CERTAIN SINGULARITY OF ORDINARY DIFFERENTIAL EQUATIONS OF THREE VARIABLES

By

Minoru URABE

(Received Dec. 17, 1951)

Introduction.

We consider the system of differential equations as follows:

(0.1)
$$\frac{dx_1}{\xi_1}=\frac{dx_2}{\xi_2}=\frac{dx_3}{\xi_3},$$

where ξ_λ $(\lambda=1,2,3)$ are expanded as follows:

(0.2)
$$\xi_\lambda=\sum_\mu c_{\lambda\mu}x_\mu+\cdots\cdots.^{(1)}$$

Let the eigen values of the matrix $||c_{\lambda\mu}||$ be μ_λ. In this paper, we consider the case where, on a complex plane,

(i) *any two of μ_λ's do not lie on a straight line passing through the origin;*

(ii) *there does not exist any straight line passing through the origin such that all μ_λ's lie on the same side of it.*

§1. Transformation of the differential equations.

By our assumption, μ_λ are different from one another. Therefore, by means of a suitable linear transformation of the variables x_λ, we can transform the differential equations (0.1) to those of the same form where

(1.1)
$$\xi_\lambda=\mu_\lambda x_\lambda+\cdots\cdots,$$

consequently, in the following, we assume that ξ_λ in (0.1) are of the form (1.1). Put $\mu_\lambda=r_\lambda e^{i\omega_\lambda}$, then, by our assumption, we may assume that

(1.2)
$$0<\omega_2-\omega_1,\quad \omega_3-\omega_2,\quad (\omega_1+2\pi)-\omega_3<\pi.$$

Consequently there exist three straight lines L_λ's passing through the origin such that, for L_λ, μ_λ lies on one side and the other eigen values lie on another side. Then it is known[2] that there exist three regular functions $f_1(x_2,x_3)$, $f_2(x_1,x_3)$, $f_3(x_1,x_2)$ such that

1) The unwritten terms are of higher orders than those written explicitly.

2) M. H. Dulac, Bull. Soc. Math. France (1912).
M. Urabe, Jour. Sci. Hiroshima Univ. (1950), pp. 195-207.

$$
(1.3)\qquad \begin{cases} \xi_2 \dfrac{\partial f_1}{\partial x_2} + \xi_3 \dfrac{\partial f_1}{\partial x_3} = \xi_1 \,; \\ \xi_1 \dfrac{\partial f_2}{\partial x_1} + \xi_3 \dfrac{\partial f_2}{\partial x_3} = \xi_2 \,; \\ \xi_1 \dfrac{\partial f_3}{\partial x_1} + \xi_2 \dfrac{\partial f_3}{\partial x_2} = \xi_3 \,, \end{cases}
$$

where the expansions of f_λ are sums of the terms of the second and higher orders. If we consider the group $\mathfrak{G}$ of transformations with the operator functions ξ_λ, then, from (1.3), it is seen that in the 6-dimensional space E_6 of complex numbers x_λ, the 4-dimensional sub-spaces defined by one of the equations as follows:

$$
(1.4)\qquad x_1 = f_1(x_2, x_3)\,; \quad x_2 = f_2(x_1, x_3)\,; \quad x_3 = f_3(x_1, x_2)\,,
$$

are the invariant sub-spaces under the transformations of $\mathfrak{G}$.

We transform the variables x_λ to $\bar{x}_\lambda$ by the equations as follows:

$$
(1.5)\qquad \begin{cases} \bar{x}_1 = x_1 - f_1(x_2, x_3)\,, \\ \bar{x}_2 = x_2 - f_2(x_1, x_3)\,, \\ \bar{x}_3 = x_3 - f_3(x_1, x_2)\,. \end{cases}
$$

Then, with regard to $\bar{x}_\lambda$-system, the sub-spaces defined by $\bar{x}_1=0$, $\bar{x}_2=0$, $\bar{x}_3=0$ are respectively the invariant sub-spaces, therefore it must be $\xi_\lambda=\bar{x}_\lambda(\mu_\lambda+\cdots)$. By this transformation of the variables, the form of the differential equations (0.1) are not altered. Thus, without loss of generality, we may assume that, in the equations (0.1), the functions ξ_λ are of the forms as follows:

$$
(1.6)\qquad \xi_\lambda = x_\lambda(\mu_\lambda + \cdots\cdots)\,.
$$

§ 2. Characters of integral curves.

Put

$$
(2.1)\qquad \xi_\lambda = x_\lambda(\mu_\lambda + \Phi_\lambda)\,.
$$

We write the differential equations (0.1) as follows:

$$
(2.2)\qquad \frac{dx_\lambda}{dt} = \xi_\lambda = x_\lambda(\mu_\lambda + \Phi_\lambda)\,.
$$

If we consider (2.2) as the equations which defines the group $\mathfrak{G}$ with the operator functions ξ_λ and with a canonical parameter t, then, the finite transformations of $\mathfrak{G}$ are obtained as follows:

$$
(2.3)\qquad x'_\nu = \varphi_\nu(x, t) = e^{tX}(x_\nu)\,,
$$

were $X\equiv\sum\xi_\lambda\frac{\partial}{\partial x_\lambda}$ and $\varphi_\nu(x,0)=x_\nu$. In another point of view, we see that, in the space E_6, the equations (2.3) furnish an integral curve of (2.2) passing through the point x_ν with the parameter t, which varies from the origin along a curve in the t-plane. Now, since $\varphi_\nu(x,t)$ are one-valued and regular with respect to t, it is sufficient to consider only the radial variation of t.

