ON THE LINEARIZATION OF A FORM OF HIGHER DEGREE AND ITS REPRESENTATION

By

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In the theory of invariants, an n-ary form of degree m $\sum a_{i_1 i_2 \dots i_m} x^{i_1} x^{i_2} \dots x^{i_m}$ $(i_1, i_2, \dots, i_m = 1, 2, \dots, n)$ is treated, only symbolically, in the linearized form

$$\sum_{i_1, i_2, \dots, i_m = 1}^{n} a_{i_1 i_2 \dots i_m} x^{i_1} x^{i_2} \dots x^{i_m} = \left(\sum_{i=1}^{n} a_i x^i\right)^m ,$$

where a_i are mere symbols, satisfying the relations

$$a_i a_k = a_k a_i \,, \quad a_{i_1} a_{i_2} \dots a_{i_m} = a_{i_1 i_2 \dots i_m} \text{ [1]}.$$

And, in the theory of spinors, an n-ary quadratic form $a_{ij} x^i x^j (a_{ij} = a_{ji})$ is linearized by the quantities γ_i satisfying $\gamma_{(i} \gamma_{j)} = a_{ij}$, in the form

$$\sum_{i,j=1}^{n} a_{ij} x^i x^j = \left(\sum_{i=1}^{n} \gamma_i x^i\right)^2 ,$$

and the structure and representation of the Clifford algebra generated by these γ_i have been investigated by many authors.[2]

We wish to extend the theory of spinors, by linearizing the n-ary form of degree m $\sum a_{i_1 i_2 \dots i_m} x^{i_1} x^{i_2} \dots x^{i_m}$ by the quantities p_i in the form $\sum a_{i_1 \dots i_m} x^{i_1} \dots x^{i_m}$ $= (\sum_{i=1}^{n} p_i x^i)^m$, and by investigating the structure and representation of the algebra generated by these p_i. In this paper, we shall define the generalized Clifford algebra by extending the concept of the ordinary Clifford algebra, and consider the linearization of $\sum_{i=1}^{n} (x^i)^m$ and its representation, by means of the particular case of this algebra.

However, the above quantities p_i satisfy the relations $p_i p_k = \omega p_k p_i$, different from the Weitzenböck's symbols a_i. But, from the standpoint of the theory of invariants, this fact does not come into question.

§ 1. Generalized Clifford Algebra

We shall define a generalized Clifford algebra (briefly G. C. algebra), by extending the concept of the ordinary Clifford algebra.[3]

1) R. Weitzenböck, Invarianten Theorie, (1923), p. 3.

2) R. Brauer and H. Weyl, Spinors in n dimensions, Amer. J. Math., vol. 57 (1935), pp. 425-449; C. Chevalley, Theory of Lie groups, (1946), p. 61.

3) C. Chevalley, ibid.

Let G be the direct product of n groups $G_1, G_2, \dots, G_n$, and let a symbol e_A correspond to each element $A \equiv (\alpha_1, \alpha_2, \dots, \alpha_n)$, $(\alpha_i \in G_i)$ of G. We define the following linear associative algebra $\mathfrak{o}$ with the basic elements e_A, $A \in G$ on a field K of characteristic zero.

i). $\mathfrak{o}$ is a left and right linear space on a field K.

ii). There is a mapping $a \to a^A$ in K, such that

$$e_A \cdot a = a^A \cdot e_A \quad \text{for any} \quad e_A, \ A \in G \quad \text{and} \quad a \in K. \tag{1.1}$$

iii). There is defined an associative multiplication in $\mathfrak{o}$, such that, for any two basic elements e_A and e_B: $A \equiv (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $B \equiv (\beta_1, \beta_2, \dots, \beta_n)$,

$$e_A e_B = \zeta(A, B) e_{AB}, \quad AB \equiv (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_n\beta_n), \tag{1.2}$$

where [1)]

$$\begin{aligned} \zeta(A, B) &= \prod_{k>i}^{*} \rho_{(\alpha_k, \beta_i)}^{(\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_{i-1}\beta_{i-1}\alpha_i\alpha_{i+1}\dots\alpha_{k-1})} \\ &\equiv \rho_{(\alpha_n, \beta_1)}^{(\alpha_1\alpha_2\dots\alpha_{n-1})} \cdot \rho_{(\alpha_{n-1}, \beta_1)}^{(\alpha_1\dots\alpha_{n-2})} \cdots \rho_{(\alpha_2, \beta_1)}^{(\alpha_1)} \\ &\quad \cdot \rho_{(\alpha_n, \beta_2)}^{(\alpha_1\beta_1\alpha_2\alpha_3\dots\alpha_{n-1})} \cdot \rho_{(\alpha_{n-1}, \beta_2)}^{(\alpha_1\beta_1\alpha_2\dots\alpha_{n-2})} \cdots \rho_{(\alpha_3, \beta_2)}^{(\alpha_1\beta_1\alpha_2)} \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad \cdot \rho_{(\alpha_n, \beta_{n-1})}^{(\alpha_1\beta_1\alpha_2\beta_2\dots\alpha_{n-2}\beta_{n-2}\alpha_{n-1})} \end{aligned} \tag{1.3}$$

in terms of $\rho(\alpha_k, \beta_i) \in K$, $(k > i)$, and

$$\rho(\alpha_k, \varepsilon_i) = \rho(\varepsilon_k, \beta_i) = 1 \quad (k > i), \tag{1.4}$$

for the unit element ε_i of G_i and the unit element 1 of K.

iv). For any basic elements e_A, e_B and any element $a \in K$,

$$e_A \cdot (a e_B) = (e_A a) \cdot e_B. \tag{1.5}$$

In the following, such a linear associative algebra $\mathfrak{o}$ shall be called a generalized Clifford algebra (G. C. algebra), and its elements, generalized Clifford numbers. And moreover we shall call $G_1, \dots, G_n$ the basic groups of $\mathfrak{o}$, and $\rho(\alpha_i, \beta_j)$ the structure numbers of $\mathfrak{o}$.

Now we state some properties of $\mathfrak{o}$ derived directly from this definition. From (i) and (ii) we get

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1) $\rho_{(\alpha_k, \beta_i)}^{(\alpha_1\beta_1\dots\alpha_{i-1}\beta_{i-1}\alpha_i\alpha_{i+1}\dots\alpha_{k-1})}$, lacking the upper indices $\alpha_k, \dots, \alpha_n$; $\beta_i, \dots, \beta_n$ in (1.3), means $\rho^A(\alpha_k, \beta_i)$ for $A \equiv (\alpha_1\beta_1, \alpha_2\beta_2, \dots, \alpha_{i-1}\beta_{i-1}, \alpha_i, \dots, \alpha_{k-1}, \varepsilon_k \dots \varepsilon_n)$. The form of $\zeta(A, B)$ is calculated, conversely, by the associative law of multiplication, from (1.1) and (1.12)-(1.15); for example, for $A \equiv (\alpha_1, \dots, \alpha_n)$ and $B \equiv (\beta_1, \varepsilon_2, \dots, \varepsilon_n)$,

$$\begin{aligned} e_A e_B &= e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_n} \cdot e_{\beta_1} = e_{\alpha_1} \dots e_{\alpha_{n-1}} \rho(\alpha_n, \beta_1) e_{\beta_1} e_{\alpha_n} = \rho_{(\alpha_n, \beta_1)}^{(\alpha_1 \dots \alpha_{n-1})} e_{\alpha_1} \dots e_{\alpha_{n-1}} e_{\beta_1} e_{\alpha_n} \\ &= \dots = \rho_{(\alpha_n, \beta_1)}^{(\alpha_1, \dots, \alpha_{n-1})} \dots \rho_{(\alpha_2, \beta_1)}^{(\alpha_1)} e_{AB}. \end{aligned}$$

$$(a+b)^A = a^A + b^A \quad \text{for} \quad a, b \in K, \ A \in G, \tag{1.6}$$

and

$$(ab)^A = a^A b^A \quad \text{for} \quad a, b \in K, \ A \in G. \tag{1.7}$$

Hence, from (1. 6) and (1. 7), the mapping $a \to a^A$ is an automorphism of the field K, and therefore

$$1^A = 1. \tag{1.8}$$

Next if we write $e_0 \equiv e_{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)}$, then, from the definition of multipication and (1. 8), we have

$$e_0 e_0 = e_0. \tag{1.9}$$

And also if we write $a^0 \equiv a^{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)}$, then we get

$$a^0 = a \quad \text{for any} \quad a \in K. \tag{1.10}$$

For, from (1. 9) it follows $e_0 e_0 a = e_0 a$, but $e_0 e_0 a = e_0 a^0 e_0 = (a^0)^0 e_0 e_0 = (a^0)^0 e_0 = e_0 a^0$, hence we have

$$e_0 a = e_0 a^0, \tag{1.11}$$

since $\mathfrak{o}$ is a linear argebra on a field of characteristic zero, from (1. 11) it follows $a^0 = a$.

