# INVARIANT VARIETIES FOR FINITE TRANSFORMATION

By

Minoru URABE

(Received Dec., 17, 1951)

## § 1. Introduction.

In the previous paper,[1] we have considered the finite transformation of the form as follows:

(1.1) $$T: {}'x^\nu = \varphi^\nu(x) = a^\nu_\mu x^\mu + \cdots\cdots,$$ [2]

where the absolute values of all the eigen values of $\| a^\nu_\lambda \|$ are either less or greater than unity. In this paper, we consider the transformation of the same form, which does not satisfy the condition on the eigen values. By the same idea as in the differential equations,[3] we seek for the invariant varieties for the given transformation. The method is analogous to that which is adopted for solving the equations of Schröder for the given transformation.

G. Birkhoff[4] has sought for invariant curves for the real transformation as follows:

$$\begin{cases} x' = \rho x \quad + \cdots\cdots, \\ y' = \dfrac{1}{\rho} y + \cdots\cdots, \end{cases}$$

where $\rho \neq 0, 1$. His method is considerably artistic, and seems not to admit of the extension for the case of three and more variables. Our results are general, and contain Birkhoff's result as a special case.

By suitable linear transformation of the variables, without loss of generality, we may assume that the given transformation is of the form as follows:

(1.2) $$T: {}'x^\nu = {}'x^i_{l_p} = \varphi^\nu(x) = \varphi^i_{l_p}(x) = \lambda_i x^i_{l_p} + \delta x^i_{l_{p-1}} + \cdots\cdots,$$ [5]

---

1) M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 113-131.

2) $a^\nu_\mu x^\mu$ means $\sum_{\mu=1}^{n} a^\nu_\mu x^\mu$. Hereafter we use this convention of tensor calculus. The unwritten terms are of the higher order than those written explicitly. In the following, we use also this convention.

3) M. Urabe, Jour. Sci. Hiroshima Univ. (1950), pp. 195-207. In this paper, for the differential equations $\frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \cdots\cdots = \frac{dx^n}{\xi^n}$, we have sought for the functions $x^\lambda = f^\lambda(x^\alpha)$ such that $\xi^\alpha \frac{\partial f^\lambda}{\partial x^\alpha} = \xi^\lambda$ for suitable $\alpha$ and $\lambda$. The sub-space defined by $x^\lambda = f^\lambda(x^\alpha)$ is an invariant variety for the transformations of the group with the operator functions $\xi^\nu$.

4) G. Birkhoff, Acta Math. (1922).

5) For the notations of indices, cf.: M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 113-131.

where $\delta$ is an arbitrary given positive number. In the following, we consider the transformation of the form (1. 2).

## § 2. Formal determination of the invariant variety.

We assume that $\lambda_i\,(i=1, 2, \dots, R)$ are classified into two sets of $\lambda_a$ and $\lambda_x$ such that

$$(2.1)\qquad \begin{cases} 0<|\lambda_a|<1 & (a=1, 2, \dots S), \\ |\lambda_x|\geqq 1 & (x=S+1, \dots R). \end{cases}$$

By effecting a linear transformation of the variables $x^\nu$ of the form as follows:

$$(2.2)\qquad S: \begin{cases} S_1: & \bar{x}^\alpha = s^\alpha_\beta x^\beta, \\ S_2: & \bar{x}^\lambda = s^\lambda_\mu x^\mu, \end{cases}$$ [1]

we can make $\delta$ in (1. 2) as small as we desire. Thus, in the following (in this and the next paragraph), we assume that $\delta$ in (1. 2) is an arbitrary small positive number.

Now, for the given transformation $T$ of the form (1. 2), we seek for the invariant variety passing through the origin defined by the equations of the form $x^\lambda = f^\lambda(x^\alpha)$. The condition that the sub-space defined by $x^\lambda = f^\lambda(x^\alpha)$ be invariant, is that

$$(2.3)\qquad f^\lambda[\varphi^\alpha(x, f^\mu)] = \varphi^\lambda(x, f^\mu).$$

In this paragraph, we seek for the formal solutions $f^\lambda(x^\alpha)$ of the functional equations (2. 3).

