FUNDAMENTAL GROUP OF TRANSFORMATIONS IN SPECIAL RELATIVITY AND QUANTUM MECHANICS

By

Takashi SHIBATA

(Received Jan. 10, 1952)

§ 1. Introduction and purpose of the paper.

Special Lorentz transformations which are generated by the infinitesimal transformations with the operators:

$$Q_i = x^i \frac{\partial}{\partial x^4} + x^4 \frac{\partial}{\partial x^i} \quad (i=1, 2, 3) \tag{1.1}$$

do not form a 3-parameter group.[1] Here we denote coordinates x, y, z and ct by $x^\lambda (\lambda=1, \dots, 4)$. In order that they constitute a group we must add the rotation group generated by the operators:

$$R_i = \varepsilon_{ijk} x^j \frac{\partial}{\partial x^k} \quad (i, j, k = 1, 2, 3) \tag{1.2}$$

where ε_{ijk} are such that

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } i, j, k \text{ is an even permutation of } 1, 2, 3, \\ -1 \text{ if } i, j, k \text{ is an odd permutation of } 1, 2, 3, \\ 0 \text{ in any other cases,} \end{cases}$$

and double suffixes j and k are summed from 1 to 3 respectively. Usually, by rotating the axes of coordinates choosing x-axis in the direction of uniform motion, we have considered special Lorentz transformations. This is possible when we can assume that the properties considered are invariant by rotation. However, in the case where we can not assume rotation invariant, it may be impossible to take always the special Lorentz transformations.[1] In this case, in place of general Lorentz transformation group as the fundamental group of transformations, new group of transformations must be considered as representing the relations between the coordinates of two systems, one of which is moving with uniform velocity to the other. In this paper, we shall obtain such group of transformations under the following two postulates.[1]

Postulate I. *Invariancy of velocity of light.* For coordinate system moving with uniform velocity, the velocity of light is constant$=c$.

Postulate II. *Principle of uniform motion.* The relations between the coordinates of two systems, one of which is moving with uniform velocity to the other, should be determined uniquely in terms of x-, y-, z-components of the velocity in a definite functional form containing the components of the velocity as parameters, and satisfy the following condition: If for a coordinate system K the second system K' is moving with uniform velocity and further the third system K'' is moving with uniform velocity with respect to K', then K'' is also moving with certain uniform velocity with respect to K (i.e. the relations between the coordinates of K and K'' are expressed by the same functional form as K and K' (K' and K'') regarding the components of velocities as parameters).

§ 2. Operators of the fundamental group and the conclusion.

Let us denote the coordinates of K and K' by x^λ and x'^λ ($\lambda=1, \ldots, 4$) respectively and suppose that x-, y-, z-components of the uniform velocity of K' with respect to K are u^1, u^2, u^3. And we assume that the functional form of the relations between x^λ and x'^λ is given by

$$x'^\lambda = f^\lambda(x^1, x^2, x^3, x^4 \,;\, u^1, u^2, u^3) \quad (\lambda = 1, \ldots, 4) \tag{2.1}$$

where f^λ's are functions of x's and u's to be determined under the postulates I and II. By the postulate II, the transformations (2.1) must form a 3-parameter group, u^1, u^2, u^3 being regarded as parameters. We represent the postulate I by the condition that the equation $(\Delta-\partial^2/c^2\partial t^2)\varphi=0$ is invariant by (2.1). Then it is known that (2.1) form a continuous transformation group which is generated by the infinitesimal transformations with the operators (1.1), (1.2), $Q=x^\lambda\partial/\partial x^\lambda$ and $\partial/\partial x^\mu$(λ, $\mu=1, \ldots, 4$). But the last operators $\partial/\partial x^\mu$ merely generate translations of x^μ. We leave them out of consideration by taking the case coordinate systems are chosen such that $x=y=z=t=0$ corresponds to $x'=y'=z'=t'=0$. Hence (2.1) must be the transformations of the group which is generated by the operators P_i of the form:

$$P_i = Q_i + c_{ij}R_j + c_iQ \quad (i, j = 1, 2, 3) \tag{2.2}$$

where c_{ij} and c_i are certain constants to be determined, such that P_i generate a group, i.e. the commutators of P_i and P_j are expressed by linear combinations of P_k with constant coefficients: $[P_iP_j]=d_{ijk}P_k$ ($i, j, k=1, 2, 3$). From the last condition, confining ourselves to real transformations, the constants c_{ij}, c_i and d_{ijk} are determined, having the values:

$$\begin{aligned} c_{ij} &= \varepsilon_{ijk}d^k + hd_id_j\,, \quad c_i = kd_i \\ d_{ijk} &= hd^l(d_j\varepsilon_{lik} - d_i\varepsilon_{ljk}) + d_i\delta_{jk} - d_j\delta_{ik} \end{aligned} \tag{2.3}$$

where $d_i(=d^i)$ are any constants satisfying the condition $d_i d^i=1$ and h and k are arbitrary constants. (Calculations are omitted in this paper.) So, substituting (2.3) into (2.2), we have the result: *The operators*:

$$P_i = Q_i + \varepsilon_{ijl} d^l R_j + d_i(hd^j R_j + kQ) \qquad (i, j, l = 1, 2, 3) \qquad (2.4)$$

$$= x^i \frac{\partial}{\partial x^4} + x^4 \frac{\partial}{\partial x^i} + d_i x^p \frac{\partial}{\partial x^i} - x^i d^p \frac{\partial}{\partial x^p} \qquad (p, q = 1, 2, 3)$$

$$+ d_i\left(hd^j \varepsilon_{jpq} x^p \frac{\partial}{\partial x^q} + kx^\lambda \frac{\partial}{\partial x^\lambda}\right) \qquad (\lambda = 1, \dots, 4)$$

where $d_i(=d^i)$ are any constants subject to the condition $d_i d^i=1$ and h and k are arbitrary constants, generate the continuous transformation group satisfying the postulates I *and* II.

In order to investigate the properties of (2.4) we put $S_i=Q_i+\varepsilon_{ijl}d^l R_j$ and $R_a=d^j R_j$. Then we see that any two independent linear combinations of S_i with constant coefficients, constitute a two-parameter group. For, if we put $T_a=c^i_a S_i (a=1, 2;\ i=1, 2, 3)$, c^i_a being arbitrary constants, using the relations $[S_i S_j]=d_i S_j - d_j S_i$, we have $[T_a T_b]=(c^i_a d_i)T_b-(c^j_b d_j)T_a$. Further we see that $[R_a S_i]=d^j \varepsilon_{ijk} S_k$. Therefore S_i and R_a together form a group G_4, and the group G_3 generated by S_i is invariant by R_a. The operator R_a generates one-parmeter rotation group, the direction cosine of the axis of rotation being d_1, d_2, d_3. Thus the group generated by Q_i, R_i and Q (general Lorentz transformation group) is divided into the following sub-groups: G_3 (or G_4), rotation group in x, y, z, and dilatation group. Therefore we have the following **conclusion**: *Only in the case where the properties considered are spherically symmetric, Lorentz transformation group may be taken as the fundamental group of transformations. In general case, whether spherically symmetric or not, the relations between the coordinates of two systems, one of which is moving with uniform velocity to the other, should be expressed by the transformations of the one of the following groups*: *The group generated by P_i defined by* (2.4), *the group G_3 or G_4 (together with dilatation group). The group G_4 corresponds to a case of axial symmetry, the direction cosine of the axis of axial symmetry being d_1, d_2, d_3*. On such groups of transformations new mechanics may be considered.

§ 3. Finite form of the transformations of the group.

