{ or } {}^{-}a^{1}_{0} \geqq a_{\omega} \geqq {}^{+}a^{1}_{0}. $$

And ${}^{-}a^{1}_{0}$, a_{ω} and ${}^{+}a^{1}_{0}$ are distinct, so we have

$$ {}^{-}a^{1}_{0} < a_{\omega} < {}^{+}a^{1}_{0} \text{ or } {}^{-}a^{1}_{0} > a_{\omega} > {}^{+}a^{1}_{0}. $$

Then we consider about two cases:

Case 1. ${}^{-}a^{1}_{0} < a_{\omega} < {}^{+}a^{1}_{0}$. Case 2. ${}^{-}a^{1}_{0} > a_{\omega} > {}^{+}a^{1}_{0}$.

Case 1 ${}^{-}a^{1}_{0} < a_{\omega} < {}^{+}a^{1}_{0}$.

For any element ${}^{-}a^{1} \in {}^{-}A^{1}$ and ${}^{+}a^{1} \in {}^{+}A^{1}$, from (5.4), we have that $\{{}^{-}a^{1}_{0}\, a_{\omega}\, {}^{+}a^{1}\}$ and $\{{}^{-}a^{1}\, a_{\omega}\, {}^{+}a^{1}_{0}\}$ are contained in $R[\mathfrak{B}]$.

So, $|{}^{-}a^{1}_{0}\, a_{\omega}\, {}^{+}a^{1}|$ and ${}^{-}a^{1}_{0} < a_{\omega} \Rightarrow a_{\omega} < {}^{+}a^{1}$,

and $|{}^{-}a^{1}\, a_{\omega}\, {}^{+}a^{1}_{0}|$ and $a_{\omega} < {}^{+}a^{1}_{0} \Rightarrow {}^{-}a^{1} < a_{\omega}$.

Hence, ${}^{-}a^{1} < a_{\omega} < {}^{+}a^{1}$ i.e.

$$ {}^{-}A^{1} < a_{\omega} < {}^{+}A^{1}. \qquad \text{.........(8.1)} $$

Next, by the mathematical induction we show how the order for ${}^{-}A^{s-1}$ and ${}^{-}A^{s}$ may be. First, we consider about the order for ${}^{-}A^{1}$ and ${}^{-}A^{2}$. When ${}^{-}a^{2} \in {}^{-}A^{2}$, there exists at least one ${}^{-}a^{1} (\in {}^{-}A^{1})$ such that $\{{}^{-}a^{1}\, {}^{-}a^{2}\, {}^{-}a^{2}\}$ is contained in $R[\mathfrak{B}]$ from the definition of ${}^{-}A^{2}$. Let ${}^{-}a^{1}$ and ${}^{-}a^{2}$ be element of ${}^{-}A^{1}$ and of ${}^{-}A^{2}$ respectively such that $\{{}^{-}a^{1}\, {}^{-}a^{2}\, {}^{-}a^{2}\}$ is contained in $R_{\mu}[\mathfrak{B}]$.

So, $|{}^{-}a^{1}\, {}^{-}a^{2}\, {}^{-}a^{2}| \Rightarrow {}^{-}a^{1} < {}^{-}a^{2}$ ⓞⓡ ${}^{-}a^{1} > {}^{-}a^{2}$. (with respect to the order introducd from $R_{\mu}[\mathfrak{B}]$)

Only the case ${}^{-}a^{1} < {}^{-}a^{2}$ may occur among them.

For: Let us suppose ${}^{-}a^{1} > {}^{-}a^{2}$.

From (8.1) and ${}^{-}a^{1} > {}^{-}a^{2}$, we have $a_{\omega} > {}^{-}a^{1} > {}^{-}a^{2}$.

$$ a_{\omega} > {}^{-}a^{1} > {}^{-}a^{2} \Rightarrow |a_{\omega}\, {}^{-}a^{1}\, {}^{-}a^{2}|. $$

This contradicts Lemma 4.3, hence ${}^{-}a^{1} > {}^{-}a^{2}$ may not occur.

Therefore $|{}^{-}a^{1}\, {}^{-}a^{2}\, {}^{-}a^{2}| \Rightarrow {}^{-}a^{1} < {}^{-}a^{2}$; i.e. it must be

$$ {}^{-}A^{1} \otimes {}^{-}A^{2} \qquad \text{.........(8.2)} $$

Assuming ${}^{-}A^{s-1} \otimes {}^{-}A^{s}$ (s is even), we shall prove ${}^{-}A^{s} \otimes {}^{-}A^{s+1}$.

For the element ${}^{-}a^{s+1} \in {}^{-}A^{s+1}$, there exists at least one element ${}^{-}a^{s}$ for

which $\{^-a^s\,^-a^{s+1}\,^-a^{s+1}\}$ is contained in $R[\mathfrak{B}]$ from the definition of $^-A^{s+1}$.[1)]
......(8.3)

Let $^-a^s \in {}^-A^s$, $^-a^{s+1} \in {}^-A^{s+1}$ be two elements such that $\{^-a^s\,^-a^{s+1}\,^-a^{s+1}\}$ is contained in $R[\mathfrak{B}]$.

$$|^-a^s\,^-a^{s+1}\,^-a^{s+1}| \Rightarrow {}^-a^s < {}^-a^{s+1} \text{ (or) } {}^-a^s > {}^-a^{s+1}.$$

We shall prove that only the case $^-a^s > {}^-a^{s+1}$ may occur among them.
For; Suppose $^-a^s < {}^-a^{s+1}$.
From the assumption: $^-A^{s-1} \lessdot {}^-A^s$ (s is even) and (8.3), there exists at least one element $^-a^{s-1}$ ($\in {}^-A^{s-1}$) such that $^-a^{s-1} < {}^-a^s$. Therefore we have $^-a^{s-1} < {}^-a^s < {}^-a^{s+1}$, so $|^-a^{s-1}\,a^s\,^-a^{s+1}|$. This contradicts Lemma 4.3. Hence $^-a^s < {}^-a^{s+1}$ may not occur. Therefore $^-a^s > {}^-a^{s+1}$, i.e. it must be

$$^-A^s \gtrdot {}^-A^{s+1} \ (s\text{=even}) \qquad \text{.........(8.4)}$$

By the same manner as the above, we have:

$$^-A^{t-1} \gtrdot {}^-A^t \ (t\text{=odd}) \rightarrow {}^-A^t \lessdot {}^-A^{t+1}. \qquad \text{.........(8.4)}'$$

From the above results, the order for $^-A^i$ (i=1, 2, ...) must be as follows:

$$^-A^1 \lessdot {}^-A^2 \gtrdot \ldots\ldots \gtrdot {}^-A^{2m-1} \lessdot {}^-A^{2m} \gtrdot {}^-A^{2m+1} \lessdot \ldots\ldots \qquad \text{...(8.5)}$$

By the same process as the above, we may conclude that the order for $^+A^i$ (i=1, 2, ...) must be as follows:

$$^+A^1 \gtrdot {}^+A^2 \lessdot \ldots\ldots \lessdot {}^+A^{2l-1} \gtrdot {}^+A^{2l} \lessdot {}^+A^{2l+1} \gtrdot \ldots\ldots \qquad \text{...(8.6)}$$

Next, we consider about the order for elements of $^-A^s$ (s is fixed).

Let $^-a_1^s$ and $^-a_2^s$ be two elements of $^-A^s$ such that $\{^-a_1^s\,^-a_2^s\,^-a_2^s\}$ is contained in $R[\mathfrak{B}]$.

$$|^-a_1^s\,^-a_2^s\,^-a_2^s| \xRightarrow{\text{lemma 6.5}} |^-a^{s-1}\,^-a_1^s\,^-a_2^s| \text{ (or) } |^-a_1^s\,^-a_2^s\,^-a^{s-1}|.$$

So, by (8.5) and (8.6) the order for $^-a_1^s$ and $^-a_2^s$ must be as follows:

When s is even $\quad |^-a^{s-1}\,^-a_1^s\,^-a_2^s| \rightarrow {}^-a_1^s < {}^-a_2^s$ ....(8.7)
When s is odd and $s \geqq 3$ $\quad |^-a^{s-1}\,^-a_1^s\,^-a_2^s| \rightarrow {}^-a_1^s > {}^-a_2^s$ ....(8.8)
(for s=1, see (8.1) considering $^-a^{s-1}$ as a_ω).