Moreover, from (1. 3) we get a system of the following relations equivalent to (1. 3), writing $e_{\alpha_i} \equiv e_{(\varepsilon_1, \dots, \varepsilon_{i-1}, \alpha_i, \varepsilon_{i+1}, \dots, \varepsilon_n)}$:

$$e_A \equiv e_{(\alpha_1, \alpha_2, \dots, \alpha_n)} = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_n}, \tag{1.12}$$

$$e_0 e_{\alpha_i} = e_{\alpha_i} e_0 = e_{\alpha_i}, \quad e_0 e_0 = e_0, \tag{1.13}$$

$$e_{\alpha_i} e_{\beta_i} = e_{\alpha_i \beta_i}, \tag{1.14}$$

and

$$e_{\alpha_i} e_{\beta_j} = \rho(\alpha_i, \beta_j) e_{\beta_j} e_{\alpha_i} \quad (i > j). \tag{1.15}$$

Furthermore we must consider the condition for the associativity of multiplication. This condition is written in terms of $\zeta(A, B)$, as follows:

$$\zeta(A, B)\zeta(AB, C) = \zeta^A(B, C)\zeta(A, BC). \tag{1.16}$$

And this condition (1. 16) becomes the following form, by means of (1. 3),

$$\prod_{i>j}^{*} \rho_{(\alpha_i, \beta_j)}^{(\alpha_1\beta_1\alpha_2\dots\alpha_{j-1}\beta_{j-1}\alpha_j\alpha_{j+1}\dots\alpha_{i-1})} \cdot \prod_{i>k}^{*} \rho_{(\alpha_i\beta_i, \gamma_k)}^{(\alpha_1\beta_1\gamma_1\alpha_2\beta_2\gamma_2\dots\alpha_{k-1}\beta_{k-1}\gamma_{k-1}\alpha_k\beta_k\dots\alpha_{i-1}\beta_{i-1})}$$
$$= \prod_{j>k}^{*} \left(\rho_{(\beta_j, \gamma_k)}^{(\beta_1\gamma_1\dots\beta_{k-1}\gamma_{k-1}\beta_{k+1}\dots\beta_{j-1})}\right)^{(\alpha_1\dots\alpha_n)} \cdot \prod_{i>j}^{*} \rho_{(\alpha_i, \beta_j\gamma_j)}^{(\alpha_1\beta_1\gamma_1\alpha_2\beta_2\gamma_2\dots\alpha_{j-1}\beta_{j-1}\gamma_{j-1}\alpha_j\dots\alpha_{i-1})}. \tag{1.17}$$

In particular, if we substitute in (1. 17) $A \equiv (\varepsilon_1, \dots, \varepsilon_{i-1}, \alpha_i, \varepsilon_{i+1}, \dots, \varepsilon_n)$, $B \equiv (\varepsilon_1, \dots, \varepsilon_{j-1}, \beta_j, \varepsilon_{j+1}, \dots, \varepsilon_n)$ and $C \equiv (\varepsilon_1, \dots, \varepsilon_{j-1}, \gamma_j, \varepsilon_{j+1}, \dots, \varepsilon_n)$, $(i > j)$, we get

$$\rho(\alpha_i, \beta_j\gamma_j) = \rho(\alpha_i, \beta_j)\rho^{(\beta_j)}(\alpha_i, \gamma_j), \quad (i > j), \tag{1.18}$$

and also, substituting $A \equiv (\varepsilon_1, \dots, \varepsilon_{i-1}, \alpha_i, \varepsilon_{i+1}, \dots, \varepsilon_n)$, $B \equiv (\varepsilon_1, \dots, \varepsilon_{i-1}, \beta_i, \varepsilon_{i+1}, \dots, \varepsilon_n)$ and $C \equiv (\varepsilon_1, \dots, \varepsilon_{j-1}, \gamma_j, \varepsilon_{j+1}, \dots, \varepsilon_n)$, we have

$$\rho(\alpha_i\beta_i, \gamma_j) = \rho^{(\alpha_i)}(\beta_i, \gamma_j)\rho(\alpha_i, \gamma_j), \quad (i > j). \tag{1.19}$$

Hence, as a necessary condition for the associativity of $\circ$, we obtain (1.18) and (1.19). We shall consider again this condition later in § 3, (3.8).

Now, let two basic groups G_i and $G_j (i > j)$ be fixed, then as we shall see from (1.18), the mapping $\beta_j \rightarrow \rho(\alpha_i, \beta_j)$, ($\alpha_i$ fixed in G_i), from G_j onto $\Lambda_j(\alpha_i) \equiv \{\rho(\alpha_i, \beta_j); \beta_j \in G_j\} \subset K$ is not always an ordinary homomorphism. But this mapping has the following properties.

(1) The set $N_j(\alpha_i) \equiv \{\beta_j; \beta_j \in G_j, \rho(\alpha_i, \beta_j) = 1\}$ is a subgroup of G_j.

For, let us take $\overset{1}{\beta}_j, \overset{2}{\beta}_j \in N_j(\alpha_i)$, then we have

$$\rho(\alpha_i, \overset{1}{\beta}_j) = 1, \quad \rho(\alpha_i, \overset{2}{\beta}_j) = 1. \tag{1.20}$$

But, by (1.18), it holds

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, \overset{1}{\beta}_j)\, \rho^{(\overset{1}{\beta}_j)}(\alpha_i, \beta_j),$$

hence, by means of (1.8) and (1.20), this becomes

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = 1, \quad \text{i.e.,} \quad \overset{1}{\beta}_j\overset{2}{\beta}_j \in N_j(\alpha_i).$$

And, since $\rho(\alpha_i, \varepsilon_j) = 1$, obviously $\varepsilon_j \in N_j(\alpha_i)$. Moreover we have

$$\rho(\alpha_i, \beta_j)\, \rho^{(\beta_j)}(\alpha_i, \beta_j^{-1}) = \rho(\alpha_i, \beta_j\beta_j^{-1}) = \rho(\alpha_i, \varepsilon_j) = 1,$$

so it holds $\rho^{(\beta_j)}(\alpha_i, \beta_j^{-1}) = 1$ for β_j such that $\beta_j \in N_j(\alpha_i)$ i.e., $\rho(\alpha_i, \beta_j) = 1$, and therefore $\rho(\alpha_i, \beta_j^{-1}) = 1$, i.e., $\beta_j^{-1} \in N_j(\alpha_i)$. Thus $N_j(\alpha_i)$ is a subgroup of G_j.