We shall find the finite form of the transformations of the group generated by

$$e_iP_i = e_ix^i\partial/\partial x^4 + [x^4e_i + d_px^pe_i - e_px^pd^i]/\partial x^i + ke_id^ix^4/\partial x^4 + e_pd^p[h\varepsilon_{jpi}d^jx^p + kx^i]/\partial x^i \tag{3.1}$$

where e_i are arbitrary constants. We have the system of differential equations to be solved:

$$\begin{cases} dx'^i/d\tau = x'^4e_i + d_px'^pe_i - e_px'^pd^i + d^pe_p(h\varepsilon_{ipq}d^px'^q + kx'^i), \\ dx'^4/d\tau = e_px'^p + kd^pe_px'^4, \qquad (i, p, q = 1, 2, 3) \end{cases} \tag{3.2}$$

with the initial condition that $x'^\lambda = x^\lambda$ when $\tau=0$. For brevity, we use the notations (de), (ee), (dx), etc. in place of d^pe_p, e_pe_p, d_px^p, etc. Then the solution of (3.2) is given by:

$$\begin{aligned}(1+h^2)x'^i = {}& e^{(k+1)(de)\tau}\Big\{(dx)+x^4\Big\}\Big[\frac{e_i}{(de)} - \frac{d^i(ee)}{2(de)^2} - \frac{h}{(de)}\varepsilon_{ijk}e_jd^k + \frac{h^2}{2}d^i\Big] \\ & + e^{(k-1)(de)\tau}\Big[\frac{(ex)}{(de)} - \frac{(ee)}{2(de)^2}\Big\{(dx)+x^4\Big\} + \frac{h}{(de)}\varepsilon_{jki}x^jd^ke_i - \frac{h^2}{2}\Big\{x^4-(dx)\Big\}\Big]d^i \\ & + e^{k(de)\tau}\cos h(de)\tau\left[\begin{aligned}&\Big\{(dx)+x^4\Big\}\Big\{\frac{(ee)}{(de)^2}d^i - \frac{e_i}{(de)} + \frac{h}{(de)}\varepsilon_{ijk}e_jd^k\Big\} \\ &+ x^i - \frac{(ex)}{(de)}d^i + \frac{hd^i}{(de)}\varepsilon_{jkl}x^je_kd^l + h^2\Big\{x^i - d^i(dx)\Big\}\end{aligned}\right] \\ & + e^{k(de)\tau}\sin h(de)\tau\left[\begin{aligned}&\frac{1}{(de)}\varepsilon_{ijk}e_j(x^k + d^kx^4) + \frac{he_i}{(de)}\Big\{(dx)+x^4\Big\} \\ &- hd^i\Big\{\frac{(ex)}{(de)} + x^4\Big\} - h^2\varepsilon_{ijk}x^jd^k\end{aligned}\right]\end{aligned} \tag{3.3 (a)}$$

(3.3)

$$\begin{aligned}(1+h^2)x'^4 = {}& e^{(k+1)(de)\tau}\Big\{(dx)+x^4\Big\}\Big[\frac{(ee)}{2(de)^2} + \frac{h^2}{2}\Big] \\ & + e^{(k-1)(de)\tau}\Big[\frac{(ee)}{2(de)^2}\Big\{(dx)+x^4\Big\} - \frac{(ex)}{(de)} - \frac{h}{(de)}\varepsilon_{ijk}x^id^je_k + \frac{h^2}{2}\Big\{x^4-(dx)\Big\}\Big] \\ & + e^{k(de)\tau}\cos h(de)\tau\Big[x^4 - \frac{(ee)}{(de)^2}\Big\{(dx)+x^4\Big\} + \frac{(ex)}{(de)} - \frac{h}{(de)}\varepsilon_{ijk}x^ie_jd^k\Big] \\ & + e^{k(de)\tau}\sin h(de)\tau\Big[\frac{1}{(de)}\varepsilon_{ijk}x^ie_jd^k + h\Big\{\frac{(ex)}{(de)} - (dx)\Big\}\Big].\end{aligned} \tag{3.3 (b)}$$

(calculations being omitted). These are the finite form of the equations of transformations generated by (3.1) provided that $(de)\neq 0$, $e_i\tau$ being parameters. When $(de)=0$, the group generated by (3.1) is two-parameter sub-group of G_3. For, if we denote by e_i^1 and e_i^2 the two independent solutions of $(de)=0$, the commutator of $e_i^1S_i$ and $e_j^2S_j$ vanishes: $[e_i^1S_i\ e_j^2S_j] = e_i^1e_j^2(d_iS_j - d_jS_i) = 0$. And in this case the solution of (3.2) is given by

$$\begin{aligned}x'^i &= x^i - d^i(ee)\{(dx) + x^4\}\tau^2/2 + [e_i\{(dx) + x^4\} - d^i(ex)]\tau \\ x'^4 &= x^4 + (ee)\{(dx) + x^4\}\tau^2/2 + (ex)\tau\,.\end{aligned} \tag{3.4}$$

We can easily see that (3. 4) is included in (3. 3) as the limiting case of (3. 3) when $(de)\to 0$.