Similarly we have that the order for elements of $^+A^s$ must be dual to to the order in (8.7) and (8.8). ....(8.9)

Furthermore, we consider about the order for $^\ominus A^s$ and $^\oplus A^s$.

Let $^\ominus a^s$ and $^\oplus a^s$ be two elements such that $\{^\ominus a^s\,^\oplus a^s\,^\oplus a^s\}$ is contained in $R[\mathfrak{B}]$.

1) By this fact (8.5): $^-A^s \gtrdot {}^-A^{s+1}$ is not trivial.

$$|{}^{\ominus}a^s\ {}^{\oplus}a^s\ {}^{\oplus}a^s| \rightarrow {}^{\ominus}a^s > {}^{\oplus}a^s \quad \textcircled{or} \quad {}^{\ominus}a^s < {}^{\oplus}a^s$$

We shall prove that only the case ${}^{\ominus}a^s > {}^{\oplus}a^s$ (for s is even) may occur among them.

Suppose $${}^{\ominus}a^s < {}^{\oplus}a^s \qquad \text{.........(i)}$$

From (8.3), we have that there exists one element ${}^{-}a^{s-1}$ such that $\{{}^{-}a^{s-1}\ {}^{\ominus}a^s\ {}^{\ominus}a^s\}$ is contained in $R[\mathfrak{B}]$. From (8.5), we have that

$$\{{}^{-}a^{s-1}\ {}^{\ominus}a^s\ {}^{\ominus}a^s\} \in R[\mathfrak{B}], \text{ so } |{}^{-}a^{s-1}\ {}^{\ominus}a^s\ {}^{\ominus}a^s| \rightarrow {}^{-}a^{s-1} < {}^{\ominus}a^s. \qquad \text{.........(ii)}$$

From (i) and (ii), $\{{}^{-}a^{s-1}\ {}^{\ominus}a^s\ {}^{\oplus}a^s\} \in R[\mathfrak{B}]$.

So, $|{}^{-}a^{s-1}\ {}^{\ominus}a^s\ {}^{\oplus}a^s| \rightarrow |{}^{-}a^{s-1}\ {}^{\oplus}a^s\ {}^{\oplus}a^s|$, hence from the definition of ${}^{-}A^s$ we have ${}^{\oplus}a^s \in {}^{-}A^s$.

This contradicts ${}^{\oplus}a^s \in {}^{+}A^s$, so ${}^{-}a^s < {}^{\oplus}a^s$ (s is even) may not occur.

Hence, it must be that ${}^{-}a^s > {}^{\oplus}a^s$, i.e.

$${}^{\ominus}A^s \textcircled{>} {}^{\oplus}A^s \quad (\text{for } s = \text{even}). \qquad \text{.........(8.10)}$$

Similarly, it must be that

$${}^{\ominus}A^s \textcircled{<} {}^{\oplus}A^s \quad (\text{for } s = \text{odd.}) \qquad \text{.........(8.11)}$$

Moreover, by the Lemmas 4.3 and 6.9, the A^t which may have the relation with ${}^{-}A^s$ is ${}^{\ominus}A^{s-1}$, ${}^{\oplus}A^{s-1}$, ${}^{\oplus}A^s$, ${}^{-}A^s$ and ${}^{-}A^{s+1}$. So, the above discussions exhaust all cases which must be considered.

From the above discussions, we conclude that for the case 1 if a partial order may be introduced by any manner in A from $R[\mathfrak{B}]$, then the order must satisfy the conditions (8.1), (8.5), (8.6), (8.7), (8.8), (8.9), (8.10) and (8.11). (This system of conditions (8.1)–(8.11) is the same as the system of (I) (II) (III) (V) (VI) (VII) in the definition of order in the previous section.)

Case 2. ${}^{-}a^1{}_0 > a_\omega > {}^{+}a^1{}_0$

For this case, similarly, the conditions (8.1)′......(8.11)′ which are obtained by conversing the order in (8.1)—(8.11). So, in this case: if a partial order may be introduced in A_μ, then the order must be dual to that in Case 1.

Thus, we have:

Theorem 5. *The partial order which may be introduced by any manner in A_μ from $R_\mu[\mathfrak{B}]$, must be that obtained in* § 7 *or its dual.*

The case when $R[\mathfrak{B}]$ is not connected: $A[R] = \sum_\mu A_\mu[R]$.

There are no elements x and y such that $x \in A_\mu$, $y \in A_\nu$ and $\| x, y, z \| \in R[\mathfrak{B}]$ as we have seen in § 4 and (5.2). So, in partial order which may be introduced in A from $R[\mathfrak{B}]$, there must be no order for elements $x \in A_\mu$

and $y \in A_\nu$ ($\mu \neq \nu$). (This condition is (IX) in § 7.)
And the orders in A_μ and A_ν are independent of each other.

So, from the above and Theorem 5, we have:

Theorem 6. *We can introduce one and only one order in A from* $R[\mathfrak{B}]$ *except for equivalent.*

Equivalent orders in connected essential set[1] $A_\mu[R]$ *are, clearly, some one and its dual,* so we have:

Corollary: *We may introduce just* 2^c*=kinds of orders in A from* $R[\mathfrak{B}]$ *when c is power of the index set* $\{\kappa\}$ *of essential connected sets* $A_\kappa[R]$ *in the decomposition:* $A[R]=\sum_\mu A_\mu[R]$.

§ 9. Other complete systems and remark.

We shall investigate the systems $\mathfrak{B}'$ and $\mathfrak{B}''$ which are equivalent to the system $\mathfrak{B}$ (≡B1, B2, B3, B4, B5, B6, B7).

I) The system $\mathfrak{B}$ is equivalent to the system $\mathfrak{B}'$:[2] ....(9.1)

$$\mathfrak{B}' \begin{cases} \text{B1.} & |aaa| \quad \text{for all } a. \\ \text{B2.} & |abx| \rightarrow |aab|. \\ \text{B3.} & |axb| \rightarrow |bxa|. \\ \text{B4.} & |axb| \cdot |abx| \rightarrow x=b. \\ \text{B5.} & |axb| \cdot |xby| \, x \neq b \rightarrow |aby|. \\ \text{B6}'. & |a_1a_2x_1| \cdot |a_2a_3x_2| \ldots\ldots |a_{2n+1}a_1x_{2n+1}| \rightarrow |a_ia_{i+1}a_{i+2}|. \\ \text{B7}'. & |abc| \cdot |bdx| \rightarrow |abd| \text{ (or) } |dbc|. \end{cases}$$

For, from the Lemmas 3.2 and 3.3 we can easily obtain B6′ and B7′ from $\mathfrak{B}$ so that $\mathfrak{B} \supset \mathfrak{B}'$; and conversely the system $\mathfrak{B}$ follows from $\mathfrak{B}'$ by means of $\mathfrak{B} \subset \mathfrak{B}'$.

The condition B4″: $|xab|$ and $|xba|$ are not compatible, is equivalent to B4 since the former is the opposition of the latter.[3] So by the similar way we have that:

II) In the system $\mathfrak{B}$ the conditions B4, B6 and B7 may be replaced by B4″, B6″ and B7″ respectively. ....(9.2)

$$\mathfrak{X} \begin{cases} \text{B4}''. & |xab| \text{ and } |xba| \text{ are not compatible.} \\ \text{B6}''. & \| a_1, a_2, x_1 \| \; \| a_2, a_3, x_2 \| \ldots\ldots \| a_{2n+1}, a_1, x_{2n+1} \| \rightarrow |a_ia_{i+1}a_{i+2}|. \\ \text{B7}''. & |abc| \cdot \| b, d, x \| \rightarrow |abd| \text{ (or) } |dbc|. \end{cases}$$

1) Essential connected set means the set centaining more than one element.