Differentiating both sides of (2. 3) with respect to $x^\alpha$, we have

$$\frac{\partial f^\lambda}{\partial \varphi^\beta}\left(\frac{\partial \varphi^\beta}{\partial x^\alpha} + \frac{\partial \varphi^\beta}{\partial f^\mu}\frac{\partial f^\mu}{\partial x^\alpha}\right) = \frac{\partial \varphi^\lambda}{\partial x^\alpha} + \frac{\partial \varphi^\lambda}{\partial f^\mu}\frac{\partial f^\mu}{\partial x^\alpha}.$$

Putting $x^\alpha=0$, we have

$$\frac{\partial f^\lambda}{\partial x^\beta}\frac{\partial \varphi^\beta}{\partial x^\alpha} = \frac{\partial \varphi^\lambda}{\partial x^\mu}\frac{\partial f^\mu}{\partial x^\alpha},$$

because $\dfrac{\partial \varphi^\beta}{\partial x^\mu} = \dfrac{\partial \varphi^\lambda}{\partial x^\alpha} = 0$. By means of (1. 2), these relations are written as follows:

$$(2.4)\qquad \frac{\partial f^x_{mq}}{\partial x^a_{lp}}\lambda_a + \frac{\partial f^x_{mq}}{\partial x^a_{lp+1}}\delta = \lambda_x \frac{\partial f^x_{mq}}{\partial x^a_{lp}} + \delta\frac{\partial f^x_{mq-1}}{\partial x^a_{lp}}.$$

For $q=1$, it follows that

$$\frac{\partial f^x_{m1}}{\partial x^a_{lp}}\lambda_a + \frac{\partial f^x_{m1}}{\partial x^a_{lp+1}}\delta = \lambda_x \frac{\partial f^x_{m1}}{\partial x^a_{lp}}.$$

1) $\alpha$, $\beta$ denote the indices of $x^\nu$ which correspond to $\lambda_a$, and $\lambda$, $\mu$ denote those of $x^\nu$ which correspond to $\lambda_x$.

Because $\lambda_a \neq \lambda_x$, we have

$$\frac{\partial f^x_{m1}}{\partial x^a_{lP^a_l}} = \frac{\partial f^x_{m1}}{\partial x^a_{lP^a_l-1}} = \cdots\cdots = \frac{\partial f^x_{m1}}{\partial x^a_{l2}} = \frac{\partial f^x_{m1}}{\partial x^a_{l1}} = 0, \quad \text{i.e.} \qquad \frac{\partial f^x_{m1}}{\partial x^a_{lp}} = 0.$$

For $q=2$, because of the above results, we have

$$\frac{\partial f^x_{m2}}{\partial x^a_{lp}} \lambda_a + \frac{\partial f^x_{m2}}{\partial x^a_{lp+1}} \delta = \lambda_x \frac{\partial f^x_{m2}}{\partial x^a_{lp}}.$$

Consequently, we have $\frac{\partial f^x_{m2}}{\partial x^a_{lp}}=0$. Thus, by induction, it is readily seen that, for any $q$,