Next we shall express (3. 3) and (3. 4) in terms of the x-, y-, z-components of uniform velocity. The components u^i of the velocity of the points fixed to K' with respect to K, are given by the values of $[dx^i/dt]$ when $dx'^i/dt'=0$. From the inverse transformation of (3. 3), which is obtained by interchanging $x^\lambda\to x'^\lambda$, $x'^\lambda\to x^\lambda$ and $\tau\to-\tau$ in (3. 3), putting $dx'^i/dx'^4=0$ we can find the values of $[dx^i/dx^4]$ i. e. u^i/c :

$$\frac{u^i}{c}=\frac{[\text{coefficient of } x^4 \text{ in (3. 3)(a), replacing } \tau \text{ by } -\tau]}{[\text{coefficient of } x^4 \text{ in (3. 3)(b), replacing } \tau \text{ by } -\tau]} \tag{3. 5}$$

From the above, we have

$$1-\frac{(uu)}{c^2}=\frac{(1+h^2)^2}{\left[\frac{1}{2}\left\{\frac{(ee)}{(de)^2}+h^2\right\}\left\{e^{-(de)\tau}+e^{(de)\tau}\right\}+\cosh(de)\tau\left\{1-\frac{(ee)}{(de)^2}\right\}\right]^2} \tag{3. 6}$$

which shows that $(uu)<c^2$. Using (3. 5) and (3. 6), we can express (3. 3) in terms of u^i instead of $e_{i\tau}$. The resulting equations are given by

$$x'^i=\left(\frac{\sqrt{\quad}}{1+\frac{(du)}{c}}\right)^k\left(\begin{aligned}&d^i\left[\frac{(ux)/c}{\sqrt{\quad}}+\frac{(dx)\sqrt{\quad}}{1+(du)/c}-\frac{(uu)/c+(du)}{\{1+(du)/c\}\sqrt{\quad}}t\right]\\&+\cos\left(h\log\frac{\sqrt{\quad}}{1+(du)/c}\right)\left[x^i-\frac{\{d^i+u^i/c\}}{1+(du)/c}(dx)+\frac{d^i(du)-u^i}{1+(du)/c}t\right]\\&+\sin\left(h\log\frac{\sqrt{\quad}}{1+(du)/c}\right)\left[\varepsilon_{ijk}d^j\left\{x^k-\frac{\{(dx)+ct\}}{1+(du)/c}\frac{u^k}{c}\right\}\right]\end{aligned}\right)$$

$$t'=\left(\frac{\sqrt{\quad}}{1+(du)/c}\right)^k\frac{t-(ux)/c^2}{\sqrt{\quad}},\qquad \left(\sqrt{\quad}\equiv\sqrt{1-(uu)/c^2}\right) \tag{3. 7}$$

These equations give the *relations between the coordinates of* K *and* K', *the latter of which i.e.* K' *is moving with uniform velocity* u^1, u^2, u^3 *with respect to the other* K. In the case of $(de)=0$, we can show that the relation $1+(du)/c=\sqrt{1-(uu)/c^2}$ holds. Hence, substituting this relation into (3. 7), we can express (3. 4) in terms of u^i as follows:

$$\begin{aligned}x'^i&=x^i+[\{d^i(du)/c-u^i/c\}(dx)+d^i(ux)/c]/[1+(du)/c]\\&\quad+[2(du)d^i-u^i]t/[1+(du)/c]\\t'&=[t-(ux)/c^2]/\sqrt{1-(uu)/c^2}\,.\end{aligned} \tag{3. 8}$$

If we put $h=k=0$ in (3. 7), we have the equations of the transformations of the group G_3. Moreover, (3. 7) in the case where $u^i=d^iu$ corresponds to special Lorentz transformations.