2) $|xyz|$ means that $\{xyz\}$ belongs to $R[Z]$ in which Z is a system of conditions considered in that place.

3) If we write the condition B4″ as $|xef| \cdot |xfe|$ being not compatible, we can easily see that fact.

III) And the system $\mathfrak{B}$ is equivalent to the system $\mathfrak{B}''$: ......(9.3)

$$\mathfrak{B}''\begin{cases} \text{B1.} & |aaa| \text{ for all } a. \\ \text{B2}''. & \| x, y, z \| \to |xyy|. \\ \text{B3.} & |axb| \to |bxa|. \\ \text{B4}''. & |xab| \text{ and } |xba| \text{ are not compatible.} \\ \text{B6}''. & \| a_1, a_2, x_1 \| \cdot \| a_2 a_3 x_2 \| \ldots\ldots \| a_{2n+1}, a_1, x_{2n+1} \| \to |a_i a_{i+1} a_{i+2}|. \\ \text{B7}''. & |abc| \cdot \| b, d, x \| \to |abd| \text{ (or) } |dbc|. \end{cases}$$

Proof. Since the condition B2″ is the same as Lemma 3.2 it follows that $\mathfrak{B} \supset \mathfrak{B}''$ from (9.2). And conversely, B2″ contains B2 and B5 follows from $\mathfrak{B}''$. For;

(i) In the case where x, b and y are all distinct we have $|axb| \cdot |xby| \to |xby| \cdot \| a, x, b \| \overset{B7''}{\Longrightarrow} |aby|$ (or) $|xba|$, on the other hand $|xba|$ (i.e. $|abx|$ from B3) and $|axb|\cdot x \neq b$ are not compatible from B4″. So we have $|axb| \cdot |xby| x \neq b \to |aby|$.

(ii) In the case $x=y$ we have $|xby| \to |xbx| \overset{\text{lemma 3.1}}{\Longrightarrow} x=b$, hence the assumption of B5 may not occur.

(iii) In the case $y=b$, we have $|axb| \to \| a, x, b \| \overset{B2''}{\Longrightarrow} |abb|$, so $|axb| \cdot |xby| x \neq b \to |aby|$.

Hence we have $\mathfrak{B}'' \supset$ B5. Therefore $\mathfrak{B}'' \supset \mathfrak{B}$. So we have $\mathfrak{B} = \mathfrak{B}''$.

Furthermore from the examples 2, 4, 6, 7 and the following examples 10 and 11, the conditions in $\mathfrak{B}''$ are mutually independent.

Example 10. Set A consists of a and b, set $R : \emptyset$.

Example 11. Set A consists of a and b, set $R : \{abb\}\{baa\}\{aaa\}\{bbb\}$.

From (9.1), (9.3) and Theorem 4 we have *the systems* $\mathfrak{B}'$ *is complete conditions*[1] *for betweenness of set A and also* $\mathfrak{B}''$. ....(9.4)

Remark:

In §1--§8 we have investigated the system of conditions for betweenness on the set A in the case where A is given at first. The betweenness in this case is called *betweenness of set A*.

As the system of conditions of betweenness from the standpoint of discussing only the character of interval without considering the set A at the start, i.e. the standpoint of characterizing the betweenness for some set (where only the conditions for R and the order of elements in R^* are discussed), we have the following:

1) Complete conditions mean the conditions which are necessary, sufficient and mutually independent.

$$\mathfrak{G}\cdot\mathfrak{B}\begin{cases} \text{G}\cdot\text{B}(1\,2)'. & |xyz|\rightarrow|xxy|. \\ \text{B3.} & |axb|\rightarrow|bxa|. \\ \text{B4.} & |axb|\cdot|abx|\rightarrow x=b. \\ \text{B5.} & |axb|\cdot|xby|x\neq b\rightarrow|aby|. \\ \text{B6.} & |a_1a_2a_2|\cdot|a_2a_3a_3|\ldots\ldots|a_{2n+1}a_1a_1|\rightarrow|a_ia_{i+1}a_{i+2}|. \\ \text{B7.} & |abc|\cdot|bdd|\rightarrow|abd| \text{ (or) } |dbc|. \end{cases}$$

This system is called *the system of conditions for betweenness.*

Theorem 7. *This system of conditions characterizes the betweenness of some set.*

Proof. First we prove that in R^* the system $\mathfrak{G}\cdot\mathfrak{B}$ is equivalent to the system $\mathfrak{B}$. The system $\mathfrak{B}$ follows from the system $\mathfrak{G}\cdot\mathfrak{B}$. For, $|xyz| \xrightarrow{\text{G.B(12)}'} |xxy| \xRightarrow{\text{G.B(12)}'} |xxx|$. And $|xxy| \overset{\text{B3}}{\Rightarrow} |yxx|$, $|xyz| \overset{\text{B3}}{\Rightarrow} |zyx|$. From $|yxx|$ and $|zyx|$ we have $|yyy|$ and $|zzz|$ by the similar way as the above. Hence for any element x of R^* there exists $|xxx|$. The condition B2 is clear from G·B(12)'. Therefore we have $\mathfrak{G}\cdot\mathfrak{B}\rightarrow\mathfrak{B}$.

Conversely, the system $\mathfrak{G}\cdot\mathfrak{B}$ follows from the system $\mathfrak{B}$. The condition G·B(12)' follows from the system $\mathfrak{B}$. For, (i) $|xyz|x=y \xrightarrow{\text{B1}} |xxx| \equiv |xxy|$. (ii) $|xyz|x\neq y \xrightarrow{\text{B2}} |xxy|$. So we have $\mathfrak{B}\rightarrow\mathfrak{G}\cdot\mathfrak{B}$. Therefore the system $\mathfrak{G}\cdot\mathfrak{B}$ is equivalent to the system $\mathfrak{B}$.

Thus, from Theorem 4, the system of conditions $\mathfrak{G}\cdot\mathfrak{B}$ is a complete system for betweenness (of R^*). So we have theorem.

From (9.4), *the system of conditions* $\mathfrak{G}\cdot\mathfrak{B}''$ ($\equiv$B2'', B3, B4'', B6'', B7'') *is also a complete system ofconditions for betweenness.*

§ 10. On the axioms which have been discussed by G. Birkhoff, E. Pitcher and M. F. Smiley.

In this section, we shall consider the conditions which have been discussed by G. Birkhoff, E. Pitcher and M. F. Smiley.

The necessary conditions for betweenness stated by G. Birkhoff[1] are as follows:

(1) (axb) implies (bxa).
(2) (axb) and (abx) imply $x=b$.
(3) (axb) and (ayx) imply (ayb).
(4) (axb), (xby) and $x\neq b$ imply (aby).
(5) (abc) and (acd) imply (bcd).

1) G. Birkhoff: Lattice Theory, Am. Math. Soc. Co. Pub. (1948).

And, E. Pitcher and M. F. Smiley obtained the three conditions which are independent of (1), (2), (3), (4) and (5),[1]

(6) (abc), (adc) and (bxd) imply (axc).

(7) (abc), (abd) and (cxd) imply (abx).

(8) (abc), (abd) and (xbc) imply (xbd).

Adding to (1)—(8), the following condition is necessary.

(9) (abc), (bcd), (abx) and (xcd) imply (bxc).

The above conditions (1), (2) and (4) are the conditions B3, B4 and B5 respectively: (1)≡B3, (2)≡B4 and (4)≡B5. ....(9.1)

We shall prove that the condition (9) is necessary and that the nine conditions (1)—(9) are independent of each other.

Proof. If (abc), (bcd), (abx) and (xcd), then only one case among the following two may occur, because a, b, c and d are distinct.

(1) $a>b>c$, $b>c>d$, $a>b\geqq x$ and $x\geqq c>d$.

(2) $a<b<c$, $b<c<d$, $a<b\leqq x$ and $x\leqq c<d$

Then in the case (1) $b\geqq x\geqq c$ and in the case (2) $b\leqq x\leqq c$, so in all cases, (bxc) must be contained in $R[\mathfrak{B}]$.

Next, we shall prove that nine conditions (1)—(9) are mutually independent.