(2) The necessary and sufficient condition for the subgroup $N_j(\alpha_i)$ to be a normal subgroup of G_j is that

$$\rho^{(\beta_j)}(\alpha_i, \theta_j) = \rho(\alpha_i, \theta_j), \quad \text{for any} \quad \beta_j \in N_j(\alpha_i), \theta_j \in G_j. \tag{1.21}$$

For, the necessary and sufficient condition for $N_j(\alpha_i)$ to be a normal subgroup of G_j is that $\rho(\alpha_i, \theta_j^{-1}\beta_j\theta_j) = 1$ for any $\beta_j \in N_j(\alpha_i)$, $\theta_j \in G_j$. But by (1.18) it holds

$$\begin{aligned}\rho(\alpha_i, \theta_j^{-1}\beta_j\theta_j) &= \rho(\alpha_i, \theta_j^{-1})\rho^{(\theta_j^{-1})}(\alpha_i, \beta_j\theta_j) \\ &= \rho(\alpha_i, \theta_j^{-1})\rho((\alpha_i, \beta_j)\rho^{(\beta_j)}(\alpha_i, \theta_j))^{(\theta_j^{-1})}\end{aligned}$$

and

$$\rho(\alpha_i, \theta_j^{-1})\rho^{(\theta_j^{-1})}(\alpha_i, \theta_j) = 1,$$

and moreover we have $\rho(\alpha_i, \beta_j)=1$ for $\beta_j \in N_j(\alpha_i)$. Therefore the above condition is equivalent to

$$\rho^{(\beta_j)}(\alpha_i, \theta_j) = \rho(\alpha_i, \theta_j) \quad \text{for any} \quad \beta_j \in N_j(\alpha_i), \theta_j \in G_j.$$

(3) The necessary and sufficient condition for the subgroup $N_j(\alpha_i)$ to contain the commutator group Q_j of G_j is that

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, \overset{2}{\beta}_j\overset{1}{\beta}_j) \quad \text{for any} \quad \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j. \tag{1.22}$$

For, the necessary and sufficient condition for $N_j(\alpha_i)$ to contain Q_j is $\rho(\alpha_i, \overset{1}{\beta}_j{}^{-1}\overset{2}{\beta}_j{}^{-1}\overset{1}{\beta}_j\overset{2}{\beta}_j)=1$ for any $\overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j$. But by (1.18) we have

$$\rho(\alpha_i, \overset{1}{\beta}_j{}^{-1}\overset{2}{\beta}_j{}^{-1}\overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, (\overset{2}{\beta}_j\overset{1}{\beta}_j)^{-1})(\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j))^{(\overset{2}{\beta}_j\overset{1}{\beta}_j)^{-1}}$$

and

$$\rho(\alpha_i, (\overset{2}{\beta}_j\overset{1}{\beta}_j)^{-1})(\rho(\alpha_i, \overset{2}{\beta}_j\overset{1}{\beta}_j))^{(\overset{2}{\beta}_j\overset{1}{\beta}_j)^{-1}} = 1.$$

Hence the above condition is equivalent to

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, \overset{2}{\beta}_j\overset{1}{\beta}_j) \quad \text{for any} \quad \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j.$$

Collecting the above results we obtain the theorem.

Theorem 1. *Let G_i and G_j $(i>j)$ be the two basic groups of a G. C. algebra $\mathfrak{o}$, and let $\rho(\alpha_i, \beta_j)$ be the structure numbers of $\mathfrak{o}$. Then we have the relations*

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, \overset{1}{\beta}_j)\cdot\rho^{(\overset{1}{\beta}_j)}(\alpha_i, \overset{2}{\beta}_j) \quad \textit{for any} \quad \alpha_i \in G_i; \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j,$$
$$\rho(\overset{1}{\alpha}_i\overset{2}{\alpha}_i, \beta_j) = \rho^{(\overset{1}{\alpha})}(\overset{2}{\alpha}_i, \beta_j)\cdot\rho(\overset{1}{\alpha}_i, \beta_j) \quad \textit{for any} \quad \overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i; \beta_j \in G_j.$$

The set $N_j(\alpha_i) \equiv \{\beta_j; \beta_j \in G_j, \rho(\alpha_i, \beta_j)=1\}$ is a subgroup of G_j. The necessary and sufficient condition for $N_j(\alpha_i)$ to be a normal subgroup of G_j is that $\rho^{(\beta_j)}(\alpha_i, \theta_j)=\rho(\alpha_i, \theta_j)$ for any $\beta_j \in N_j(\alpha_i)$, $\theta_j \in G_j$. The necessary and sufficient condition for $N_j(\alpha_i)$ to contain the commutator group Q_j of G_j is that $\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j)=\rho(\alpha_i, \overset{2}{\beta}_j\overset{1}{\beta}_j)$ for any $\overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j$.

As for $N_i'(\beta_j)=\{\alpha_i; \alpha_i \in G_i, \rho(\alpha_i, \beta_j)=1\}$, the similar results are obtained; that is, $N_i'(\beta_j)$ is a subgroup of G_i, the necessary and sufficient condition for $N_i'(\beta_j)$ to be normal is that $\rho^{(\alpha_i)}(\theta_i, \beta_j)=\rho(\theta_i, \beta_j)$ for any $\alpha_i \in N_i'(\beta_j)$, $\theta_i \in G_i$, and the necessary and sufficient condition for $N_i'(\beta_j)$ to contain Q_i is that $\rho(\overset{1}{\alpha}_i\overset{2}{\alpha}_i, \beta_j)=\rho(\overset{2}{\alpha}_i\overset{1}{\alpha}_i, \beta_j)$ for any $\overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i$.

§ 2. G. C. Algebra Reducible to G. C. Algebra Each of Whose Basic Groups is Commutative

The structure of G. C. algebra is determined by the properties of the set $\Lambda\equiv\{\rho(\alpha_i, \beta_j)\,;\, \alpha_i\in G_i, \beta_j\in G_j, i>j, i, j=1, 2, \dots, n\}$ and the basic groups G_i. Now, we consider the case, in particular, where Λ has the following properties:

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j)=\rho(\alpha_i, \overset{2}{\beta}_j\overset{1}{\beta}_j) \quad \text{for any} \quad \alpha_i\in G_i,\ \overset{1}{\beta}_j, \overset{2}{\beta}_j\in G_j, \tag{2.1}$$

$$\rho(\overset{1}{\alpha}_i\overset{2}{\alpha}_i, \beta_j)=\rho(\overset{2}{\alpha}_i\overset{1}{\alpha}_i, \beta_j) \quad \text{for any} \quad \overset{1}{\alpha}_i, \overset{2}{\alpha}_i\in G_i,\ \beta_j\in G_j. \tag{2.2}$$

These conditions (2.1) and (2.2), by Theorem 1, are equivalent to

$$N_j(\alpha_i)\supset Q_j \quad \text{for any} \quad \alpha_i\in G_i, \tag{2.3}$$

$$N'_i(\beta_j)\supset Q_i \quad \text{for any} \quad \beta_j\in G_j \tag{2.4}$$

respectively.

Let $\tilde{G}_i$ be the factor group G_i/Q_i of G_i by Q_i, then $\tilde{G}_i$ is commutative. We write $\{\tilde{\alpha}_i\}$ the represetative system of $\tilde{G}_i$ in G_i, then any element α_i of G_i is expressed in the form: $\alpha_i=\alpha_i\tilde{\sigma}_i, \tilde{\sigma}_i\in Q_i$. Similarly and element β_j of G_j is written in the form: $\beta_j=\tilde{\beta}_j\tau_j, \tau_j\in Q_j$, where $\{\tilde{\beta}_j\}$ denotes the representative system of $\tilde{G}_j$ in G_j.

