THEORY OF THE SPHERICALLY SYMMETRIC SPACE-TIMES. II GROUP OF MOTIONS[1]

By

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§ 1. Introduction.

In his previous paper[2] the present writer determined the group of motions of the s.t. (space-time) of relativistic cosmology which is defined by

$$L': \ ds^2 = -f(r, t)(dx^2+dy^2+dz^2)+dt^2\,, \quad f=e^{2g(t)}(1+r^2/4R^2)^{-2}\,, \qquad (1.1)$$

where $r=\sqrt{(x^2+y^2+z^2)}$ and R^2 is a constant. And it was shown that this s.t. $S(L)$, where L denotes ds^2 in polar coordinate form, is classified into the following five types:

[A] de Sitter type s.t., which belongs to both $S(L_a)$ and $S(L_b)$,

[B] Minkowski type s.t., which also belongs to both $S(L_a)$ and $S(L_b)$,

[C] Einstein type s.t., which belongs to $S(L_b)$,

[D] $S(\overline{L_a})$ type s.t., which is the $S(L_a)$ type s.t. other than [A] and [B].

[E] $S(\overline{L_b})$ type s.t., which is the $S(L_b)$ type s.t. other than [A], [B] and [C],

where $S(L_a)$ and $S(L_b)$ are $S(L)$ in which $1/R^2=0$ and $\neq 0$ respectively. The numbers of the parameters of the groups are 10, 10, 7, 6 and 6 respectively and corresponding infinitesimal motions were all obtained. Especially detailed discussions were made by Sibata[3),4)] and the writer.[2)] On the other hand in (I) the writer introduced a new definition of the s.s. space-time S_0 and showed that some peculiar sets of vectors and scalars called characteristic systems belong to every S_0.

In this paper, in the first place, in the s.s. coordinate system where

$$ds^2 = -A(r, t)dr^2-B(r, t)(d\theta^2+\sin^2\theta d\phi^2)+C(r, t)dt^2 \qquad (1.2)$$

holds, we shall determine all the types of the groups of motions of S_0 by solving Killing's equation and then classify it from the standpoint of these groups. Then by using the theory of (I) we shall express this classification in an invariant form. The special positions occupied by the above mentioned five groups shall be clarified in due course. Lastly we shall obtain some results concerning the form-invariancy of the c.s. under the group of motions.

§ 2. Killing's equation for S_{I} and its solution.

By a suitable transformation of (r, t), ds^2 of S_{I} becomes

$$ds^2 = -A(r,t)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + C(r,t)dt^2 . \tag{2.1}$$

ξ^i, the vector of motion, is given by solving Killing's equation which becomes

$$\xi^1 A' + \xi^4 \dot{A} + 2A\partial_r \xi^1 = 0, \quad (E_{11}), \qquad A\partial_t \xi^1 - C\partial_r \xi^4 = 0, \quad (E_{14})$$
$$\xi^1 C' + \xi^4 \dot{C} + 2C\partial_t \xi^4 = 0, \quad (E_{44}), \qquad \xi^1 + r\partial_\theta \xi^2 = 0, \quad (E_{22})$$
$$\xi^1 \sin\theta + \xi^2 r\cos\theta + r\sin\theta \partial_\phi \xi^3 = 0, \quad (E_{33}), \qquad \partial_\phi \xi^2 + \sin^2\theta \partial_\theta \xi^3 = 0, \quad (E_{23})$$
$$A\partial_\theta \xi^1 + r^2 \partial_r \xi^2 = 0, \quad (E_{12}), \qquad A\partial_\phi \xi^1 + r^2 \sin^2\theta \partial_r \xi^3 = 0, \quad (E_{13})$$
$$r^2 \partial_t \xi^2 - C\partial_\theta \xi^4 = 0, \quad (E_{24}), \qquad r^2 \sin^2\theta \partial_t \xi^3 - C\partial_\phi \xi^4 = 0, \quad (E_{34})$$

for (2.1), and as the general solution of (E_{12}), (E_{13}), (E_{23}), (E_{22}) and (E_{33}), we have