(2.5) $$\frac{\partial f^x_{mq}}{\partial x^a_{lp}} = 0.$$

Differentiating both sides of (2.3) with respect to $x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_N}$, we have

$$\frac{\partial^N f^\lambda}{\partial\varphi^{\beta_1}\partial\varphi^{\beta_2}\dots\partial\varphi^{\beta_N}}\left(\frac{\partial\varphi^{\beta_1}}{\partial x^{\alpha_1}}+\frac{\partial\varphi^{\beta_1}}{\partial f^{\mu_1}}\frac{\partial f^{\mu_1}}{\partial x^{\alpha_1}}\right)\cdots\cdots\left(\frac{\partial\varphi^{\beta_N}}{\partial x^{\alpha_N}}+\frac{\partial\varphi^{\beta_N}}{\partial f^{\mu_N}}\frac{\partial f^{\mu_N}}{\partial x^{\alpha_N}}\right)$$
$$+\sum\frac{\partial^{N-1}f^\lambda}{\partial\varphi^{\beta_2}\dots\partial\varphi^{\beta_N}}\left(\frac{\partial^2\varphi^{\beta_2}}{\partial x^{\alpha_2}\partial x^{\alpha_1}}+\frac{\partial^2\varphi^{\beta_2}}{\partial x^{\alpha_2}\partial f^{\mu_1}}\frac{\partial f^{\mu_1}}{\partial x^{\alpha_1}}+\frac{\partial^2\varphi^{\beta_2}}{\partial f^{\mu_2}\partial x^{\alpha_1}}\frac{\partial f^{\mu_2}}{\partial x^{\alpha_2}}\right.$$
$$\left.+\frac{\partial^2\varphi^{\beta_2}}{\partial f^{\mu_2}\partial f^{\mu_1}}\frac{\partial f^{\mu_2}}{\partial x^{\alpha_2}}\frac{\partial f^{\mu_1}}{\partial x^{\alpha_1}}+\frac{\partial\varphi^{\beta_2}}{\partial f^{\mu_2}}\frac{\partial^2 f^{\mu_2}}{\partial x^{\alpha_2}\partial x^{\alpha_1}}\right)\times$$
$$\times\left(\frac{\partial\varphi^{\beta_3}}{\partial x^{\alpha_3}}+\frac{\partial\varphi^{\beta_3}}{\partial f^{\mu_3}}\frac{\partial f^{\mu_3}}{\partial x^{\alpha_3}}\right)\cdots\cdots\left(\frac{\partial\varphi^{\beta_N}}{\partial x^{\alpha_N}}+\frac{\partial\varphi^{\beta_N}}{\partial f^{\mu_N}}\frac{\partial f^{\mu_N}}{\partial x^{\alpha_N}}\right)$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{\partial f^\lambda}{\partial\varphi^\beta}\left(\frac{\partial^N\varphi^\beta}{\partial x^{\alpha_1}\dots\partial x^{\alpha_N}}+\cdots\cdots+\frac{\partial\varphi^\beta}{\partial f^\mu}\frac{\partial^N f^\mu}{\partial x^{\alpha_1}\dots\partial x^{\alpha_N}}\right)$$
$$=\frac{\partial^N\varphi^\lambda}{\partial x^{\alpha_1}\partial x^{\alpha_2}\dots\partial x^{\alpha_N}}+\cdots\cdots+\frac{\partial\varphi^\lambda}{\partial f^\mu}\frac{\partial^N f^\mu}{\partial x^{\alpha_1}\partial x^{\alpha_2}\dots\partial x^{\alpha_N}}.$$

When $x^\alpha$ are put zero, the terms containing the derivatives of $N$-th order of $f^\lambda$ are as follows:

$$\frac{\partial^N f^\lambda}{\partial x^{\beta_1}\partial x^{\beta_2}\dots\partial x^{\beta_N}}\frac{\partial\varphi^{\beta_1}}{\partial x^{\alpha_1}}\cdots\cdots\frac{\partial\varphi^{\beta_N}}{\partial x^{\alpha_N}}-\frac{\partial\varphi^\lambda}{\partial x^\mu}\frac{\partial^N f^\mu}{\partial x^{\alpha_1}\partial x^{\alpha_2}\dots\partial x^{\alpha_N}},$$

and these are polynomials of the derivatives of at most $(N-1)$-th order of $f^\lambda$. Now, by means of (1.2), the above expressions are written as follows:

$$\frac{\partial^N f^x_{mq}}{\partial x^{a_1}_{l_1p_1}\partial x^{a_2}_{l_2p_2}\cdots\partial x^{a_N}_{l_Np_N}}\lambda_{a_1}\lambda_{a_2}\cdots\lambda_{a_N}+\sum\frac{\partial^N f^x_{mq}}{\partial x^{a_1}_{l_1p_1+1}\partial x^{a_2}_{l_2p_2}\cdots\partial x^{a_N}_{l_Np_N}}\delta\lambda_{a_2}\cdots\lambda_{a_N}$$

$$+\cdots\cdots+\frac{\partial^N f^x_{mq}}{\partial x^{a_1}_{l_1p_1+1}\cdots\partial x^{a_N}_{l_Np_N+1}}\delta\delta\ldots\delta$$

$$-\frac{\partial^N f^x_{mq}}{\partial x^{a_1}_{l_1p_1}\cdots\partial x^{a_N}_{l_Np_N}}\lambda_x-\frac{\partial^N f^x_{mq-1}}{\partial x^{a_1}_{l_1p_1}\cdots\partial x^{a_N}_{l_Np_N}}\delta$$

$$=(\lambda_{a_1}\lambda_{a_2}\ldots\lambda_{a_N}-\lambda_x)\frac{\partial^N f^x_{mq}}{\partial x^{a_1}_{l_1p_1}\cdots\partial x^{a_N}_{l_Np_N}}+\cdots\cdots.$$

From our assumption that $|\lambda_a|<1$ and $|\lambda_x|\geq 1$, $\lambda_{a_1}\lambda_{a_2}\ldots\lambda_{a_N}-\lambda_x\neq 0$. Thus the values of the derivatives of $f^\lambda$ are successively determined.

Thus we have seen that *the formal solutions $f^\lambda(x^\alpha)$ of (2.3) are uniquely determined and moreover $f^\lambda(x^\alpha)$ are sums of the terms of the second and higher orders.*

## § 3. Convergence of the formal solutions.

We write (1.2) briefly as follows:

$$\text{(3.1)}\qquad \begin{cases} 'x^\alpha=\varphi^\alpha(x)=\lambda_a x^\alpha+\delta x^{\alpha+1}+\cdots\cdots, \\ 'x^\lambda=\varphi^\lambda(x)=\lambda_x x^\lambda+\delta x^{\lambda-1}+\cdots\cdots=\lambda_x x^\lambda+\Phi^\lambda(x). \end{cases}$$

We make $\delta$ so small that $2\delta<1-|\lambda_a|$. Then $|\lambda_a|+\delta<1-\delta$. We take $\Lambda$ such that $|\lambda_a|+\delta<\Lambda<1-\delta$. Put $\max.|x^\nu|=|x|'$.

If we take sufficiently small $r$, then, for $|x|'\leq r$, we have

$$\text{(3.2)}\qquad \begin{cases} |\varphi^\alpha(x)|\leq\Lambda|x|'<|x|', \\ |\Phi^\lambda(x)|\leq\delta|x^{\lambda-1}|+K|x|'^2, \end{cases}$$

where $K$ is a suitable constant. We take $L$ so that

$$\text{(3.3)}\qquad L\geq\max.\left(\frac{1}{r},\ \frac{K}{1-(\Lambda+\delta)}\right),$$

and we put $\rho=1/L$ and $\max.|x^\alpha|=|x|$. We consider the family $\mathfrak{F}$ of the functions $f^\lambda(x^\alpha)$ such that, for $|x|\leq\rho$, they are regular and

$$\text{(3.4)}\qquad |f^\lambda(x^\alpha)|\leq L|x|^2.$$

Then, for $|x|\leq\rho$, it follows that

$$|f^\lambda(x^\alpha)|\leq L|x|^2\leq|x|\leq\rho,$$

consequently $\varphi^\alpha(x, f^\mu)$ are defined and

$$|\varphi^\alpha(x, f^\mu)|\leq\Lambda|x|<|x|\leq\rho,$$

consequently $f^{\lambda}\{\varphi^{\alpha}(x, f^{\mu})\}$ are also defined and

$$|f^{\lambda}\{\varphi^{\alpha}(x, f^{\mu})\}| \leq L\Lambda^2|x|^2. \tag{3.5}$$