§ 4. Sum of Velocities.

In order to obtain the expression for sum of velocities we shall acquire the inverse transformation of (3. 7), which is given by

$$
\begin{aligned}
x^i &= \left(\frac{1+\frac{(du)}{c}}{\sqrt{\qquad}}\right)^{k}\left(\begin{array}{l}
\{(dx')u^i/c+u^i t'\}/\sqrt{\qquad}+d^i(dx')\sqrt{\qquad}/\{1+(du)/c\} \\
+\cos\left(h\log\frac{\sqrt{\qquad}}{1+(du)/c}\right)\left[x'^i-\frac{d^i\{(dx')+(ux')/c\}}{1+(du)/c}\right] \\
-\sin\left(h\log\frac{\sqrt{\qquad}}{1+(du)/c}\right)\left[\varepsilon_{ijk}d^j x'^k+\varepsilon_{jkl}\frac{d^i x'^j d^k u^l/c}{1+(du)/c}\right]
\end{array}\right) \\
t &= \frac{\left(\frac{1+\frac{(du)}{c}}{\sqrt{\qquad}}\right)^{k}}{1+\frac{(du)}{c}}\left(\begin{array}{l}
[\{(du)/c+(uu)/c^2\}(dx')/c+\{1+(du)/c\}t']/\sqrt{\qquad} \\
+\cos\left(h\log\frac{\sqrt{\qquad}}{1+(du)/c}\right)\left\{(ux')-(du)(dx')\right\}/c^2 \\
+\sin\left(\qquad ,, \qquad\right)\varepsilon_{jkl}x'^j d^k u^l/c^2
\end{array}\right).
\end{aligned}
\tag{4. 1}
$$

From the above, putting $dx'^i/dt'=v^i$, $dx^i/dt=w^i$, we have

$$
w^i=\frac{\left(\begin{array}{l}
u^i\{1+(dv)/c\}\{1+(du)/c\}/\sqrt{\qquad}+d^i(dv)\sqrt{\qquad} \\
+\cos\left(h\log\frac{\sqrt{\qquad}}{1+(du)/c}\right)\left[v^i\left\{1+\frac{(du)}{c}\right\}-d^i\left\{(dv)+\frac{(uv)}{c}\right\}\right] \\
-\sin\left(\qquad ,, \qquad\right)\left[\varepsilon_{ijk}d^j v^k\{1+(du)/c\}+d^i\varepsilon_{jkl}v^j d^k u^l/c\right]
\end{array}\right)}{\left(\begin{array}{l}
[\{(du)/c+(uu)/c^2\}(dv)/c+\{1+(du)/c\}]/\sqrt{\qquad} \\
+\cos\left(\qquad ,, \qquad\right)\{(uv)-(du)(dv)\}/c^2 \\
+\sin\left(\qquad ,, \qquad\right)\varepsilon_{jkl}v^j d^k u^l/c^2
\end{array}\right)}
\tag{4. 2}
$$

which gives the expression for *sum of velocities* u^i (*with respect to* K) *and* v^i (*with respect to* K'). When $u^i=d^i u$ and $v^i=d^i v$, (4.2) gives the same expression as in special Lorentz transformations, viz. $w^i=d^i w$ where $w=[u+v]/[1+uv/c^2]$. In the case of $(de)=0$, viz. $1+(du)/c=\sqrt{1-(uu)/c^2}$, (4. 2) is reduced to the simpler form:

$$
w^i=\frac{u^i\{1+(dv)/c\}+v^i\{1+(du)/c\}+d^i\{(du)(dv)-(uv)\}/c}{1+(uv)/c^2-2(du)(dv)/c^2}
$$

In our future paper, we shall show that *mass* is defined from the postulate: *Invariancy of momentum mass under the new fundamental group of transformations.*

Reference

1) T. Shibata: Some properties of Lorentz transformations, this journal Vol. **16**, No. 2 (1952).