$$\begin{aligned} \xi^1 &= \omega_1 r\sin\theta\cos\phi + \omega_2 r\sin\theta\sin\phi + \omega_3 r\cos\theta, \\ \xi^2 &= \omega_1 \cos\theta\cos\phi + \omega_2\cos\theta\sin\phi - \omega_3\sin\theta - b_1\sin\phi + b_2\cos\phi, \\ \xi^3 &= -\omega_1\sin\phi/\sin\theta + \omega_2\cos\phi/\sin\theta - b_1\cot\theta\cos\phi - b_2\cot\theta\sin\phi + b_3, \end{aligned} \tag{2.2}$$

where b_a, $(a=1, 2, 3)$, are arbitrary constants and ω_a are functions of (r, t) satisfying $\omega_a A + r\omega_a' = 0$. The terms of b_a give the ordinary rotations.

I. When $\dot{A}=0$. From (2.2) and the remaining (E_{ij})'s, we have $\omega_a = p_a(t)/r\sqrt{A}$ and

$$\xi^4 = r\left\{\dot{p}_1\sin\theta\cos\phi + \dot{p}_2\sin\theta\sin\phi + \dot{p}_3\cos\theta\right\}/C\sqrt{A} + a(t), \tag{2.3}$$

$$\begin{aligned} &\dot{p}_a\left\{r(C'A + CA'/2) + AC(A-1)\right\} = 0, \quad p_a C' + (r\dot{C}/C)\dot{p}_a + 2C(r\dot{p}_a/C)^{\cdot} = 0, \\ &2C\dot{a} + \dot{C}a = 0. \end{aligned} \tag{2.4}$$

By solving these equations we can prove:

Theorem 1. *If we classify S_{I} whose A is static in the coordinate system of (2.1), from the standpoint of the group of motions, we have [A], [B], [C], [F] (the s.t. in which, in this coordinate system, we can take $C=C(r)$ by a suitable transformation of t, excluding [A], [B] and [C]), and [G] (the remaining s.t.). The groups of [F] and [G] are given by (G_3 and ∂_t) and (G_3) respectively, where G_3 is the group of the ordinary rotations.*

Therefore if there exists at least one motion besides G_3, the s.t. becomes one of [A], [B], [C] and [F]. And especially, if we assume the existence of a motion whose $\xi^1 \neq 0$, (i.e. a motion which moves the spatial origin), it becomes one of [A], [B] and [C]. Sibata obtained this result by putting a further assumption $C=C(r)$.[4]

II. When $\dot{A}\neq 0$. Now, if we assume that $\xi^1=0$ and $\xi^4\neq 0$, instead of putting any assumption on A and C, we have $\dot{A}=0$ from (E_{11}). And similarly under the assumption $\xi^1=\xi^4=0$, we obtain G_3 as the general solution. Hence in the following we shall assume that $\xi^1\neq 0$. Using (2.2) we have

$$\xi^4 = -M(\omega_1 \sin\theta\cos\phi+\omega_2\sin\theta\sin\phi+\omega_3\cos\theta), \tag{2.5}$$

where $M=(A'r+2A-2A^2)/\dot{A}$. Putting $\omega_a=\exp\lambda_a(r,t)$, we have

$$\lambda_a' = -A/r, \quad \dot{\lambda}_a = -CM/r, \tag{2.6}$$

and

$$M'=2AM/r, \quad C'r-\dot{C}M-2C(\dot{M}-CM^2/r^2)=0, \quad r^2\dot{A}=MrC'-2CM(1-A) \tag{2.7}$$

from the remaining (E_{ij})'s. Conversely if (2.7) holds, from (2.6), we obtain $\omega_a=e_a\overset{\circ}{\omega}$, where e_a are arbitrary constants, and we know that the group of this S_{I} is of 6 parameters. Then by using the following lemmas we can prove that this S_{I} is either $S(L_a)$ or $S(L_b)$:[5] (i) *This s.t. is conformally flat.* (We can prove this by using the condition that (2.1) be conformally flat.[6]) (ii) *There exists a non-null vector v_i satisfying $\nabla_i v_j=\overset{\circ}{\beta}g_{ij}$, ($\overset{\circ}{\beta}=\nabla_s v^s/4$).* (We can prove this by showing that the integrability condition of this equation is satisfied by virtue of (2.7).) (iii) *The s.t. is neither [A] nor [B] nor [C].* (The proof is evident by the fact that in these s.t.'s $\eta\equiv K_{23}{}^{23}$ must be constant.) Hence we have:

Theorem 2. *If we classify the S_{I} whose $\dot{A}\neq 0$ in the coordinate system of (2.1), from the standpoint of the group of motions, we have [D], [E] and [H] (the remaining s.t.). The group of [H] is G_3.*

ξ^i of [D] and [E] in the coordinate system of L' are given in the writer's paper[2] and those in the coordinate system of (2.1) are easily obtained from these ξ^i by using the transformation equations which transform L_a and L_b into the form of (2.1).[2] To determine whether an $S(L)$[7] is $S(L_a)$ or $S(L_b)$ in any coordinate system, we have several methods, e.g.; (i) The method using the c.s. (See § 4). (ii) $S(L)$ is $S(L_a)$ or $S(L_b)$ according as $\nabla_i K$ is proportional to v_i or not, respectively.

§ 3. Killing's equation for S_{II} and its solutions.

Killing's equation (F_{ij}) for S_{II} in the coordinate system of (1.2) in which B=const. is given by

$$\partial_\theta\xi^2=0, \quad (F_{22}), \qquad \xi^2\cos\theta+\sin\theta\partial_\phi\xi^3=0, \quad (F_{33})$$
$$\partial_\phi\xi^2+\sin^2\theta\partial_\theta\xi^3=0, \quad (F_{23}), \qquad A\partial_\theta\xi^1+B\partial_r\xi^2=0, \quad (F_{12})$$
$$A\partial_\phi\xi^1+B\sin^2\theta\partial_r\xi^3=0, \quad (F_{13}), \qquad -B\partial_t\xi^2+C\partial_\theta\xi^4=0, \quad (F_{24})$$
$$-B\sin^2\theta\partial_t\xi^3+C\partial_\phi\xi^4=0, \quad (F_{34}),$$

and the remaining (F_{11}), (F_{14}) and (F_{44}) are identical with (E_{11}), (E_{14}) and (E_{44}) respectively. In this s.t., (r, t)-space and (θ, ϕ)-space are mutually independent and by solving (F_{ij}) we can easily show that ξ^i is also decomposed into two parts belonging to these spaces respectively. The group of motions in (θ, ϕ)-space is G_3, and $\xi^1(r, t)$ and $\xi^4(r, t)$ give the motion in (r, t)-space. Hence by using the theorem concerning the group of motions in 2-dimensional space, we have:

Theorem 3. *If we classify S_{II} from the standpoint of the group of motions, we have [I] (the s.t. in which (r, t)-space is of constant curvature including the flat case), [J] (the s.t. in which the fundamental form of (r, t)-space is reducible to*

$$[J_1]\ \ ds^2=-A(t)dr^2+dt^2, \quad or \quad [J_2]\ \ ds^2=-dr^2+C(r)dt^2,$$

excluding [I]), and [K] (the remaining s.t.). The groups of [I], $[J_1]$, $[J_2]$ and [K] are given by (G_3 and a group of 3 parameters in (r, t)-space), (G_3 and ∂_r), (G_3 and ∂_t) and (G_3) respectively.

ξ^i of [I] is given in the text books on differential geometry.

§ 4. Invariant expression of the classification.

Summarizing the theorems 1, 2 and 3 we have a complete classification of the s.s. space-time S_0 from the standpoint of the group of motions. The groups of [G], [H] and [K] are the same, i.e. G_3. But, since the physical meanings of these s.t.'s are different to each other, they are treated as different types. Their differences become clearer in n-dimensional case. At first sight this classification seems to depend upon the coordinate system, but it has an invariant significance. To prove this, we shall express this classification in an invariant form by using the theory of c.s. developed in (I).