By the examples 3, 4 and the following examples 12, 13, 14, 15, 16, 17 and 18, each of which satisfies the eight of the nine conditions respectively, except one, we prove the above proposition.

Example	set A	system R	
12	a, b, c	$\{aab\}$ $\{baa\}$ $\{aca\}$	only (3) is not satisfied.
13	a,b,c,d	$\{abc\}$ $\{cba\}$ $\{bcd\}$ $\{dcb\}$	(4) 〃
14	a,b,c,d	$\{abc\}$ $\{cba\}$ $\{abd\}$ $\{dba\}$ $\{acd\}$ $\{dca\}$	(5) 〃
15	a, b, c d, e	$\{abc\}$ $\{cba\}$ $\{adc\}$ $\{cda\}$ $\{bed\}$ $\{deb\}$	(6) 〃
16	a, b, c d, e	$\{abc\}$ $\{cba\}$ $\{abd\}$ $\{dba\}$ $\{ced\}$ $\{dec\}$	(7) 〃
17	a, b, c d, e	$\{abc\}$ $\{cba\}$ $\{abd\}$ $\{dba\}$ $\{ebc\}$ $\{cbe\}$	(8) 〃
18	a, b, c d, e	$\{abc\}$ $\{cba\}$ $\{bcd\}$ $\{dcb\}$ $\{acd\}$ $\{dca\}$ $\{abd\}$ $\{dba\}$ $\{abe\}$ $\{eba\}$ $\{ecd\}$ $\{dce\}$	(9) 〃

1) E. Pitcher and M. F. Smiley, Transitivities of betweenness: Trans. Am. Math. Soc. (1942).

Now, we have the following result:

[1] The system of conditions (1), (2), ... , (9) follows from the system of conditions B1, B2, ... , B7.

For: since the system of conditions B1, B2, ... , B7 is necessary and sufficient for betweenness of set A, so the system (1), (2), ... , (9) must be obtained from the system of conditions B1, B2, ... , B7.

[2] Each condition of B1, B2, B6 and B7 is independent of the conditions (1), (2), ... , (9) and the remaining three conditions of B1, B2, B6 and B7.

By the examples 1, 2, 6 and 7 (page 183) we may show the above.

From the above discussion, we have the conclusion:

The nine conditions (1), (2), ... , (9) *are necessary and independent of each other, but are not sufficient for betweenness of set A.*

Chapter II

On betweenness of special cases

In this chapter, we shall investigate the betweenness of set A when the system $A[\rho]$ is the following special system: [1] bounded, [2] being connected and having center,[1] [3] having one extreme element, [4] having two extreme elements [5] simply ordered.

We shall obtain the following result:

The systems of axioms for these special cases are obtained by replacing merely B6[1], B6[2], B6[3], B6[4] and B6[5] respectively for B6 in the system of axioms in general case. (Cf. $\mathfrak{B}$[S. I], $\mathfrak{B}$[S. II], $\mathfrak{B}$[S. III] $\mathfrak{B}$[S. IV] and $\mathfrak{B}$[S. V]).

In this chapter, $|xyz|$ means that $\{xyz\}$ belongs to $R[Z]$ in which Z is a system of conditions considered in that place.

§1. Case I. Bounded.

In the decomposition of A: $A=\sum A_\mu=\sum\limits_{\mu}(\sum\limits_{s} A_\mu^s+a_\omega(\mu))$, we can easily see from the definition of A_μ^s

$$A_\mu^{t_1}=0 \rightarrow A_\mu^{t_2}=0 \ (t_2>t_1).$$

When, in particular, $A_\mu=\sum\limits_{s=1}^{m_\mu} A_\mu^s+a_\omega(\mu)$ (m_μ is finite), A_μ is said to be *bounded*. This is eqivalent to that the distance of any two elements in A_μ

1) Definition of center is different from that being used in Lattice.

is bounded, so that it is independent of choosing of $a_\omega(\mu)$. But, the value of m_μ depends on choosing of $a_\omega(\mu)$. The minimum value m_μ^0 of m_μ for all $a_\omega(\mu)(\in A_\mu)$ is said to be the *radius* of connected set $A_\mu[R]$, and an $a_\omega^0(\mu)$ for the m_μ^0 is said to be *pseudo center*, and specially the pseudo center for the radius one is said to be *center*. When the set of radii of connected sets A_μ in $A=\sum A_\mu$ is bounded, A is said to be *inner bounded*, and the maximal radius $A_\mu(\mu)$, is said to be *inner radius* of A.

When the inner radius of A is equal to r, in the proof of the sufficiency in § 7, Chapter I, the condition B6′ has been used for only $n \leqq r$[1]: i.e. $|a_1a_2x_1|\cdot|a_2a_3x_2|\ldots\ldots|a_{2n+1}a_1x_{2n+1}|\rightarrow|a_ia_{i+1}a_{i+2}|$ (for $n=1,2,\ldots,r$ and at least one i, $(1\leqq i\leqq 2n+1)$). We shall write this condition by $\overset{r}{\mathrm{B}}6\equiv$B6[I]. And by [I, examples 1, 2, 3, 4, 5, 6, 7, 8][2] (page 183) we can easily see that the conditions B1, B2, B3, B4, B5, $\overset{r}{\mathrm{B}}6$ and B7 are mutually independent.

Therefore, from [I, Theorem 4] we have:

Theorem 1. *When the inner radius of A is equal to r, then the system of conditions*: $\mathfrak{B}$[S. I]$\equiv$(*B1, B2, B3, B4, B5, $\overset{r}{B}6$, B7*) *is the complete condition for betweenness of set A, where*

$\overset{r}{B}6$: $|a_1a_2x_1|\cdot|a_2a_3x_2|\ldots\ldots|a_{2n+1}a_1x_{2n+1}|\rightarrow|a_ia_{i+1}a_{i+2}|$,
for $n=1,2,\ldots,r$ *and at least one* i $(1\leqq i\leqq 2n+1)$.

§ 2. Case II. Being connected and having center.

In particular, from the Theorem 1, the condition for betweenness of set A such that A is connected and has center, is the system: B0, B1, B2, B3, B4, B5, $\overset{1}{\mathrm{B}}6$, B7, where

B0: $|a_0xx|$ for all x and some one a_0,

$\overset{1}{\mathrm{B}}6$: $|a_1a_2x_1|\cdot|a_2a_3x_2|\cdot|a_3a_1x_3|\rightarrow\|a_1,a_2,a_3\|$.

By taking b as center a_0 in the following example and [I, examples 1, 2, 3, 4, 5, 6, 7] (page 183), each of which does not satisfy only one of the the conditions B0, ..., B7 respectively, we see that the conditions B1, B2, B3, B4, B5, $\overset{1}{\mathrm{B}}6$ and B7 are mutually independent.

Example 1. Set A consists of a and b, system R: $\{aaa\}\{bbb\}$
(only B0 is not satisfied)

1) When the inner radius of A is equal to r, the number of elements of cyclic path is at most $2r+1$.

2) [I, example 1] means the example 1 in Chapter I.

Next, we shall investigate the other complete conditions for betweenness of set A such that A is connected and has center.

Theorem 2. *The system of conditions:* $\mathfrak{B}[\text{S. II}]\equiv(B1, B2, B3, B4, B5, B6[2], B7)$ *is the complete condition for betweenness of set* A *such that* A *is connected and has center, where*

B6[2]: $\| x, y, z \| \rightarrow \| a_0, x, y \|$ *for some one element* a_0.

Proof. First, from B0, there exists $|a_0xx|$, $|a_0yy|$, so by the lemma 3.2

$$|a_0xx| \cdot \|x, y, z\| \cdot |yya_0| \overset{\overset{1}{B6}}{\Longrightarrow} \| a_0, x, y \| .$$

Hence condition B6[2] is necessary.

Conversely, we shall prove that conditions B0 and $\overset{1}{\text{B6}}$ follow from conditions B1, B2, B3, B4, B5, B6[2] and B7.

