DECOMPOSITION OF FINITE TRANSFORMATION INTO INFINITESIMAL TRANSFORMATIONS

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Given the regular transformation

$$\mathfrak{T}:\ 'x^{\nu} = \varphi^{\nu}(x) = a^{\nu}_{\mu}x^{\mu} + \cdots\cdots ,^{(1)} \tag{1}$$

where $\det.|a^{\nu}_{\mu}| \neq 0$ and the unwritten terms are those of the second and higher orders.[2] In the previous paper[3], we have seen that, when the eigen values λ_{ν} of the matrix $\|a^{\nu}_{\mu}\|$ are all either less or greater than unity, the transformation $\mathfrak{T}$ can be expressed as the iteration of one infinitesimal transformation as follows:

$$'x^{\nu} = \varphi^{\nu}(x) = e^{X}(x^{\nu}), \tag{2}$$

where $X \equiv \xi^{\mu}\dfrac{\partial}{\partial x^{\mu}}$. When, among λ_i, there does not hold any of the relations of the forms as follows:

$$\lambda_{\nu} = \lambda_1^{p_1}\lambda_2^{p_2}\cdots\cdots\lambda_n^{p_n}, \tag{3}$$

where n is a number of the variables x^{ν} and $p_1, p_2, \ldots, p_n$ are non-negative integers such that $p_1+p_2+\cdots+p_n \geqq 2$, we have seen that the operator functions ξ^{μ} are sought in the following way:

At first, we solve the equations of Schröder for $\mathfrak{T}$ as follows:

$$f^{i}_{lp}(\varphi) = \lambda_i f^{i}_{lp}(x) + f^{i}_{lp-1}(x).^{(4)} \tag{4}$$

We take the matrix $K = \|k^{\mu}_{\nu}\|$ such that $K = \sum_{i=1}^{R}\sum_{l=1}^{L_i} \oplus K^{i}_{l}$, where

$$K^{i}_{l} \begin{pmatrix} \lambda_i & 0 & \cdots\cdots & 0 & 0 \\ \frac{1}{1!}\lambda_i & \lambda_i & \ddots & & \vdots \\ \frac{1}{2!}\lambda_i & \frac{1}{1!}\lambda_i & \ddots & & \vdots \\ \vdots & \vdots & & \lambda_i & 0 \\ \frac{1}{(P^{i}_{l}-1)!}\lambda_i & \frac{1}{(P^{i}_{l}-2)!}\lambda_i & & \frac{1}{1!}\lambda_i & \lambda_i \end{pmatrix} (K^{i}_{l})^{-1} = \begin{pmatrix} \lambda_i & 0 & \cdots\cdots\cdots\cdots & 0 \\ 1 & \lambda_i & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots\cdots 0 & 1 & \lambda_i \end{pmatrix}.$$

1) $a^{\nu}_{\mu}x^{\mu}$ means $\sum_{\mu} a^{\nu}_{\mu}x^{\mu}$. In the following, as here, we use this convention of tensor calculus.

2) In the following, we agree that the unwritten terms in the expansion formulae denote the terms of the higher orders than those written explicitly.

3) M. Urabe, *Equations of Schröder (Continued)*, Jour. Sci. Hiroshima Univ. 15 (1952), pp. 203-233.

4) For the notations of indices, cf., M. Urabe, ibid.

Put $f^\mu = k^\mu_\nu g^\nu$, then ξ^ν are determined by the equations as follows:

(5) $$Xg^i_{i_p} \equiv \xi^\mu \frac{\partial g^i_{i_p}}{\partial x^\mu} = \mu_i g^i_{i_p} + g^i_{i_{p-1}},$$

where $\mu_i = \log \lambda_i$. Consequently ξ^ν become of the forms as follows:

(6) $$\xi^\nu = c^\nu_\mu x^\mu + \cdots\cdots .$$

In this paper, we consider the general case where the conditions on the eigen values λ_ν of $||a^\mu_\nu||$ are not assumed.

We take a sufficiently large positive number c' and put $\lambda'_\nu = e^{-c'}\lambda_\nu$, then we may assume that $0 < |\lambda'_\nu| < 1$ ($\nu = 1, 2, \dots, n$). Besides, if we take a sufficiently large positive number c'' and put $\lambda''_\nu = e^{-c''}\lambda'_\nu$, then, among λ''_ν there does not hold any of the relations of the form (3). For, if there holds some one of the relations of the form (3), then we have

$$e^{-c''}\lambda'_\nu = (e^{-c''}\lambda'_1)^{p_1}(e^{-c''}\lambda'_2)^{p_2}\cdots\cdots(e^{-c''}\lambda'_n)^{p_n}.$$

Then it follows that

$$e^{c''(p_1+p_2+\cdots+p_n-1)}\lambda'_\nu = \lambda'^{p_1}_1\lambda'^{p_2}_2\cdots\cdots\lambda'^{p_n}_n,$$

consequently

$$|e^{c''}\lambda'_\nu| \leqq |e^{c''(p_1+p_2+\cdots+p_n-1)}\lambda'_\nu| < 1.$$

However, this does not hold for a sufficiently large positive number c''. Thus, putting $c = c' + c''$, we see that, if we take a sufficiently large positive number c and put $\mathring{\lambda}_\nu = e^{-c}\lambda_\nu$, then $0 < |\mathring{\lambda}_\nu| < 1$ and, among $\mathring{\lambda}_\nu$, there does not hold any of the relations of the form (3).

Taking such c, we consider the linear transformation $\mathfrak{L}$ as follows:

$$\mathfrak{L}: \qquad 'x^\nu = e^c x^\nu.$$

Put $\mathfrak{T}\mathfrak{L}^{-1} = \mathfrak{T}_0$, then it follows that

$$\mathfrak{T}_0: \qquad 'x^\nu = \varphi^\nu_0(x) = \varphi^\nu(e^{-c}x) = e^{-c}a^\nu_\mu x^\mu + \cdots\cdots .$$

Now the eigen values of the matrix $||e^{-c}a^\nu_\mu||$ are evidently $\mathring{\lambda}_\nu = e^{-c}\lambda_\nu$. Then, from the above results, we see that

(7) $$'x^\nu = \varphi^\nu_0(x) = e^{X_0}(x^\nu),$$

where $X_0 \equiv \xi^\mu_0 \dfrac{\partial}{\partial x^\mu}$ and ξ^μ_0 are determined by (5) corresponding to $\mathfrak{T}_0$, consequently they are of the forms (6). Now it is easily seen that $\mathfrak{L}$ can be expressed as follows:

(8) $$'x^\nu = e^c x^\nu = e^{cD}(x^\nu),$$

where D is the operator of the group of dilatations, i.e. $D = x^{\mu}\frac{\partial}{\partial x^{\mu}}$. Thus, from $\mathfrak{T}=\mathfrak{T}_0\mathfrak{L}$, we see that

$$\mathfrak{T}: \qquad 'x^{\nu} = \varphi^{\nu}(x) = \varphi_0^{\nu}(e^{c}x) = e^{cD}\varphi_0^{\nu}(x) = e^{cD}e^{X_0}(x^{\nu}).$$

Thus we have

Theorem. *Any finite transformation $\mathfrak{T}$ of the form* (1) *can be expressed as the product of two transformations, each of which is an iteration of the infinitesimal transformation, as follows:*

$$\mathfrak{T}: \qquad 'x^{\nu} = \varphi^{\nu}(x) = e^{cD}e^{X_0}(x^{\nu}),$$

where c is a suitable large positive number and D is the operator of the group of dilatations and X_0 is the operator of the suitable group with the regular operator functions.

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