In the first place, we can express the distinction between S_I and S_{II} in invariant form in several ways, e.g., $F\neq$const. and =const. are the characteristic properties for S_I and S_{II} respectively. Eight s.t.'s [A], ..., [H] belong to S_I and the remaining three [I], [J] and [K] to S_{II}. Evidently [A] and [B] are characterized by $(\overset{1}{\rho}=\overset{2}{\rho}=\overset{3}{\rho}=0,\ \overset{4}{\rho}=\text{const.}\neq 0)$ and $(\overset{a}{\rho}=0,$

$a=1, \dots, 4$) respectively. We shall denote S_{15} (i.e. S_{I} whose $\overset{2}{\rho}=\overset{3}{\rho}$) excluding [A] and [B] by [F_1]. Accordingly [F_1] is a part of [F] and the well-known s.t. of Schwarzschild belongs to this [F_1].

Next, we shall consider S_b, i.e. S_{I} whose $\overset{2}{\rho}\neq\overset{3}{\rho}$. As is proved in (I), $\bar{\kappa}=0$ is the condition that $\dot{A}=0$ in the coordinate system of (2.1), so this holds in [C], [G] and [F_2] (i.e. [F] other than [F_1]), and $\bar{\kappa}\neq 0$ holds in [D], [E] and [H]. As is easily seen, ([C], [F_2]) and [G] are characterized by $\beta^s\overset{3}{\rho}_s=0$ and $\neq 0$ respectively.[8] Further, since [C] is characterized by $\overset{3}{\rho}=-2\overset{4}{\rho}=\text{const.}\ (=4/R^2)$, we can distinguish [C] from [F_2] by this condition.

When $\bar{\kappa}\neq 0$, we can classify the three s.t.'s [D], [E] and [H] invariantly by using the following three lemmas: (i) *A necessary and sufficient condition that S_0 be conformally flat is given by $\overset{1}{\rho}=0$.* (This lemma is given in (I)). (ii) *$S(L)$ is characterized by $\overset{1}{\rho}=\bar{\sigma}=\sigma+\bar{\kappa}=0$.* ($\beta_i$ is a gradient vector and defines a geodesic congruence. By taking this as parametric curves of x^4 and their arc length as x^4, we can prove this.). (iii) *$S(L_a)$ and $S(L_b)$ are characterized by $\overset{4}{\rho}+2\sigma^2=0$ and $\neq 0$ respectively.* (We can prove this by using the formulae of c.s. given in (I).) And these equations are invariant under ε- and m-transformations.

Lastly, with respect to S_{II}, we can easily prove that necessary and sufficient conditions that S_{II} be [I], [J_1] and [J_2] are given by ($\overset{1}{\rho}=\text{const.}$), ($\overset{1}{\rho}\neq\text{const.}$ and the existence of a c.s. such that $\bar{\sigma}=\alpha^i\sigma_i=0$) and ($\overset{1}{\rho}\neq\text{const.}$ and the existence of a c.s. such that $\sigma=\beta^i\bar{\sigma}_i=0$), respectively.

- S_0 (General s.s. space-times)
  - S_{I} ($\kappa=\bar{\kappa}=0$ does not hold)
    - $\overset{2}{\rho}=\overset{3}{\rho}$ (S_{15})
      - $\overset{1}{\rho}=\overset{2}{\rho}=\overset{3}{\rho}=0$
        - $\overset{4}{\rho}=\text{const.}\neq 0$ .... [A] 10
        - $\overset{4}{\rho}=0$ .... [B] 10
      - other ones .... [F_1] 4
    - $\overset{2}{\rho}\neq\overset{3}{\rho}$
      - $\bar{\kappa}=0$ (static A)
        - $\beta^s\overset{3}{\rho}_s=0$
          - $\overset{3}{\rho}=-2\overset{4}{\rho}=\text{const.}$ [C] 7
          - other ones...... [F_2] 4
        - ,, $\neq 0$ .... [G] 3
      - $\bar{\kappa}\neq 0$ (non-static A)
        - $\overset{1}{\rho}=\bar{\sigma}=\sigma+\bar{\kappa}=0$
          - $\overset{4}{\rho}+2\sigma^2=0$. [D] 6
          - ,, $\neq 0$. [E] 6
        - other ones .... [H] 3
  - S_{II} ($\kappa=\bar{\kappa}=0$)
    - $\overset{1}{\rho}=\text{const.}$ .... [I] 6
    - $\overset{1}{\rho}\neq\text{const.}$
      - when c.s. whose $\bar{\sigma}=\alpha^i\sigma_i=0$ exists. [J_1] 4
      - ,, ,, $\sigma=\beta^i\bar{\sigma}_i=0$,, [J_2] 4
      - other ones .... [K] 3