Putting $x=y$ in the condition B6[2], we have B0, and we shall prove that condition $\overset{1}{\text{B6}}$ follows.

$$|a_1a_2x_1| \xrightarrow{B6[2]} \| a_0, a_1, a_2 \| , \qquad \text{.........(2.1)}$$

$$|a_2a_3x_2| \xrightarrow{B6[2]} \| a_0, a_2, a_3 \| . \qquad \text{.........(2.2)}$$

$$|a_3a_1x_3| \xrightarrow{B6[2]} \| a_0, a_3, a_1 \| . \qquad \text{.........(2.3)}$$

Combining (2.1) and (2.2), we have the cases:

1) $|a_0a_1a_2| \cdot |a_0a_2a_3|$: $|a_0a_1a_2| \cdot |a_0a_2a_3| \xrightarrow{[\text{I, lemma 3.6}]} |a_1a_2a_3|$
2) $|a_0a_1a_2| \cdot |a_0a_3a_2|$: According to (2.3), we have the cases:
   (i) $|a_0a_3a_1|$: $|a_0a_3a_1| \cdot |a_0a_1a_2| \xrightarrow{[\text{I, lemma 3.6}]} |a_3a_1a_2|$.
   (ii) $|a_0a_1a_3|$: $|a_0a_1a_3| \cdot |a_0a_3a_2| \xrightarrow{[\text{I, lemma 3.6}]} |a_1a_3a_2|$.
   (iii) $|a_1a_0a_3|$: $|a_1a_0a_3| \cdot |a_0a_3a_2| \xrightarrow{B5} |a_1a_3a_2|$.
3) $|a_0a_1a_2| \cdot |a_2a_0a_3|$: $|a_2a_1a_0| \cdot |a_2a_0a_3| \xrightarrow{[\text{I, lemma 3.6}]} |a_1a_0a_3|$,
   $|a_2a_1a_0| \cdot |a_1a_0a_3| \xrightarrow{B3,\, B5} |a_2a_1a_3|$.
4) $|a_0a_2a_1|$: In this case, in the similar way as 1), 2) and 3), we can prove that it follows that $\| a_1, a_2, a_3 \| \in R[\mathfrak{B}[\text{S. II}]]$.
5) $|a_1a_0a_2| \cdot |a_0a_2a_3|$: $|a_1a_0a_2| \cdot |a_0a_2a_3| \xrightarrow{B5} |a_1a_2a_3|$.
6) $|a_1a_0a_2| \cdot |a_0a_3a_2|$: By the similar way as 3), we have $|a_1a_3a_2|$.
7) $|a_1a_0a_2| \cdot |a_2a_0a_3|$: According to (2.3), we have the following cases:
   (i) $|a_0a_3a_1|$: $|a_2a_0a_3| \cdot |a_0a_3a_1| \xrightarrow{B5} |a_2a_3a_1|$,
   (ii) $|a_0a_1a_3|$: $|a_2a_0a_1| \cdot |a_0a_1a_3| \xrightarrow{B5} |a_2a_1a_3|$,
   (iii) $|a_1a_0a_3|$: $|a_1a_0a_2| \cdot |a_2a_0a_3| \xrightarrow{B7} |a_1a_0a_3|$ (or) $|a_3a_0a_2|$.

But both $|a_1a_0a_3|$ and $|a_3a_0a_2|$ must be contained in $R[\mathfrak{B}[\text{S. II}]]$, so this is contradiction. Hence, this case (iii) may not occur. Thus we have

$\overset{1}{\mathrm{B}}6$. And [I, Lemma 3.6] follows from conditions B2, B3, B4 and B7 (page 185). So that, the conditions B1, B2, B3, B4, B5, B6[2] and B7 are sufficient.

By taking b as center a_0 in [I, Examples 1, 2, 3, 4, 5, 6, 7] (page 183), each of which does not satisfy only one of the conditions B1, B2, B3, B4, B5, B6[2] and B7 respectively, we see that these conditions are mutually independent. Thus we have Theorem 2.

§ 3. Case III. Having one extreme element.

In this case, A is connected and has center, and so we have:

The condition for betweenness of set A such that A has one extreme element, is the system B0, B0′, B1, B2, B3, B4, B5, $\overset{1}{\mathrm{B}}6$ and B7, where

$$\mathrm{B}0': \quad |xa_0y| \to x = a_0 \text{ or } y = a_0 \text{ for some one element } a_0;$$

and a_0 is the extreme element.

We see that the conditions B0, B0′, B1, B2, B3, B4, B5, $\overset{1}{\mathrm{B}}6$ and B7 are mutually independent, by taking b as a_0 in the Example 1 and [I, Examples 1, 2, 3, 6] (page 183), each of which does not satisfy only one of the conditions B0, B1, B2, B3 and $\overset{1}{\mathrm{B}}6$ respectively, and from the following examples.

Example	set A	system R	
2	a_0, a, b c, d	$\{aa_0b\}$ $\{ba_0a\}$, $\{aa_0d\}$ $\{da_0a\}$, $\{ba_0c\}$ $\{ca_0b\}$, $\{ca_0d\}$ $\{da_0c\}$ $\{a_0a_0a\}\{aa_0a_0\}\{a_0aa\}\{aaa_0\}$, $\{a_0a_0b\}\{ba_0a_0\}\{a_0bb\}\{bba_0\}$ $\{a_0a_0c\}\{ca_0a_0\}\{a_0cc\}\{cca_0\}$, $\{a_0a_0d\}\{da_0a_0\}\{a_0dd\}\{dda_0\}$ $\{aab\}$ $\{baa\}$ $\{abb\}$ $\{bba\}$, $\{aad\}$ $\{daa\}$ $\{add\}$ $\{dda\}$ $\{bbc\}$ $\{cbb\}$ $\{bcc\}$ $\{ccb\}$, $\{ccd\}$ $\{dcc\}$ $\{cdd\}$ $\{ddc\}$ $\{a_0a_0a_0\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$ $\{ddd\}$	only B0′ is not satisfied.
3	a_0, a, b c	$\{a_0ab\}$ $\{a_0ba\}$ $\{baa_0\}$ $\{aba_0\}$, $\{a_0a_0a\}\{aa_0a_0\}\{a_0aa\}\{aaa_0\}$, $\{a_0a_0b\}\{ba_0a_0\}\{a_0bb\}\{bba_0\}$ $\{a_0a_0c\}$ $\{ca_0a_0\}$, $\{a_0cc\}$ $\{cca_0\}$ $\{aac\}$ $\{caa\}$ $\{acc\}$ $\{cca\}$, $\{bbc\}$ $\{cbb\}$ $\{bcc\}$ $\{ccb\}$, $\{aab\}$ $\{baa\}$ $\{abb\}$ $\{bba\}$, $\{abc\}$ $\{cba\}$, $\{a_0ac\}$ $\{caa_0\}$, $\{a_0bc\}$ $\{cba_0\}\{a_0aa_0\}$ $\{a_0ba_0\}$ $\{a_0a_0a_0\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$	only B4 is not satisfied.