Then, from (1.18) we get

$$\rho(\alpha_i, \beta_j)=\rho(\alpha_i, \tilde{\beta}_j\tau_j)=\rho(\alpha_i, \tilde{\beta}_j)\, \rho^{(\tilde{\beta}_j)}(\alpha_i, \tau_j),$$

but, from $\tau_j\in Q_j\subset N_j(\alpha_i)$ it follows that $\rho(\alpha_i, \tau_j)=1$, and therefore $\rho^{(\tilde{\beta}_j)}(\alpha_i, \tau_j)=1$. Hence we have

$$\rho(\alpha_i, \beta_j)=\rho(\alpha_i, \tilde{\beta}_j). \tag{2.5}$$

And similarly it follows that

$$\rho(\alpha_i, \tilde{\beta}_j)=\rho(\tilde{\alpha}_i\sigma_i, \tilde{\beta}_j)=\rho^{(\tilde{\alpha}_i)}(\sigma_i, \tilde{\beta}_j)\cdot\rho(\tilde{\alpha}_i, \tilde{\beta}_j).$$

Since $\sigma_i\in Q_i\subset N'_i(\beta_j)$, we have $\rho(\sigma_i, \tilde{\beta}_j)=1$, and therefore $\rho^{(\tilde{\alpha}_i)}(\sigma_i, \tilde{\beta}_j)=1$. Hence we get

$$\rho(\alpha_i, \tilde{\beta}_j)=\rho(\tilde{\alpha}_i, \tilde{\beta}_j). \tag{2.6}$$

Therefore from (2.5) and (2.6) it follows that

$$\rho(\alpha_i, \beta_j)=\rho(\tilde{\alpha}_i, \tilde{\beta}_j), \quad (i>j). \tag{2.7}$$

So we may define a G. C. algebra $\mathfrak{o}$ which has the basic groups $\tilde{G}_i$ and the structure numbers $\tilde{\rho}(\tilde{\alpha}_i, \tilde{\beta}_j)=\rho(\alpha_i, \beta_j)$. Denote by $\tilde{e}_A$ the basic element of $\tilde{\mathfrak{o}}$ which corresponds to an element $\tilde{A}\equiv(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)$ of $G\equiv\tilde{G}_1\times\tilde{G}_2\times\cdots\times G_n$.

And we make $\tilde{e}_{\tilde{A}}$ correspond to the basic element e_A of $\mathfrak{o}$. By this correspondence between e_A and $\tilde{e}_{\tilde{A}}$ we may give a correspondence between the linear spaces $\mathfrak{o}$ and $\tilde{\mathfrak{o}}$. Then, since $e_A a = a^A e_A$, we get $\tilde{e}_{\tilde{A}} a = a^A \tilde{e}_{\tilde{A}}$, hence we have

$$a^{\tilde{A}} = a^A. \tag{2.8}$$

Therefore, by (2.7), (2.8) and (1.3), we obtain

$$\zeta(A, B) = \tilde{\zeta}(\tilde{A}, \tilde{B}). \tag{2.9}$$

Thus we see that the G. C. algebra $\mathfrak{o}$ is homomorphic to the G. C. algebra $\tilde{\mathfrak{o}}$.

Next let $\mathfrak{o}'$ be a G. C. algebra which has the basic groups G'_i $(i=1, 2, \dots n)$, and has the structure numbers $\rho'(\alpha'_i, \beta'_j)$. And we shall define that the G. C. algebra $\mathfrak{o}$ is homomorphic to $\mathfrak{o}'$ when the basic groups G_i are homomorphic onto G'_i respectively, and in the homomorphisms: $\alpha_i \to \alpha'_i$ it holds $\rho(\alpha_i, \beta_j) = \rho'(\alpha'_i, \beta'_j)$. Then by the correspondence from the basic element e_A, $A \in G \equiv G_1 \times G_2 \times \cdots G_n$ to $e'_{A'}$, $A' \in G' \equiv G'_1 \times G'_2 \times \cdots \times G'_n$, the G. C. algebra $\mathfrak{o}$ is homomorphic to $\mathfrak{o}'$ in the ordinary sense, (see (2.8) and (2.9)).

Now let us suppose that the G. C. algebra $\mathfrak{o}$ is homomorphic to a G. C. algebra $\mathfrak{o}'$ each of whose basic groups is commutative. Then, from the above definition, we have

$$\rho(\alpha_i, \beta_j) = \rho'(\alpha'_i, \beta'_j) \tag{2.10}$$

Since G'_i is commutative, $\overset{1}{\beta'_j}\overset{2}{\beta'_j} = \overset{2}{\beta'_j}\overset{1}{\beta'_j}$ for any $\overset{1}{\beta'_j}, \overset{2}{\beta'_j} \in G'_j$, hence we have

$$\rho'(\alpha'_i, \overset{1}{\beta'_j}\overset{2}{\beta'_j}) = \rho'(\alpha'_i, \overset{2}{\beta'_j}\overset{1}{\beta'_j}) \text{ for any } \alpha'_i \in G'_i, \overset{1}{\beta'_j}, \overset{2}{\beta'_j} \in G'_j. \tag{2.11}$$

From (2.10) and (2.11) it follows that

$$\rho(\alpha_i, \overset{1}{\beta_j}\overset{2}{\beta_j}) = \rho(\alpha_i, \overset{2}{\beta_j}\overset{1}{\beta_j}) \text{ for any } \alpha_i \in G_i, \overset{1}{\beta_j}, \overset{2}{\beta_j} \in G_j. \tag{2.1}$$

and similarly we have

$$\rho(\overset{1}{\alpha_i}\overset{2}{\alpha_i}, \beta_j) = \rho(\overset{2}{\alpha_i}\overset{1}{\alpha_i}, \beta_j) \text{ for any } \overset{1}{\alpha_i}, \overset{2}{\alpha_i} \in G_i, \beta_j \in G_j. \tag{2.2}$$

Thus we have the following theorem.

Theorem 2. *Let G. C. algebra $\mathfrak{o}$ be defined by the basic groups G_i and the structnre numbers $\rho(\alpha_i, \beta_j)$, $(i>j, i, j=1, 2, \dots, n)$, Then the necessary and sufficient condition in order that the G. C. algebra $\mathfrak{o}$ is homomorphic to a G. C. algebra $\mathfrak{o}'$ each of whose basic groups is commutative is that*

$$\rho(\alpha_i, \overset{1}{\beta_j}\overset{2}{\beta_j}) = \rho(\alpha_i, \overset{2}{\beta_j}\overset{1}{\beta_j}) \quad \textit{for any} \quad \alpha_i \in G_i, \overset{1}{\beta_j}, \overset{2}{\beta_j} \in G_j$$
$$\rho(\overset{1}{\alpha_i}\overset{2}{\alpha_i}, \beta_j) = \rho(\overset{2}{\alpha_i}\overset{1}{\alpha_i}, \beta_j) \quad \textit{for any} \quad \overset{1}{\alpha_i}, \overset{2}{\alpha_i} \in G_i, \beta_j \in G_j.$$

§ 3. G. C. Algebra Whose Basic Elements are Commutative with its Structure Numbers

In this section we shall consider the G. C. algebra whose basic elements e_A, $A \in G$ are commutative with its structure numbers $\rho(\alpha_i, \beta_j)$, $i > j$, $i, j = 1, 2, \dots, n$. In this case, we have

$$e_A \cdot \rho(\alpha_i, \beta_j) = \rho(\alpha_i, \beta_j) \cdot e_A \quad \text{for any} \quad A \in G. \tag{3.1}$$

And it follows from (1.12) that (3.1) is equivalent to

$$e_{\gamma_k} \cdot \rho(\alpha_i, \beta_j) = \rho(\alpha_i, \beta_j) \cdot e_{\gamma_k},$$

in other words,

$$\rho^{(\gamma_k)}(\alpha_i, \beta_j) = \rho(\alpha_i, \beta_j) \quad \text{for any} \quad \gamma_k \in G_k, \quad k = 1, 2, \dots, n. \tag{3.2}$$

That is, the mapping $a \to a^A$ is an identical mapping on the set Λ.