We define the operation $T$ as follows:

$$Tf^{\lambda} = \frac{1}{\lambda_x}[f^{\lambda}\{\varphi^{\alpha}(x, f^{\mu})\} - \Phi^{\lambda}(x, f^{\mu})]. \tag{3.6}$$

Then, for $f^{\mu}(x^{\alpha}) \in \mathfrak{F}$, it is easily seen that, for $|x| \leq \rho$, $Tf^{\lambda}$ are also regular. Now, for $|x| \leq \rho$, from (3.2), it follows that

$$|\Phi^{\lambda}(x, f^{\mu})| \leq \delta|f^{\lambda-1}| + K|x|^2 \leq (\delta L + K)|x|^2. \tag{3.7}$$

Then, for $|x| \leq \rho$, from (3.5) and (3.7), it follows that

$$\begin{aligned} |Tf^{\lambda}| &\leq \frac{1}{|\lambda_x|}\{(\Lambda^2+\delta)L+K\}|x|^2 \\ &< \{(\Lambda+\delta)L+K\}|x|^2 \\ &\leq L|x|^2. \qquad \text{(because of (3.3))} \end{aligned}$$

Thus we see that, when $f^{\lambda}(x^{\alpha})$ belongs to the family $\mathfrak{F}$, then $Tf^{\lambda}$ also belongs to the family $\mathfrak{F}$. Then, by the theorem of existence of fixed points in functional space, we see that there exists a set of functions $f^{\lambda}(x^{\alpha})$ such that $Tf^{\lambda} = f^{\lambda}$. For such $f^{\lambda}(x^{\alpha})$, it is evident that (2.3) are satisfied. Thus we see that there exists a set of regular solutions of the functional equations (2.3). Combining the result of § 2, we have

**Theorem 1.** *There exists a unique set of regular solutions $f^{\lambda}(x^{\alpha})$ of the functional equations (2.3) for the given transformation (1.2), where $\delta$ is a sufficiently small positive number. These solutions $f^{\lambda}(x^{\alpha})$ are determined formally and their expansions are sums of the terms of the second and higher orders.*

## § 4. The case where $\delta$ in (1.2) is unity.

In the preceding paragraphs, by effecting a linear transformation of the form (2.2), we have assumed that $\delta$ in (1.2) is sufficiently small. If we denote the variables and the functions in § 2 and § 3 by the letters with bars, then $\bar{\varphi}^{\alpha}(\bar{x}) = s^{\alpha}_{\beta}\varphi^{\beta}(S^{-1}\bar{x})$, $\bar{\varphi}^{\lambda}(\bar{x}) = s^{\lambda}_{\mu}\varphi^{\mu}(S^{-1}\bar{x})$,[1] and the functional equations (2.3) are written as follows:

1) $S^{-1}\bar{x}$ means $S^{\nu}_{\sigma}\bar{x}^{\tau}$, where $\| S^{\nu}_{\sigma} \| = \| s^{\nu}_{\sigma} \|^{-1}$.

$$\bar{f}^{\lambda}[\bar{\varphi}^{\alpha}(\bar{x}, \bar{f}^{\mu})] = \bar{\varphi}^{\lambda}(\bar{x}, \bar{f}^{\mu}).$$