4	a_0, a, b c, d	$\{a_0ab\}$ $\{baa_0\}$, $\{abc\}$ $\{cba\}$ $\{abd\}$ $\{dba\}$ $\{a_0ac\}$ $\{caa_0\}$ $\{a_0bc\}$ $\{cba_0\}$ $\{a_0dc\}$ $\{cda_0\}$ $\{a_0bd\}$ $\{dba_0\}$ $\{bdc\}$ $\{cdb\}$ $\{aab\}$ $\{baa\}$ $\{abb\}$ $\{bba\}$, $\{bbd\}$ $\{dbb\}$ $\{bdd\}$ $\{ddb\}$ $\{aac\}$ $\{caa\}$ $\{acc\}$ $\{cca\}$, $\{ddc\}$ $\{cdd\}$ $\{dcc\}$ $\{ccd\}$ $\{a_0a_0a\}$ $\{aa_0a_0\}$ $\{a_0aa\}$ $\{aaa_0\}$, $\{a_0a_0b\}$ $\{ba_0a_0\}$ $\{a_0bb\}$ $\{bba_0\}$ $\{a_0a_0d\}$ $\{da_0a_0\}$ $\{a_0dd\}$ $\{dda_0\}$, $\{a_0a_0c\}$ $\{ca_0a_0\}$ $\{a_0cc\}$ $\{cca_0\}$ $\{bbc\}$ $\{cbb\}$ $\{bcc\}$ $\{ccb\}$, $\{a_0a_0a_0\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$ $\{ddd\}$	only B5 is not satisfied.
5	a_0, a, b c	$\{a_0ac\}$ $\{caa_0\}$ $\{a_0bc\}$ $\{cba_0\}$ $\{a_0ba\}$ $\{aba_0\}$ $\{abc\}$ $\{cba\}$ $\{aab\}$ $\{baa\}$ $\{abb\}$ $\{bba\}$, $\{bbc\}$ $\{cbb\}$ $\{bcc\}$ $\{ccb\}$ $\{aac\}$ $\{caa\}$ $\{acc\}$ $\{cca\}$, $\{a_0a_0a\}$ $\{aa_0a_0\}$ $\{a_0aa\}$ $\{aaa_0\}$ $\{a_0a_0b\}$ $\{ba_0a_0\}$ $\{a_0bb\}$ $\{bba_0\}$, $\{a_0a_0c\}$ $\{ca_0a_0\}$ $\{a_0cc\}$ $\{cca_0\}$ $\{a_0a_0a_0\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$.	only B7 is not satisfied.

Next, we shall investigate the second system of conditions for betweenness of set A such that A has one extreme element.

Theorem 3. *The system of conditions:* $\mathfrak{B}$[S. III]≡(*B1, B2, B3, B4, B5, B6[3], B7*) *is the complete condition for betweenness of set A such that A has one extreme element, where*

B6[3]: $|xyz| \to |a_0xy|$ or $|a_0yx|$ *for some one element* a_0.

Proof. From B0, B0′, and $\overset{1}{\text{B6}}$, it is clear that condition B6[3] is necessary. Conversely, we shall prove that conditions B0, B0′, and $\overset{1}{\text{B6}}$ follow from B1, B2, B3, B4, B5, B6[3] and B7. It is clear that conditions B0 and B0′ follow from B1 and B6[3], and

$$|a_1a_2x_1| \xrightarrow{B6[3]} |a_0a_1a_2| \text{ or } |a_0a_2a_1|. \qquad \text{.........(3.1)}$$
$$|a_2a_3x_2| \xrightarrow{B6[3]} |a_0a_2a_3| \text{ or } |a_0a_3a_2|. \qquad \text{.........(3.2)}$$
$$|a_3a_1x_3| \xrightarrow{B6[3]} |a_0a_3a_1| \text{ or } |a_0a_1a_3|. \qquad \text{.........(3.3)}$$

From (3.1) and (3.2), we have the following two different types of cases:

Case 1. $|a_0a_1a_2|\cdot|a_0a_2a_3|$: $|a_0a_1a_2|\cdot|a_0a_2a_3| \xRightarrow{[I,\ \text{lemma } 3.6]} |a_1a_2a_3|$.[1]

Case 2. $|a_0a_1a_2|\cdot|a_0a_3a_2|$: According to (3.3), we have two cases:

(i) $|a_0a_3a_1|$: $|a_0a_3a_1|\cdot|a_0a_1a_2| \xRightarrow{[I,\ \text{lemma } 3.6]} |a_3a_2a_1|$.

(ii) $|a_0a_1a_3|$: $|a_0a_1a_3|\cdot|a_0a_3a_2| \xRightarrow{[I,\ \text{lemma } 3.6]} |a_1a_3a_2|$.

Hence, we have $\overset{1}{\text{B6}}$.

So that the conditions B1, B2, B3, B4, B5, B6[3] and B7 are sufficient.

1) Lemma 3·6 in Chapter I is the same condition as (5) which has been discussed by G. Birkhoff.

And by taking b as a_0 in [I, examples 1, 2, 3, 4, 5, 6] (page 183), and the example 5, each of which does not satisfy only one of the conditions B1, B2, B3, B4, B5, B6[3] and B7 respectively, we see that the conditions B1, B2, B3, B4, B5, B6[3] and B7 are mutually independent.

Thus we have Theorem 3.

We shall investigate the third system of conditions for betweenness of set A in this case. In the proof of Theorem 3, we have used only B1, B2, B3, B4, B5, B6[3] and [I, Lemma 3.6] ($\equiv$(5)). And taking b as a_0 in [I, examples 1, 2, 3, 4, 5, 6], each of which does not satisfy only one of the conditions B1, B2, B3, B4, B5 and B6[3] respectively, and by the following example, we see that the conditions B1, B2, B3, B4, B5, B6[3] and (5) are mutually independent.

Example 6. Set A consists of a_0, a, b and c, set R:

$$\{a_0ab\}\ \{baa_0\}\ \{a_0bc\}\ \{cba_0\},\ \{a_0ac\}\ \{caa_0\},\ \{bac\}\ \{cab\}$$
$$\{a_0a_0a\}\ \{aa_0a_0\}\ \{a_0aa\}\ \{aaa_0\},\ \{a_0a_0b\}\ \{ba_0a_0\}\ \{a_0bb\}\ \{bba_0\}$$
$$\{a_0a_0c\}\ \{ca_0a_0\}\ \{a_0cc\}\ \{cca_0\},\ \{aac\}\ \{caa\}\ \{acc\}\ \{cca\}$$
$$\{aab\}\ \{baa\}\ \{abb\}\ \{bba\},\ \{bbc\}\ \{cbb\}\ \{bcc\}\ \{ccb\}$$
$$\{a_0a_0a_0\}\ \{aaa\}\ \{bbb\}\ \{ccc\}.$$

This example does not satisfy only (5).

So, we have Theorem:

Theorem 4. *The system of conditions $\mathfrak{L}_1 \equiv$(B1, B2, B3, B4, B5, B6[3], (5)) is the complete condition for betweenness of set A such that A has one extreme element, where*

(5): $|abc|\cdot|acd|\rightarrow|bcd|$.

§ 4. Case IV. Having two extreme elements.

This case is the special one of Case III such that the following condition must hold:

B6[4]: $\| x, y, z \| \rightarrow |a_0xa_\infty|$, for all x and some two elements a_0 and a_∞. So, from Theorem 3, we have:

The system of conditions B1, B2, B3, B4, B5, B6[3], B6[4], B7 is necessary and sufficient for betweenness of set A such that A has two extreme elements.

We shall investigate the independency of the above conditions B1, B2, B3, B4, B5, B6[3], B6[4] and B7.

Lemma 4.1. *The condition B6[3] follows from the conditions B3, B4, B6[4] and B7.*

Proof. Suppose that $\{xyz\}$ is contained is $R[\mathfrak{B}[\text{S. IV}]]$. From B6[4] there exist $|a_0xa_\infty|$ and $|a_0ya_\infty|$.

$$|a_0xa_\infty|\cdot|xyz|\xrightarrow{B7}|a_0xy| \text{ (or) } |yxa_\infty|.$$

$$|a_0ya_\infty|\cdot|xyz|\xrightarrow{B7}|a_0yx| \text{ (or) } |xya_\infty|.$$

And from B3 and B4 $|yxa_\infty|$ and $|xya_\infty|$ are not compatible, so $|a_0xy|$ or $|a_0yx|$ is contained in $R[\mathfrak{B}[\text{S. IV}]]$.

From the above Lemma, condition B6[3] follows from conditions B1, B2, B3, B4, B5, B6[4] and B7. And, by taking a as a_0 and b as a_∞ in [I, examples 1, 6], and taking c as a_∞ in the examples 3, 4, 5 and by the following examples 7 and 8, each of which does not satisfy only one of B1, B6[4], B4, B5, B7, B2 and B3 respectively, we see that the conditions B1, B2, B3, B4, B5, B6[4] and B7 are mutually independent.

Example 7. $\{a_0aa_\infty\}\{a_\infty aa_0\}\{a_0a_0a_\infty\}\{a_\infty a_0a_0\}\{a_0a_\infty a_\infty\}\{a_\infty a_\infty a_0\}\{a_0a_0a_0\}\{a_\infty a_\infty a_\infty\}$ $\{aaa\}$.