We shall prove that, in this case, Λ is a commutative set in K. We have from (1.15), for $i > j$, $k > l$,

$$(e_{\alpha_i} e_{\beta_j})(e_{\gamma_k} e_{\delta_l}) = \rho(\alpha_i, \beta_j) e_{\beta_j} e_{\alpha_i} \cdot \rho(\gamma_k, \delta_l) e_{\delta_l} e_{\gamma_k},$$

any by (3.2)

$$= \rho(\alpha_i, \beta_j) \rho(\gamma_k, \delta_l) e_{\beta_j} e_{\alpha_i} e_{\delta_l} e_{\gamma_k}. \tag{3.3}$$

On the other hand, we have from (1.15)

$$(e_{\alpha_i} e_{\beta_j}) e_{\gamma_k} e_{\delta_l} = \rho(\alpha_i, \beta_j) e_{\beta_j} e_{\alpha_i} e_{\gamma_k} e_{\delta_l},$$

by (3.2)

$$= e_{\beta_j} e_{\alpha_i} (e_{\gamma_k} e_{\delta_l}) \rho(\alpha_i, \beta_j)$$
$$= e_{\beta_j} e_{\alpha_i} \rho(\gamma_k, \delta_l) e_{\delta_l} e_{\gamma_k} \rho(\alpha_i, \beta_j),$$

and by (3.2)

$$= \rho(\gamma_k, \delta_l) \rho(\alpha_i, \beta_j) e_{\beta_j} e_{\alpha_i} e_{\delta_l} e_{\gamma_k}. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$\rho(\alpha_i, \beta_j) \rho(\gamma_k, \delta_l) = \rho(\gamma_k, \delta_l) \rho(\alpha_i, \beta_j), \quad (i > j, \ k > l). \tag{3.5}$$

Therefore Λ is a commutative set in K.

Next we shall consider the condition for the associativity of $\circ$ in this case. By means of (3.2) and (3.5), it follows from (1.3) that in this case

$$\zeta(A, B) = \prod_{k > i} \rho(\alpha_k, \beta_i). \tag{3.6}$$

And from (1.18) and (1.19) we get

$$\rho(\alpha_i, \overset{1}{\beta_j}\overset{2}{\beta_j}) = \rho(\alpha_i, \overset{1}{\beta_j}) \rho(\alpha_i, \overset{2}{\beta_j}), \tag{3.7}$$

$$\rho(\overset{1}{\alpha_i}\overset{2}{\alpha_i}, \beta_j) = \rho(\overset{1}{\alpha_i}, \beta_j) \rho(\overset{2}{\alpha_i}, \beta_j) \tag{3.8}$$

respectively. Moreover, it is easily verified that in this case (1.17) follows from (3.7) and (3.8) by using (3.5). Hence, for this case, (3.7) and (3.8) are the necessary and sufficient condition in order that the multiplication to define the G. C. algebra is associative.

Thus we get the theorem.

Theorem 3. *Let e_A be a symbol corresponding to a element $A \in G \equiv G_1 \times \cdots \times G_n$, and let $\Lambda \equiv \{\rho(\alpha_i, \beta_j)\,;\ \alpha_i \in G_i,\ \beta_j \in G_j,\ i > j,\ i, j = 1, 2, \ldots, n\}$ be a set of numbers which are commutative with e_A, $A \in G$. Then the necessary and sufficient condition that there is a G. C. algebra having G_i as the basic groups and Λ as the set of the structure numbers is that*

1°. *Λ is a commutative set, and*

2°. $$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, \overset{1}{\beta}_j)\rho(\alpha_i, \overset{2}{\beta}_j) \quad \textit{for any} \quad \alpha_i \in G_i,\ \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j \tag{3.7}$$

$$\rho(\overset{1}{\alpha}_i\overset{2}{\alpha}_i, \beta_j) = \rho(\overset{1}{\alpha}_i, \beta_j)\rho(\overset{2}{\alpha}_i, \beta_j) \quad \textit{for any} \quad \overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i,\ \beta_j \in G_j. \tag{3.8}$$

As we shall see from this theorem, in the G. C. algebra whose basic elements are commutative with its structure numbers, it is evident that

$$\rho(\alpha_i, \overset{1}{\beta}_j\overset{2}{\beta}_j) = \rho(\alpha_i, \overset{2}{\beta}_j\overset{1}{\beta}_j) \quad \text{for any} \quad \alpha_i \in G_i,\ \overset{1}{\beta}_j, \overset{2}{\beta}_j \in G_j,$$

$$\rho(\overset{1}{\alpha}_i\overset{2}{\alpha}_i, \beta_j) = \rho(\overset{2}{\alpha}_i\overset{1}{\alpha}_i, \beta_j) \quad \text{for any} \quad \overset{1}{\alpha}_i, \overset{2}{\alpha}_i \in G_i,\ \beta_j \in G_j.$$

So by Theorem 2 such a G. C. algebra $\mathfrak{o}$ is homomorphic to the G. C. algebra $\breve{\mathfrak{o}}$ constructed in Theorem 2. Therefore, in our case, the G. C. algebra $\mathfrak{o}$ is reduced to the G. C. algebra, as for the structure, each of whose basic groups $G_1, G_2, \ldots, G_n$ is commutative.

In particular, let us suppose that the basic groups $G_1, G_2, \ldots, G_n$ are the cyclic groups of order $m_1, m_2, \ldots, m_n$ respectively, i.e., $G_i \equiv [\varepsilon_i, \alpha_i, \ldots, \alpha_i^{m_i-1}]$, $\alpha_i^{m_i} = \varepsilon_i$. By Theorem 3, the set $\Lambda \equiv \{\rho(\alpha_i^{\lambda_i}, \alpha_j^{\lambda_j})\,;\ \alpha_i \in G_i,\ \lambda_i = 0, 1, \ldots, m_i - 1,\ i, j = 1, \ldots, n\}$ of the structure numbers of the G. C. algebra with the basic groups $G_1, \ldots, G_n$ is commutative and is determined by the conditions:

$$\begin{aligned} \rho(\alpha_i, \alpha_j^{\lambda+\mu}) &= \rho(\alpha_i, \alpha_j^{\lambda})\rho(\alpha_i, \alpha_j^{\mu}) \\ \rho(\alpha_i^{\lambda+\mu}, \alpha_j) &= \rho(\alpha_i^{\lambda}, \alpha_j)\rho(\alpha_i^{\mu}, \alpha_j) \quad (i > j). \end{aligned} \tag{3.9}$$

By means of (3.9) we have

$$\rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j}) = (\rho(\alpha_i^{\lambda_i}, \alpha_j))^{\mu_j} = (\rho(\alpha_i, \alpha_j))^{\lambda_i\mu_j}. \tag{3.10}$$

Since $\alpha_i^{m_i} = \varepsilon_i$, we have

$$\left.\begin{aligned} \rho(\alpha_i, \alpha_j)^{m_i} &= \rho(\alpha_i^{m_i}, \alpha_j) = \rho(\varepsilon_i, \alpha_j) = 1 \\ \rho(\alpha_i, \alpha_j)^{m_j} &= \rho(\alpha_i, \alpha_j^{m_j}) = \rho(\alpha_i, \varepsilon_j) = 1 \end{aligned}\right\}, \tag{3.11}$$

and therefore $\rho(\alpha_i, \alpha_j)$ must be a primitive l_{ij}-th root ω_{ij} of unity where l_{ij} is a factor of m_i and m_j. Then by (3.6) and (3.10) we obtain

$$\rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j}) = \omega_{ij}^{\lambda_i \mu_j} \tag{3.12}$$

and, for $A \equiv (\alpha_1^{\lambda_1}, \alpha_2^{\lambda_2}, \dots, \alpha_n^{\lambda_n})$ and $B \equiv (\alpha_1^{\mu_1}, \alpha_2^{\mu_2}, \dots, \alpha_n^{\mu_n})$, we obtain

$$\zeta(A, B) = \prod_{i>j} \rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j}) = \prod_{i>j} \omega_{ij}^{\lambda_i \mu_j}. \tag{3.13}$$

Since we suppose that e_A is commutative with the structure numbers, e_A must be commutative with ω_{ij}. Besides, ω_{ij} must be commutative mutually, Λ being commutative. And conversely, from (3.12), it is easily verified that $\rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j}) = \omega_{ij}^{\lambda_i \mu_j}$ satisfies the conditions (3.7) and (3.8) in Theorem 3.

Thus we have the theorem.

Theorem 4. *For the G. C. algebra with the basic groups* $G_i \equiv [\varepsilon_i, \alpha_i, \dots, \alpha_i^{m_i - 1}]$, $\alpha_i^{m_i} = \varepsilon_i$, *the structure numbers* $\rho(\alpha_i^{\lambda_i}, \alpha_j^{\mu_j})$ *are equal to* $\omega_{ij}^{\lambda_i \mu_j}$ *where* ω_{ij} *is a primitive* l_{ij}*-th root of unity and* l_{ij} *is a factor of* m_i *and* m_j.