If we put $\bar{f}^{\lambda}(\bar{x}^{\alpha}) = s^{\lambda}_{\mu} f^{\mu}(S_1^{-1}\bar{x})$, namely $f^{\mu}(x^{\alpha}) = S^{\mu}_{\lambda}\bar{f}^{\lambda}(S_1 x)$[2], then it is easily seen that

$$f^{\mu}[\varphi^{\alpha}(x, f)] = \varphi^{\mu}(x, f). \tag{4.1}$$

These are of the same forms as (2.3). From the relation between $f^{\mu}(x^{\alpha})$ and $\bar{f}^{\mu}(\bar{x}^{\alpha})$, we see that $f^{\mu}(x^{\alpha})$ are regular in the neighbourhood of the origin and are sums of the terms of the second and higher orders. Evidently these functions $f^{\mu}(x^{\alpha})$ are also determined formally from (4.1) in the same way as $\bar{f}^{\mu}(\bar{x}^{\alpha})$ do from (2.3).

Thus we have

**Theorem 2.** *There exists a unique set of regular solutions $f^{\lambda}(x^{\alpha})$ of the functional equations (2.3) for the given transformation (1.2), where $\delta$ is unity*[3]. *These solutions $f^{\lambda}(x^{\alpha})$ are determined formally and their expansions are sums of the terms of the second and higher orders.*

By this theorem, we see that, for the given transformation $T$ of the form (1.2) where $\delta=1$, there exists an invariant variety defined by $x^{\lambda}=f^{\lambda}(x^{\alpha})$.

## § 5. Another invariant variety.

We assume that $\lambda_i$ $(i=1, 2, \dots, R)$ are classified into two sets of $\lambda_c$ and $\lambda_z$ such that

$$\begin{cases} 0<|\lambda_c|\leqq 1, \\ \quad |\lambda_z|>1. \end{cases} \tag{5.1}$$

We write (1.2) briefly as follows:

$$T: \begin{cases} 'x^{\alpha} = \varphi^{\alpha}(x) = \lambda_c x^{\alpha} + x^{\alpha-1} + \dots\dots. \\ 'x^{\lambda} = \varphi^{\lambda}(x) = \lambda_z x^{\lambda} + x^{\lambda-1} + \dots\dots. \end{cases} \tag{5.2}$$

Then, the inverse transformation of $T$ is expressed as follows:

$$T^{-1}: \begin{cases} x^{\alpha} = \psi^{\alpha}('x) = '\lambda_c 'x^{\alpha} - '\lambda_c^2 'x^{\alpha-1} + '\lambda_c^3 'x^{\alpha-2} + \dots\dots + ['x]_2, \\ x^{\lambda} = \psi^{\lambda}('x) = '\lambda_z 'x^{\lambda} - '\lambda_z^2 'x^{\lambda-1} + '\lambda_z^3 'x^{\lambda-2} + \dots\dots + ['x]_2, \end{cases} \tag{5.3}$$

where $'\lambda_c = 1/\lambda_c$, $'\lambda_z = 1/\lambda_z$ and $['x]_2$ denotes the sum of the terms of the second and higher orders of $'x^{\nu}$. By effecting a suitable linear transforma-

2) $\| S^{\mu}_{\lambda} \| = \| s^{\mu}_{\lambda} \|^{-1}$.

3) $\delta$ may be an arbitrary given number which is not zero. But, for the sake of simplicity, we have taken unity as the value of $\delta$.

tion of the form $\bar{x}^\alpha = s^\alpha_\beta x^\beta$ and $\bar{x}^\lambda = s^\lambda_\mu x^\mu$, the transformation $T^{-1}$ is expressed as follows:

$$T^{-1}: \begin{cases} \bar{x}^\alpha = \bar{\psi}^\alpha('\bar{x}) = '\lambda_c{}'\bar{x}^\alpha + '\bar{x}^{\alpha-1} + ['\bar{x}]_2, \\ \bar{x}^\lambda = \bar{\psi}^\lambda('\bar{x}) = '\lambda_z{}'\bar{x}^\lambda + '\bar{x}^{\lambda-1} + ['\bar{x}]_2. \end{cases}$$