Example 8. $\{a_0a_\infty a_\infty\}\{a_0a_0a_\infty\}\{a_0a_0a_0\}\{a_\infty a_\infty a_\infty\}$.

Thus, we have the Theorem:

Theorem 5. *The system of conditions*: $\mathfrak{B}[\text{S. IV}]\equiv$(*B1, B2, B3, B4, B5, B6[4], B7*) *is the complete condition for betweenness of set A such that A has two extreme elements, where*

B6[4]: $\|x, y, z\| \rightarrow |a_0xa_\infty|$, *for all x and some two elements a_0 and a_∞.*

We shall investigate the other system of condition for betweenness of set A in this case.

First, by Theorem 3 and Lemma 4.1, condition $\overset{1}{B}6$ follows from B1, B2, B3, B4, B5, B6[4] and B7, and condition (5) follows from B2, B3, B4 and B7.

Coversely we shall show that condition B7 follows from conditions B1, B2, B3, B4, B5, B6[4], $\overset{1}{B}6$ and (5).

Lemma 4. 2. *The following condition follows from conditions B2, B3, B4, B5, $\overset{1}{B}6$, B6[4] and (5)*

$$|xyz|\rightarrow\{|a_0xy| \text{ and } |xya_\infty|\} \text{ or } \{|a_0yx| \text{ and } |yxa_\infty|\}.$$

Proof. Suppose that $\{xyz\}$ is contained in $R[\mathfrak{L}_2]$. By B6[4], $|a_0xa_\infty|$ and $|a_0ya_\infty|$ are contained in $R[\mathfrak{L}_2]$. So we have,

$$|a_0xa_\infty|\cdot|xyz|\cdot|a_0ya_\infty|\xrightarrow{\overset{1}{B}6}\|a_0, x, y\|.$$

According to $\|a_0, x, y\|$, we have three cases:

Case 1 $|a_0xy|$: $|a_0xy|\cdot|a_0ya_\infty|\xrightarrow{(5)}|xya_\infty|$.

Case 2 $|a_0yx|$: $|a_0yx|\cdot|a_0xa_\infty|\xrightarrow{(5)}|yxa_\infty|$.

Case 3 $|xa_0y|$: $|xa_0y|\cdot|a_0ya_\infty|\xrightarrow{B5}|xa_0a_\infty|$.

This contradicts B6[4]. So case 3 may not occur.

Therefore, $|xyz| \to \{|a_0xy|\cdot|xya_\infty|\}$ or $\{|a_0yx|\cdot|yxa_\infty|\}$.

Lemma 4. 3. *The condition B7 follows from conditions B2, B3, B4, B5, $\overset{1}{B}6$, B6[4] and (5).*

Proof. Suppose that $|abc|$ and $|bdx|$ are contained in $R[\mathfrak{L}_2]$.

$$|abc| \xRightarrow{\text{lemma 4.2}} \{|a_0ab|\cdot|aba_\infty|\} \text{ (or) } \{|a_0ba|\cdot|baa_\infty|\}, \qquad \text{.........(4.1)}$$
$$|abc| \xRightarrow{\text{lemma 4.2}} \{|a_0bc|\cdot|bca_\infty|\} \text{ (or) } \{|a_0cb|\cdot|cba_\infty|\}. \qquad \text{.........(4.2)}$$

Case 1 $|a_0ab|\cdot|aba_\infty|$:

(a) $|a_0bc|$:

$$|bdx| \xRightarrow{\text{lemma 4.2}} \{|a_0bd|\cdot|bda_\infty|\} \text{ or } \{|a_0db|\cdot|dba_\infty|\}.$$

(a_1) $|a_0bd|$: $|a_0ab|\cdot|a_0bd| \xRightarrow{(5)} |abd|$.

(a_2) $|a_0db|$: $|a_0db|\cdot|a_0bc| \xRightarrow{(5)} |dbc|$.

So, we have $|abc|\cdot|bcx| \to |abd|$ (or) $|dbc|$.

(b) $|a_0cb|\,|cba_\infty|$:

$$|abc|\cdot|cba_\infty| \xRightarrow{B7} |aba_\infty| \text{ (or) } |a_\infty bc|.$$

But $|aba_\infty|$ and $|a_\infty bc|$ must be compatible, so this is contradiction. Hence, this case (b) may not occur.

Case 2. $|a_0ba|\cdot|baa_\infty|$: In this case in the similar way as Case 1, we can prove that it follows that $|abd|$ (or) $|dbc|$.

Therefore we have B7.

And by taking a as a_0 and b as a_∞ in [I, example 1], and taking c as a_∞ in the examples 3, 4, 6 and by the examples 7, 8 and the following examples 9 and 10, we see that the conditions B1, B4, B5, (5), B2, B3, B6[4] and $\overset{1}{B}6$ are mutually independent;

Example 9. Set A consists of a_0, a_∞ and a, system R:

$\{a_0a_0a_\infty\}$ $\{a_\infty a_0a_0\}$ $\{a_0a_\infty a_\infty\}$ $\{a_\infty a_\infty a_0\}$ $\{a_0a_0a_0\}$ $\{a_\infty a_\infty a_\infty\}$ $\{aaa\}$.

Example 10. Set A consists of a_0, a, b, c and a_∞ system R:

$\{a_0aa_\infty\}$ $\{a_\infty aa_0\}$, $\{a_0ba_\infty\}$ $\{a_\infty ba_0\}$, $\{a_0ca_\infty\}$ $\{a_\infty ca_0\}$

$\{aab\}$ $\{baa\}$ $\{abb\}$ $\{bba\}$, $\{bbc\}$ $\{cbb\}$ $\{bcc\}$ $\{ccb\}$

$\{aac\}$ $\{caa\}$ $\{acc\}$ $\{cca\}$, $\{a_0a_0a\}$ $\{aa_0a_0\}$ $\{a_0aa\}$ $\{aaa_0\}$

$\{aaa_\infty\}$ $\{a_\infty aa\}$ $\{aa_\infty a_\infty\}$ $\{a_\infty a_\infty a\}$, $\{a_0a_0b\}$ $\{ba_0a_0\}$ $\{a_0bb\}$ $\{bba_0\}$

$\{bba_\infty\}$ $\{a_\infty bb\}$ $\{ba_\infty a_\infty\}$ $\{a_\infty a_\infty b\}$, $\{a_0a_0c\}$ $\{ca_0a_0\}$ $\{a_0cc\}$ $\{cca_0\}$

$\{cca_\infty\}$ $\{a_\infty cc\}$ $\{ca_\infty a_\infty\}$ $\{a_\infty a_\infty c\}$, $\{a_0a_0a_\infty\}$ $\{a_\infty a_0a_0\}$ $\{a_0a_\infty a_\infty\}$ $\{a_\infty a_\infty a_0\}$

$\{a_0a_0a_0\}$ $\{aaa\}$ $\{bbb\}$ $\{ccc\}$ $\{a_\infty a_\infty a_\infty\}$.

From the above results, we have Theorem:

Theorem 6. *The system of conditions* $\mathfrak{L}_2 \equiv (B1, B2, B3, B4, B5, \overset{1}{B6}, B6[4], (5))$ *is the complete condition for betweenness of set A such that A has two extreme elements.*

§ 5. **Case V. Simply ordered.**

In this case, $A[R]$ is connected and every element of A may be center, and the following condition must hold,

$\overline{\text{B2}}$: $|abb|$ for all a and b.

From case II, the system of conditions B1, $\overline{\text{B2}}$, B3, B4, B5, $\overset{1}{\text{B6}}$ and B7 are necessary and sufficient for betweenness of set A in this case. ...(5.1)
Now, we shall prove the following Theorem:

Theorem 7. *The system of conditions*: $\mathfrak{B}[\text{S. V}] \equiv (B1, B2, B3, B4, B5, B6[5], B7)$ *is the complete condition for betweenness of set A such that A is simply ordered, where*

$B6[5]$: $\| x, y, z \| \cdot \| u, v, w \| \rightarrow \| a_0, x, u \|$, *for some one element* a_0.