§ 4. Linearizatin of $\sum_{i=1}^{n} (x^i)^m$ [1]

Now, in particular, we shall consider the G. C. algebra whose basic groups are the same cyclic group of order m: $[\varepsilon, \alpha, \alpha^2, \dots, \alpha^{m-1}]$, $\alpha^m = \varepsilon$ and whose basic elements are commutative with its structure numbers. Suppose that the field K is commutative and contains a primitive m-th root of unity. If we take $m=2$ and K as the field of real or complex numbers, then we obtain the theory of ordinary spinors. As a special case of Theorem 4 we have

Theorem 4′. *For the G. C. algebra with the basic groups* $G_i \equiv [\varepsilon, \alpha, \alpha^2, \dots, \alpha^{m-1}]$, $\alpha^m = \varepsilon$, *the structure numbers* $\rho(\alpha^\lambda, \alpha^\mu)$ *are equal to* $\omega^{\lambda\mu}$, *where* ω *is any* m*-th root of unity in* K.

So if we write $\overset{\lambda}{e}_i \equiv e_{(\varepsilon, \dots, \underset{(i)}{\alpha^\lambda}, \varepsilon, \dots, \varepsilon)}$, then we have

$$\overset{\lambda}{e}_i \overset{\mu}{e}_j = \omega^{\lambda\mu} \overset{\mu}{e}_j \overset{\lambda}{e}_i \quad (i > j), \tag{4.1}$$

---

1) If an n-ary form of degree m: $'a_{j_1 j_2 \dots j_m} {'x}^{j_1} {'x}^{j_2} \dots {'x}^{j_m}$ is transformed to $\sum_{i=1}^{n} (x^i)^m$ by a linear transformation $'x^j = h_i^j x^i$, namely, $'a_{j_1 j_2 \dots j_m} = \sum_{k=1}^{n} \bar{h}_{j_1}^k \bar{h}_{j_2}^k \dots \bar{h}_{j_m}^k$ where $\| \bar{h}_j^k \|$ is the invers matrix of $\| h_i^j \|$, then this form is linearized as follows: $'a_{j_1 j_2 \dots j_m} {'x}^{j_1} {'x}^{j_2} \dots {'x}^{j_m} = \sum_{i=1}^{n} (x^i)^m = (x^i p_i)^m = (x^l h_l^j \cdot \bar{h}_j^k p_k)^m = ({'x}^j {'p}_j)^m$ where $'x^j = h_i^j x^i$ and $'p_j = \bar{h}_j^k p_k$. (see p. 25). Only here we use the summation convention in tensor calculus.

in particular,

$$\overset{\lambda}{e}_i\overset{\lambda}{e}_j=\omega^{\lambda^2}\overset{\lambda}{e}_j\overset{\lambda}{e}_i \quad (i>j). \tag{4.2}$$

We shall prove the following theorem.

Theorem 5. *Let $\overset{\lambda}{e}_i$ be the basic element $e_{(\varepsilon,\dots,\varepsilon,\underset{(i)}{x^\lambda},\varepsilon,\dots,\varepsilon)}$ of the G. C. algebra whose basic groups are the same cyclic group of order m and whose structure numbers are $\rho(\alpha^\lambda,\alpha^\mu)=\omega^{\lambda\mu}$, ($\omega$ is a primitive m-th root of unity in K). Then we have a identity*

$$\left(\sum_{i=1}^{n} x^i\overset{\lambda}{e}_i\right)^l=e_0\sum_{i=1}^{n}(x^i)^l, \tag{4.3}$$

for $l=m/(\lambda, m)$ if and only if $(m/(\lambda, m), (\lambda, m))=1$.

Proof. For and only for such a integer l, l which is a minimal positive integer satisfying $l\lambda^2\equiv 0\ (mod.\ m)$, satisfies $l\lambda\equiv 0\ (mod.\ m)$, and therefore ω^{λ^2} is a primitive l-th root of unity, and also $(\overset{\lambda}{e}_i)^l=e_{(\varepsilon,\dots;\underset{(i)}{\alpha^{l\lambda}},\varepsilon,\dots,\varepsilon)}=e_0$. Now we write g_0, g_i and τ for $e_0, \overset{\lambda}{e}_i$ and ω^{λ^2} respectively. And we shall prove, by the mathematical induction with respect to n, that if

$$g_ig_j=\tau g_jg_i \quad (i>j),\ (g_i)^l=g_0, \tag{4.4}$$

then it holds identically

$$\left(\sum_{i=1}^{n} x^ig_i\right)^l=g_0\sum_{i=1}^{n}(x^i)^l.$$

First we shall consider the case for $n=2$. In this case we have

$$(x^1g_1+x^2g_2)^l=\sum_{r=0}^{l}C_r(x^1)^{l-r}(x^2)^r,$$

where the quantity C_r is obtained by multiplying $l-r$ g_1's and r g_2's together in every possible way and by adding the terms so obtained. It follows from (4.4) that

$$C_0=C_l=g_0,\ C_r=\sum_{0\leq k_1\leq k_2\leq\cdots\leq k_r\leq l-r}\tau^{k_1+k_2+\cdots+k_r}\cdot(g_1)^{l-r}(g_2)^r, \quad (1\leq r\leq l-1). \tag{4.5}$$

If we write $h_1=k_1, h_2=k_2+1, \dots, h_r=k_r+(r-1)$, then we have

$$D_r\equiv\tau^{\frac{r(r-1)}{2}}\cdot C_r=\sum_{0\leq h_1<h_2<\cdots<h_r\leq l-1}\tau^{h_1+h_2+\cdots+h_r}\cdot(g_1)^{l-r}(g_2)^r, \quad (1\leq r\leq l-1). \tag{4.6}$$

To prove $D_r=0$, we shall prove, by the mathematical induction with respect to r,

$$\overset{*}{D}_{r;\nu_1,\nu_2,\dots,\nu_r}\equiv\overset{*}{\sum_{0\leq t_1,t_2,\dots,t_r\leq l-1}}\tau^{\nu_1t_1+\nu_2t_2+\cdots+\nu_rt_r}=0, \quad (\nu_1,\nu_2,\dots,\nu_r=0,1,2,\dots,l-1), \tag{4.7}$$

where $\overset{*}{\sum}$ means the sum of the terms for all distinct $t_1, t_2, \dots, t_r$. In the case for $r=1$, we have

$$\overset{*}{D}_{1\,;\,\nu_1} \equiv \sum_{t_1=0}^{l-1} \tau^{\nu_1 t_1} = 0, \tag{4.8}$$

since $(\tau^{\nu_1})^l=1$ but $\tau^{\nu_1}\neq 1$. Next, let us suppose that

$$\overset{*}{D}_{s\,:\,\nu_1, \nu_2, \dots, \nu_s} = 0 \quad \text{for} \quad r-1 \geqq s.$$

It is easily verified that

$$0 = \sum_{t_1=0}^{l-1} (\tau^{\nu_1})^{t_1} \cdot \sum_{t_2=0}^{l-1} (\tau^{\nu_2})^{t_2} \cdots \sum_{t_r=0}^{l-1} (\tau^{\nu_r})^{t_r} = \overset{*}{D}_{r\,:\,\nu_1, \dots, \nu_r} + R,^{1)} \tag{4.9}$$

where R is the sum of the terms of the same form as $\overset{*}{D}_{s\,:\,\kappa_1, \dots, \kappa_s}, (r-1\geqq s)$. By the assumption of the mathematical induction we have $R=0$. Hence we obtain from (4.9)

$$\overset{*}{D}_{r\,;\,\nu_1, \nu_2, \dots, \nu_r} = 0. \tag{4.7}$$

In particular, if we put $\nu_1=\nu_2=\cdots=\nu_r=1$ in (4.7), then we get

$$\overset{*}{D}_{r\,;\,1,1,\dots,1} \equiv \overset{*}{\sum_{0\leqq t_1, \dots t_r \leqq l-1}} \tau^{t_1+t_2+\cdots+t_r} = 0 \tag{4.10}$$