For sufficiently small r, in each x_λ-plane, we consider the circles $C_\nu: |x_\nu|\leqq r$ and also the cylindrical domain D in E_6 defined by these circles C_ν. Put

$$t=\rho e^{-i\left(\alpha+\frac{\pi}{2}\right)}, \tag{2.4}$$

and let ρ vary from 0 to $+\infty$. Then, from (2.2), we have

$$\frac{dx_\lambda}{d\rho}=e^{-i\left(\alpha+\frac{\pi}{2}\right)}x_\lambda(\mu_\lambda+\Phi_\lambda),$$

consequently, for $x_\lambda\neq0$ [1], it follows that

$$\frac{d\log x_\lambda}{d\rho}=r_\lambda e^{i\left(\omega_\lambda-\alpha-\frac{\pi}{2}\right)}+e^{-i\left(\alpha+\frac{\pi}{2}\right)}\Phi_\lambda. \tag{2.5}$$

From our assumption, for any α, at least one of $\omega_\lambda-\alpha$ is positive and less than π, and at least one of them is greater than π and less than 2π. Consequently, for example, we assume that

$$0<\omega_1-\alpha<\pi \quad \text{and} \quad \pi<\omega_2-\alpha<2\pi.$$

Then it follows that

$$-\frac{\pi}{2}<\omega_1-\alpha-\frac{\pi}{2}<\frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{2}<\omega_2-\alpha-\frac{\pi}{2}<\frac{3\pi}{2},$$

namely

$$\cos\left(\omega_1-\alpha-\frac{\pi}{2}\right)>0 \quad \text{and} \quad \cos\left(\omega_2-\alpha-\frac{\pi}{2}\right)<0.$$

For sufficiently small positive number ε, if we take r sufficiently small, then, for $|x_\lambda|\leqq r$, it is valid that

$$\begin{cases} r_1\cos\left(\omega_1-\alpha-\frac{\pi}{2}\right)-\varepsilon>|\Phi_1|, \\ -r_2\cos\left(\omega_2-\alpha-\frac{\pi}{2}\right)-\varepsilon>|\Phi_2|. \end{cases}$$

Then, taking the real parts of both sides of (2.5), we see that

1) If $x_\lambda\neq0$, then $x'_\lambda\neq0$. For, if $x'_\lambda=\varphi_\lambda(x,t)=0$, then, since $x_\lambda=\varphi_\lambda(x',-t)$, we have $x_\lambda=0$, for, the sub-space defined by $x_\lambda=0$ is an invariant sub-space.

$$\frac{d \log |x_1|}{d\rho} > \varepsilon, \quad \frac{d \log |x_2|}{d\rho} < -\varepsilon.$$

Consequently, for positive ρ, we have

$$(2.6) \qquad |x_1'| > |x_1| e^{\varepsilon\rho}, \quad |x_2'| < |x_2| e^{-\varepsilon\rho},$$

where $x_\lambda' = \varphi_\lambda(x, t)$ for t of (2.4). Then, when we vary ρ monotone from 0 to $+\infty$, then $|x_1'| = |\varphi_1(x, t)|$ increases monotone indefinitely and $|x_2'| = |\varphi_2(x, t)|$ decreases monotone and converges to 0. Consequently a point x_λ' on the integral curve passing through the point $x_\lambda \in D$, arrives at the boundary of D for a finite value of ρ and never reaches the origin. Thus we see that, for x_λ such that any of them does not vanish, the domain T of t, for which $x_\lambda' = \varphi_\lambda(x, t) \in D$, is a bounded domain containing the origin, and that, for $t \in T$, $x_\lambda' \neq 0$.[1]

When one of x_λ's is zero, for example, $x_1 = 0$, then the integral curve is given as follows:

$$\begin{cases} x_1' = 0, \\ x_\xi' = \varphi_\xi(x, t). \quad (\xi \neq 1) \end{cases}$$

In this case, from (1.2), there exists α such that

$$\pi < \omega_\xi - \alpha < 2\pi.$$

Then, from (2.6), for x_ξ lying in the sufficiently small D, it is valid that

$$|x_\xi'| < |x_\xi| e^{-\varepsilon\rho}.$$

Consequently $x_\xi' \in D$ for any ρ and, when $\rho \to +\infty$, $x_\xi' \to 0$. Namely a point of the integral curve passing through the point $(0, x_\xi)$, approaches monotone indefinitely the origin when $\rho \to +\infty$.

Summarizing the results, we see that *the origin is a saddle point in the space E_6 of x_λ and that there exist only three 4-dimensional integral subspaces passing through the origin. In the neighbourhood of the origin, the general integrals of the differential equations* (0.1) *are given by* (2.3).

Department of Mathematics,
Hiroshima University.

1) Cf. the preceding footnote.