Now $0 < |'\lambda_z| < 1$ and $|'\lambda_c| \geqq 1$, therefore, by Theorem 2, there exist regular functions $\bar{f}^\alpha(\bar{x}^\lambda)$ such that

$$\bar{f}^\alpha[\bar{\psi}^\lambda(\bar{f}, '\bar{x})] = \bar{\psi}^\alpha(\bar{f}, '\bar{x}).$$

If we put $f^\alpha(x^\lambda) = S^\alpha_\beta \bar{f}^\beta(s^\lambda_\mu x^\mu)$[1], then, by the analogous reasonings as in § 4, we see that

$$f^\alpha[\psi^\lambda(f, 'x)] = \psi^\alpha(f, 'x).$$

This means that the sub-space defined by the equations $x^\alpha = f^\alpha(x^\lambda)$ is invariant under the transformation $T^{-1}$.

We shall prove that this sub-space is also invariant under $T$. Put $g^\alpha(x) = x^\alpha - f^\alpha(x^\lambda)$, then, by invariance of the sub-space, $g^\alpha(x) = 0$ for $g^\alpha('x) = 0$. Put $g^\alpha('x) = 'x^\alpha - f^\alpha('x^\lambda) = 'g^\alpha$, then $'x^\alpha = 'g^\alpha + f^\alpha('x^\lambda)$ and

$$(5.4) \qquad g^\alpha(x) = \psi^\alpha('x^\alpha, 'x^\lambda) - f^\alpha[\psi^\lambda('x^\alpha, 'x^\lambda)]$$

$$= \psi^\alpha['g^\alpha + f^\alpha('x^\lambda), 'x^\lambda] - f^\alpha[\psi^\lambda\{'g^\alpha + f^\alpha('x^\lambda), 'x^\lambda\}].$$

These are expanded as the power series of $'g^\alpha$ and $'x^\lambda$, and moreover, from invariance of the sub-space, their expansions become as follows:

$$(5.5) \qquad g^\alpha(x) = b^\alpha_\beta('g^\alpha, 'x^\lambda)\, 'g^\beta = b^\alpha_\gamma('x^\lambda)\, 'g^\gamma + \cdots\cdots.$$

Putting $'x^\lambda = 0$ in (5.4), we see that

$$\| b^\alpha_\gamma(0) \| = \begin{pmatrix} \lambda_c & 0 & \cdots\cdots & 0 \\ 1 & \lambda_c & & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots\cdots & 1 & \lambda_c \end{pmatrix}^{-1}.$$

Therefore, solving (5.5) with respect to $'g^\beta$, we have:

$$'g^\alpha = a^\alpha_\beta(g^\alpha, 'x^\lambda) g^\beta(x).$$

Consequently, when $g^\alpha(x) = 0$, $g^\alpha('x) = 0$, namely the sub-space defined by

1) $\| S^\alpha_\beta \| = \| s^\alpha_\beta \|^{-1}$.

$g^\alpha(x)=x^\alpha-f^\alpha(x^\lambda)=0$ is invariant also under $T$. In other words, $f^\alpha(x^\lambda)$ satisfy the functional equations as follows:

$$f^\alpha[\varphi^\lambda(f, x)]=\varphi^\alpha(f, x). \tag{5.6}$$

Now, as in § 2, it is readily seen that $f^\alpha(x^\lambda)$ are formally determined from (5.6). Thus we have

**Theorem 3.** *There exists a unique set of regular solutions $f^\alpha(x^\lambda)$ of the functional equations (5.6) for the given transformation (5.2). These solutions $f^\alpha(x^\lambda)$ are determined formally and their expansions are sums of the terms of the second and higher orders of $x^\lambda$.*

The sub-space defined by $x^\alpha=f^\alpha(x^\lambda)$ is invariant under the given transformation $T$.