Summarizing the above results we have the scheme representing our classification as is seen in the last page, where 10, 10, 4, ... indicate the numbers of the parameters of the corresponding groups. The scheme, being represented in terms of the c.s., is invariant under coordinate transformations.

§ 5. Characteristic system and *m*-transformation.

Let (k) be any c.s. of an S_0. From the definition of the c.s., $T(k)$ which is obtained by an arbitrary m-transformation T from (k) must be a c.s. of the S_0 again. Excluding some special cases, however, the set of all c.s.'s of an S_0 is form-invariant under m-transformations as will be seen in the following.

Let $X \equiv \xi^i \partial_i$ be an infinitesimal motion of S_0. The conditions for a scalar v and a vector v_i to be form-invariant under X are given by

$$\xi^s \partial_s v = 0 \quad \text{and} \quad \xi^s \partial_s v_i + v_s \partial_i \xi^s = 0\,, \tag{5.1}$$

respectively. If v and v_i are s.s. in the old sense, i.e. form-invariant under G_3, (5.1) is satisfied by G_3. Hence, using the formulae concerning the c.s. obtained in (I) we know that each c.s. of [G], [H] and [K] is form-invariant individually under m-transformations. In the same way, from (5.1) and the formulae of c.s., we can easily obtain the following results:

[A] ... ○ by (G_3, U), and △ by $(\overset{a}{T}, \overset{a}{S})$, [B] ... ○ by $(G_3, \bar{U})$, and △ by $(\overset{a}{T}, \overset{a}{S})$,
[C] ... ◎ by $(G_3, \bar{U})$, and △ by $\overset{a}{V}$, [D] ... ◎ by G_3, and △ by $\overset{a}{T}$,
[E] ... ◎ by G_3, and △ by $\overset{a}{V}$, [F_1], [I], [J] ... ○ by any m-transformation,
[F_2], [G], [H], [K] ... ◎ by any m-transformation,

where $U, \bar{U}, \overset{a}{T}, \ldots$ are the operators of the motions[2] and the notations ○, ◎ and △ mean that "the set of all c.s.'s is form-invariant as a whole", "each c.s. is form-invariant individually", and "not form-invariant", respectively. In case of △-relation new types of c.s. which are not s.s. in the old sense can be obtained by m-transformation from the s.s. ones.

These results are consistent with the investigation concerning the freedom of the c.s. in (I).

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Notes.

1) This paper is the continuation of Journ. Math. Soc. Japan, **3** (1951), 317, which is cited as (I) and the same notations as in (I) are used throughout the present paper.

2) H. Takeno. This Journal, **11** (1941), 201. The same notations concerning the operators of the motions e.g., $U, \bar{U}, \overset{a}{T}, \ldots$ are also used.

3) T. Sibata. This Journal, **11** (1941), 21.

4) T. Sibata. „ „ **11** (1941), 231.

5) H. Takeno. „ „ **12** (1942), 125. (In Japanese).

6) H. Takeno. „ „ **10** (1940), 173.

7) We can also characterize $S(L)$ by

$$K_{ijlm}=\mu g_{[i[l}v_{j]}v_{m]}+\nu g_{[i[l}g_{j]m]}, \quad \nabla_i v_j=\overset{0}{\beta} g_{ij}.$$

8) We can use $\bar{\sigma}$ in place of $\overset{3}{\rho}$.