Since $\overset{*}{D}_{r\,:\,1,1,\dots,1}(g_1)^{l-r}(g_2)^r = r!D_r$, from (4.10) we have $D_r=0$. Therefore from (4.6) we obtain

$$C_r = 0. \tag{4.11}$$

Thus it follows from the former of (4.5), and (4.11) that

$$(x^1 g_1 + x^2 g_2)^l = g_0((x^1)^l + (x^2)^l). \tag{4.12}$$

Next let us suppose that

$$\left(\sum_{i=1}^{n-1} x^i g_i\right)^l = g_0 \sum_{i=1}^{n-1} (x^i)^l. \tag{4.13}$$

If we write $f_1 = \sum_{i=1}^{n-1} x^i g_i$ and $f_2 = x^n g_n$, then it follows from (4.4) that

$$f_2 f_1 = \tau f_1 f_2. \tag{4.14}$$

Therefore, as we shall see by taking f_1 and f_2 instead of g_1 and g_2 respectively in the above proof in the case $n=2$, it holds that

$$(f_1+f_2)^l = f_1^l + f_2^l, \tag{4.15}$$

1) For example, the sum of terms such that $t_1=t_2$ and t_1, t_j $(j=3, 4, \dots, r)$ are all distinct is written as follows:

$$\sum_{0\leqq t_1,\, t_3, \dots t_r \leqq l-1} \tau^{(\nu_1+\nu_2)t_1+\nu_3 t_3+\cdots+\nu_r t_r} = \overset{*}{D}_{r-1\,;\,\nu_1+\nu_2,\, \nu_3, \dots, \nu_r}.$$

and therefore we have

$$\left(\sum_{i=1}^{n} x^i g_i\right)^l = \left(\sum_{i=1}^{n-1} x^i g_i\right)^l + (x^n g_n)^l .$$

Hence, by means of (4.16) and the latter expression of (4.4) we obtain

$$\left(\sum_{i=1}^{n} x^i g_i\right)^l = g_0 \sum_{i=1}^{n} (x^i)^l :$$

§ 5. Structure of the G. C. Algebra Associated with the Linearization of $\sum_{i=1}^{n}(x^i)^m$.

In the present section we write p_0 and p_i for e_0 and $\overset{1}{e}_i$ in § 4 respectively. So we have from (4.5)

$$p_i p_j = \omega p_j p_i \quad (i > j), \tag{5.1}$$

where ω is a primitive m-th root of unity. It is easily seen that

$$p_i^m = p_0 . \tag{5.2}$$

Hence we have by Theorem 5

$$\left(\sum_{i=1}^{n} x^i p_i\right)^m = p_0 \sum_{i=1}^{n} (x^i)^m . \tag{5.3}$$

And if we write p_A for e_A in § 4, then we have

$$p_A = p_1^{\lambda_1} p_2^{\lambda_2} \dots p_n^{\lambda_n} , \tag{5.4}$$

where $A \equiv (\alpha^{\lambda_1}, \alpha^{\lambda_2}, \dots, \alpha^{\lambda_n})$. Moreover it follows from (4.3) that

$$p_A p_B = \prod_{i>j} \omega^{\lambda_i \mu_j} p_{AB} = \omega^{\sum_{i>j} \lambda_i \mu_j} p_{AB} , \tag{5.5}$$

where $B \equiv (\alpha^{\mu_1}, \alpha^{\mu_2}, \dots, \alpha^{\mu_n})$. For the sake of brevity we shall write $A \equiv (\lambda_1, \lambda_2, \dots, \lambda_n)$ in place of $A \equiv (\alpha^{\lambda_1}, \alpha^{\lambda_2}, \dots, \alpha^{\lambda_n})$. Then we have

$$AB = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots, \lambda_n + \mu_n) . \tag{5.6}$$

Now we shall investigate the center and ideal of the G. C. algebra generated by p_0 and p_i $(i=1, 2, \dots, n)$. This G. C. algebra will be called the G. C. algebra associated with the linearization of $\sum_{i=1}^{n}(x^i)^m$.

Let $x \equiv \sum_A c_A p_A$, $c_A \in K$ be any element of the center $\mathfrak{c}$ of $\mathfrak{o}$, then $\mathfrak{c}$ is characterized by the property

$$p_h x p_h^{-1} = x , \ (h = 1, 2, \dots, n) \ \text{ for } \ x \in \mathfrak{c} . \tag{5.7}$$

But we have, by (5.5)

$$p_h p_A = \omega^{\sum_{i<h} \lambda_i} \cdot p_{(\lambda_1, \dots, \lambda_{h-1}, \lambda_h + 1, \lambda_{h+1}, \dots, \lambda_n)} ,$$

and similarly

$$p_A p_h = \omega^{\sum_{j>h}\lambda_j} \cdot p_{(\lambda_1, \dots, \lambda_{h-1}, \lambda_h+1, \lambda_{h+1}, \dots, \lambda_n)},$$

so we get, eliminating $p_{(\lambda_1, \dots, \lambda_{h-1}, \lambda_h+1, \lambda_{h+1}, \dots, \lambda_n)}$,

$$p_h p_A p_h^{-1} = \omega^{\sum_{i<h}\lambda_i - \sum_{j>h}\lambda_j} \cdot p_A. \tag{5.8}$$

Hence it follows from (5.7) and (5.8) that

$$\sum_A c_A \cdot \omega^{\sum_{i<h}\lambda_i - \sum_{j>h}\lambda_j} \cdot p_A = \sum_A c_A \cdot p_A \quad \text{for} \quad h = 1, 2, \dots, n. \tag{5.9}$$

Since p_A are linearly independent over the field K of characteristic zero, (5.9) implies that

$$c_A \cdot \omega^{\sum_{i<h}\lambda_i - \sum_{h>j}\lambda_j} = c_A. \tag{5.10}$$

Therefore $A \equiv (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $c_A \neq 0$ must satisfy

$$\sum_{i<h} \lambda_i - \sum_{h>j} \lambda_j \equiv 0 \ (mod.\ m). \tag{5.11}$$

So we get from (5.11)

$$\lambda_i + \lambda_{i+1} \equiv (mod.\ m), \quad i = 1, 2, \dots, n-1, \tag{5.12}$$

and

$$\lambda_1 - \lambda_n \equiv (mod.\ m). \tag{5.13}$$

In the case (I) when n is even, from (5.12) and (5.13) we have

$$\lambda_i \equiv 0 \ (mod.\ m), \quad i = 1, 2, \dots, n,$$

therefore the center $\mathfrak{c}$ of $\mathfrak{o}$ is $\{p_0\}$.[1]

Next, in the case (II) when n is odd, the conditions (5.12) and (5.13) are reduced to

$$\lambda_1 \equiv -\lambda_2 \equiv \lambda_3 \equiv \cdots \equiv -\lambda_{n-1} \equiv \lambda_n \ (mod.\ m),$$

therefore, the center $\mathfrak{c}$ of $\mathfrak{o}$ is $\{\overset{0}{p}, \overset{1}{p}, \dots, \overset{m-1}{p}\}$, where

$$\overset{\lambda}{p} \equiv p_1^{\lambda} p_2^{-\lambda} p_3^{\lambda} \dots p_{n-1}^{-\lambda} p_n^{\lambda}, \quad \lambda = 0, 1, 2, \dots, m-1. \tag{5.14}$$

Moreover $\overset{\lambda}{p}$ satisfy the relation

$$\overset{\lambda}{p} \cdot \overset{\mu}{p} = \omega^{-\frac{n-1}{2}\lambda\mu} \cdot \overset{\lambda+\mu}{p}. \tag{5.15}$$

For, since $p_i^{\alpha} p_j^{\beta} = \omega^{\alpha\beta} p_j^{\beta} p_i^{\alpha}$ $(i > j)$, we have

$$\begin{aligned}\overset{\lambda}{p} \cdot \overset{\mu}{p} &= p_1^{\lambda} p_2^{-\lambda} p_3^{\lambda} \dots p_{n-1}^{-\lambda} p_n^{\lambda} \cdot p_1^{\mu} p_2^{-\mu} p_3^{\mu} \dots p_{n-1}^{-\mu} p_n^{\mu} \\ &= \omega^{-\frac{n-1}{2}\lambda\mu} \overset{\lambda+\mu}{p}.\end{aligned}$$

1) $\{q_1, q_2, \dots, q_s\}$ denotes the linear space over K spanned by $q_1, q_2, \dots, q_s$.