## § 6. Summary.

Combining Theorem 2 and 3, we have

**Theorem 4.** *For the given transformation (1.2) where $\delta=1$, let the eigen values whose absolute values are equal to, less and greater than unity, be $\lambda_u$, $\lambda_a$ and $\lambda_x$ respectively, and the corresponding variables be $x^\kappa$, $x^\alpha$ and $x^\omega$ respectively. Then there exists a unique set of regular solutions $\{f^\kappa(x^\alpha), f^\omega(x^\alpha)\}$ and $\{f^\alpha(x^\omega), f^\kappa(x^\omega)\}$ respectively of the functional equations as follows:*

$$\begin{cases} f^\kappa[\varphi^\alpha(x, f)]=\varphi^\kappa(x, f), \\ f^\omega[\varphi^\alpha(x, f)]=\varphi^\omega(x, f); \end{cases}$$

$$\begin{cases} f^\alpha[\varphi^\omega(f, x)]=\varphi^\alpha(f, x), \\ f^\kappa[\varphi^\omega(f, x)]=\varphi^\kappa(f, x). \end{cases}$$

*These solutions are formally determined from the above equations respectively and their expansions are sums of the terms of the second and higher orders of the arguments. The two sub-spaces defined by $x^\kappa=f^\kappa(x^\alpha)$, $x^\omega=f^\omega(x^\alpha)$ and $x^\alpha=f^\alpha(x^\omega)$, $x^\kappa=f^\kappa(x^\omega)$ are respectively invariant under the given transformation $T$.*

In each invariant variety defined in this theorem, the given transformation $T$ induces the following transformation respectively:

$$T_1^0:\ \ 'x^\alpha=\varphi^\alpha(x, f)=\lambda_a x^\alpha+x^{\alpha-1}+\cdots\cdots,\ (0<|\lambda_a|<1);$$
$$T_2^0:\ \ 'x^\omega=\varphi^\omega(f, x)=\lambda_x x^\omega+x^{\omega-1}+\cdots\cdots,\ (|\lambda_x|>1\ ).$$

These transformations are those discussed in detail in the previous paper.[1]

1) M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 113-131.

For example, we consider the sub-space $R_1$ defined by $x^\kappa=f^\kappa(x^\alpha)$, $x^\omega=f^\omega(x^\alpha)$. Then the points of $R_1$ are transformed by $T$ in the following way:

$$T_1: \quad 'x^\alpha = \varphi^\alpha(x, f), \quad 'x^\kappa = f^\kappa('x^\alpha), \quad 'x^\omega = f^\omega('x^\alpha).$$

By the results of the same paper, there exists a one-parameter group $\mathfrak{G}_1^0$ containing $T_1^0$. Let the operator functions of $\mathfrak{G}_1^0$ be $\xi^\alpha$. Then it is easily seen that there exists a one-parameter group $\mathfrak{G}_1$ containing the transformation $T_1$ of the points of $R_1$, and that the operator functions $(\xi^\alpha, \xi^\kappa, \xi^\omega)$ of $\mathfrak{G}_1$ become as follows:

$$\xi^\kappa = \frac{\partial f^\kappa}{\partial x^\alpha}\xi^\alpha \quad \text{and} \quad \xi^\omega = \frac{\partial f^\omega}{\partial x^\alpha}\xi^\alpha.$$

Thus we see that, *in each invariant variety, the given transformation is produced by iteration of one infinitesimal transformation.*

Also, by the results of the same paper, we can obtain the invariants in the invariant variety. For example, let the invariants for $T_1^0$ be $I(x^\alpha)$. Then, these $I(x^\alpha)$ and $x^\kappa-f^\kappa(x^\alpha)$, $x^\omega-f^\omega(x^\alpha)$ are invariants for the transformations $T_1$ of the points of $R_1$, and $I(x^\alpha)$ are the invariants in $R_1$.

DEPARTMENT OF MATHEMATICS,
HIROSHIMA UNIVERSITY.