Proof. From $\overline{\text{B2}}$ we have B2, and from B1, $\overline{\text{B2}}$ and $\overset{1}{\text{B6}}$ we have B6[5].

Conversely, we shall prove that conditions $\overline{\text{B2}}$ and $\overset{1}{\text{B6}}$ follow from conditions B1, B2, B3, B4, B5, B6[5] and B7.
For any a and b, we have $\| a_0, a, b \|$ from B1 and B6[5]. $\| a_0, a, b \| \Rightarrow |a_0ab|$ ⓞⓡ $|aa_0b|$ ⓞⓡ $|a_0ba|$.

$$|a_0ab| \xrightarrow{B3} |baa_0| \xRightarrow{B2} |bba| \xRightarrow{B3} |abb|.$$
$$|a_0ab| \xrightarrow{B3} |ba_0a| \xRightarrow{B2} |bba_0| \xRightarrow{B3} |a_0bb|. \quad |aa_0b| \cdot |a_0bb| \xRightarrow{B5} |abb|.$$
$$|a_0ba| \xrightarrow{B3} |aba_0| \xRightarrow{B2} |aab| \xRightarrow{B3} |baa| \xRightarrow{B2} |bba| \xRightarrow{B3} |abb|.$$

So, we have $\overline{\text{B2}}$. And the condition B6[2] follows from B6[5]. Hence, by Theorem 2 (in Case II) the condition $\overset{1}{\text{B6}}$ follows from B1, B2, B3, B4, B5, B6[5] and B7.

Therefore, from (5.1), the system of conditions: $\mathfrak{B}[\text{S. V}]$ is necessary and sufficient for betweenness of set A in this case.

And from [I, examples 1, 2, 3, 4, 5, 6] and example 5, each of which does not satisfy only one of B1, B2, B3, B4, B5, B6[5] and B7 respectively, we see that the conditions B1, B2, B3, B4, B5, B6[5] and B7 are mutually independent.

Thus we have Theorem 7.

We shall investigate the other systems of conditions: $\mathfrak{L}_3$ and $\mathfrak{L}_4$ for betweenness in this case.

Theorem 8. *The system:* $\mathfrak{L}_3 \equiv (B1, \overline{B2}, B3, B4, \overline{B6}, B7)$ *is the complete condition for betweenness of set A such that A is simply ordered, where*

$\overline{B6}$: $\| a, b, c \|$ *for all a, b and c.*

Proof. The system $\mathfrak{L}_3$ follows from the system B1, $\overline{\mathrm{B2}}$, B3, B4, B5, $\overset{1}{\mathrm{B}}6$ and B7 (page 219). For, the condition $\overline{\mathrm{B6}}$ follows from $\overline{\mathrm{B2}}$ and $\overset{1}{\mathrm{B}}6$.

Conversely, we shall prove that the system B1, $\overline{\mathrm{B2}}$, B3, B4, B5, $\overset{1}{\mathrm{B}}6$ and B7 follows from $\mathfrak{L}_3$.

First we shall prove that: The condition B5: $|axb| \cdot |xby| x \neq b \rightarrow |aby|$ follows from conditions $\overline{\mathrm{B2}}$, B3, B4 and B7. ....(5.2)

When $y=b$, it is clear that B5 is obtained from $\overline{\mathrm{B2}}$, so we shall prove the case $y \neq b$. If $x=y$, we have $x=b$ from $\overline{\mathrm{B2}}$ and B4. This contradicts the assumption $x \neq b$. So, we have $x \neq y$. Suppose $|axb|$ and $|xby|$ are contained in $R[\mathfrak{L}_3]$. Then,

$$|axb| \cdot |xby| \overset{B7}{\Longrightarrow} |aby| \text{ ⓞⓡ } |xba|.$$

But $|xba| \cdot x \neq b$ contradicts $|axb|$ by B4, so $|xba|$ may not occur. Hence we have: $|axb| \cdot |xby| x \neq b \rightarrow |aby|$.

So, we have the assertion.

And condition $\overset{1}{\mathrm{B}}6$ follows from $\overline{\mathrm{B2}}$ and $\overline{\mathrm{B6}}$. Hence the system B1, $\overline{\mathrm{B2}}$, B3, B4, B5, $\overset{1}{\mathrm{B}}6$ and B7 follows from $\mathfrak{L}_3$. Therefore, from (5.1) the system $\mathfrak{L}_3$ is necessary and sufficient.

And from [I, examples 1, 2, 11, 4, 6] and example 5, each of which does not satisfy only one of the conditions B1, $\overline{\mathrm{B2}}$, B3, B4, $\overline{\mathrm{B6}}$ and B7 respectively, we see that these conditions are mutually independent.

Thus, we have Theorem 8.

Theorem 9. *The system of conditions:* $\mathfrak{L}_4 \equiv (B1, \overline{B2}, B3, B4, B5, \overline{B6}, (5))$ *is also the complete condition for betweenness of set A such that A is simply ordered.*

Proof. We shall prove that system $\mathfrak{L}_4$ is equivalent to $\mathfrak{L}_3$. First, $\mathfrak{L}_4$ follows from $\mathfrak{L}_3$: For, by Note in Lemma 3.6 and (5.2) the conditions B5 and (5) follow from the conditions $\overline{\mathrm{B2}}$, B3, B4 and B7.

Conversely, $\mathfrak{L}_3$ follows from $\mathfrak{L}_4$: The condition B7 follows from the conditions B3, B5, $\overline{\mathrm{B6}}$ and (5). For, Suppose $|abc|$ and $|bdx|$.

By $\overline{\mathrm{B6}}$, there exists $\| b, c, d \|$ i.e. $|bcd|$ or $|cdb|$ or $|dbc|$ for b, c and d.[1)] ....(5.3)

1) Because b, c and d are different elements, we have $\| b, c, d \|$ means that $|bcd|$ or $|cdb|$ or $|dbc|$.

Case 1. $|bcd|$: $|abc|\cdot|bcd|\xrightarrow{B5}|abd|$. ....(5. 4)

Case 2. $|cdb|$: $|cdb|\cdot|cba|\xrightarrow{(5)}|dba|\xrightarrow{B3}|abd|$. ....(5. 5)

From (5. 3), (5. 4) and (5. 5), we have B7.

Therefore, the conditions B1, $\overline{B2}$, B3, B4, B5, $\overline{B6}$ and (5) are necessary and sufficient for betweenness of set A in this case. And from [I, examples 1, 2, 11, 4, 6] and example 6 and the following example 11, we see that the conditions B1, $\overline{B2}$, B3, B4, B5, $\overline{B6}$ and (5) are mutually independent;

Example 11. Set A consists of a, b, c and d, set R:

$\{abc\}$ $\{cba\}$ $\{bcd\}$ $\{dcb\}$ $\{adc\}$ $\{cda\}$ $\{bad\}$ $\{dab\}$

$\{aab\}$ $\{baa\}$ $\{abb\}$ $\{bba\}$ $\{aac\}$ $\{caa\}$ $\{acc\}$ $\{cca\}$

$\{aad\}$ $\{daa\}$ $\{add\}$ $\{dda\}$ $\{bbc\}$ $\{cbb\}$ $\{bcc\}$ $\{ccb\}$

$\{bbd\}$ $\{dbb\}$ $\{bdd\}$ $\{ddb\}$ $\{ccd\}$ $\{dcc\}$ $\{cdd\}$ $\{ddc\}$

$\{aaa\}$ $\{bbb\}$ $\{ccc\}$ $\{ddd\}$.

Thus we have Theorem 9.

Remark :

I) The systems of conditions: $\mathfrak{L}_3$ and $\mathfrak{L}_4$ correspond to the systems No. 12 and No. 1 respectively which have been obtained by E. V. Huntington and J. R. Kline for betweenness such that A is simply ordered.[1]

II) In the above special cases I......V, the systems of conditions for betweenness are obtained by replacing G.B(12)′ for B1 and B2. (Cf. p. 209)

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1) E. V. Huntington and J. R. Kline, Sets of Independent Postulates for Betweenness, with proof of Complete Independence, Trans. Am. Math. Soc. vol. 26 (1924).