In the case (II_1) when m is odd, and the case (II_2) when m is even and $n\equiv 1\ (mod.\ 4)$, we shall write $\overset{\lambda}{\underset{0}{p}}=\omega^{-\frac{1}{4}(n-1)(\ -1)}\overset{\lambda}{p}$; and in the case ($\mathrm{II}_3$) when m is even, $n\equiv 3\ (mod.\ 4)$ and there exists an element κ in K such that $\omega=\kappa^2$, we shall write $\overset{\lambda}{\underset{0}{p}}=\omega^{-\frac{1}{4}(n-1)\lambda^2}\cdot\overset{\lambda}{p}$. Then we have from (5.15)

$$\overset{\lambda}{\underset{0}{p}}\cdot\overset{\mu}{\underset{0}{p}}=\overset{\lambda+\mu}{\underset{0}{p}}. \tag{5.16}$$

Furtheremore if we write

$$\overset{\lambda}{z}=\frac{1}{m}\sum_{\mu=0}^{m-1}\omega^{\lambda\mu}\overset{\mu}{\underset{0}{p}},\quad \lambda=0,1,2,\dots,m-1, \tag{5.17}$$

then we can prove, as follows, that

$$\overset{\lambda}{z}\cdot\overset{\mu}{z}=\delta_{\lambda\mu}\cdot\overset{\lambda}{z}. \tag{5.18}$$

By (5.16) and (5.17) we have

$$\begin{aligned}\overset{\lambda}{z}\cdot\overset{\mu}{z}&=\frac{1}{m^2}\sum_{\nu=0}^{m-1}\omega^{\lambda\nu}\overset{\nu}{\underset{0}{p}}\cdot\sum_{\kappa=0}^{m-1}\omega^{\mu\kappa}\overset{\kappa}{\underset{0}{p}}\\&=\frac{1}{m^2}\sum_{\nu,\kappa=0}^{m-1}\omega^{\lambda\nu+\mu\kappa}\cdot\overset{\nu+\kappa}{\underset{0}{p}}.\end{aligned}$$

If $\lambda=\mu$, then

$$\overset{\lambda}{z}\cdot\overset{\lambda}{z}=\frac{1}{m^2}\sum_{\nu,\kappa=0}^{m-1}\omega^{\lambda(\nu+\kappa)}\overset{\nu+\kappa}{\underset{0}{p}}=\frac{1}{m}\sum_{\pi=0}^{m-1}\omega^{\lambda\pi}\overset{\pi}{\underset{0}{p}}=\overset{\lambda}{z}.$$

and if $\lambda\neq\mu$, then

$$\begin{aligned}\overset{\lambda}{z}\cdot\overset{\mu}{z}&=\frac{1}{m^2}\sum_{\pi=0}^{m-1}\Bigg(\sum_{\substack{0\leqq\nu,\kappa\leqq m-1\\ \nu+\kappa=\pi}}\omega^{\lambda\nu+\mu\kappa}\Bigg)\overset{\pi}{\underset{0}{p}}\\&=\frac{1}{m^2}\sum_{\nu=0}^{m-1}(\omega^{\lambda-\mu})^{\nu}\cdot\sum_{\pi=0}^{m-1}\omega^{\mu\pi}=0,\end{aligned}$$

since ω is a primitive m-th root of unity and $\omega^{\lambda-\mu}\neq 1$. Moreover we shall see easily that $\overset{\lambda}{z}$ constitute a linearly independent basis of the center $\mathfrak{c}$ of $\mathfrak{o}$, and $p_0=\sum_{\lambda=0}^{m-1}\overset{\lambda}{z}$.

Finally, we shall consider the case (II_4) when m is even: $m=2m_0$, $n\equiv 3\ (mod.\ 4)$ and there does not exist an element κ in K such that $\omega=\kappa^2$. If we write, in this case

$$\overset{\mu}{\underset{0}{p}}=\omega^{-\frac{1}{4}(n-1)\mu^2}\cdot\overset{\mu}{p}\qquad\text{for any even numbers}\quad \mu=0,2,\dots,m-2,$$

and

$$\overset{\nu}{\underset{0}{q}}=\omega^{-\frac{1}{4}(n-1)\nu^2+\frac{1}{2}}\cdot\overset{\nu}{p}\quad\text{for any odd numbers}\quad \nu=1,3,\dots,m-1,$$

then we have from (5.15)

$$\left.\begin{array}{ll} \overset{\mu}{\underset{0}{p}}\overset{\mu'}{\underset{0}{p}} = \overset{\mu+\mu'}{p} & \text{for any even numbers } \mu \text{ and } \mu', \\ \overset{\mu}{\underset{0}{p}}\overset{\nu}{\underset{0}{q}} = \overset{\mu+\nu}{\underset{0}{q}} & \text{for any even number } \mu \text{ and odd number } \nu, \\ \text{and} & \\ \overset{\nu}{\underset{0}{q}}\overset{\nu'}{\underset{0}{q}} = \omega \overset{\nu+\nu'}{\underset{0}{p}} & \text{for any odd numbers } \nu \text{ and } \nu'. \end{array}\right\} \quad (5.19)$$

Furtheremore if we write

$$\text{and} \quad \left.\begin{array}{l} \overset{\alpha}{u} = \dfrac{2}{m} \sum\limits_{\substack{\mu=2 \\ (\text{even})}}^{m-2} \omega^{\alpha\mu} \overset{\mu}{\underset{0}{p}} \\ \overset{m_0+\alpha}{u} = \dfrac{2}{m} \sum\limits_{\substack{\nu=1 \\ (\text{odd})}}^{m-1} \omega^{\alpha\nu} \overset{\nu}{\underset{0}{q}} \end{array}\right\}, \quad \alpha = 0, 1, 2, \dots, m_0-1, \quad (5.20)$$

then we obtain, by the same calculation as the above calculation for z

$$\overset{\alpha}{u}\overset{\beta}{u} = \delta_{\alpha\beta}\overset{\alpha}{u}, \; \overset{\alpha}{u}\overset{m_0+\beta}{u} = \delta_{\alpha\beta}\overset{m_0+\alpha}{u}, \; \overset{m_0+\alpha}{u}\overset{m_0+\beta}{u} = \delta_{\alpha\beta}\omega\overset{\alpha}{u}, \quad \alpha, \beta = 0, 1, 2, \dots, m_0-1. \quad (5.21)$$

Moreover, $\overset{\alpha}{u}$ and $\overset{m_0+\alpha}{u}$ constitute a linearly independent basis of the center of $\mathfrak{o}$. For, suppose that there exist a linear relation

$$\sum_{\alpha=0}^{m_0-1} a_\alpha \overset{\alpha}{u} + \sum_{\alpha=0}^{m_0-1} a_{m_0+\alpha} \overset{m_0+\alpha}{u} = 0, \quad a_\alpha, a_{m_0+\alpha} \in K \quad (5.22)$$

then, by multiplying $\overset{\beta}{u}$ and $\overset{m_0+\beta}{u}$ on (5.22), we have by (5.21)

$$a_\beta \overset{\beta}{u} + a_{m_0+\beta} \overset{m_0+\beta}{u} = 0 \quad \text{and} \quad \omega a_{m_0+\beta} \overset{\beta}{u} + a_\beta \overset{m_0+\beta}{u} = 0 \quad (5.23)$$

respectively, and therefore

$$(a_\beta^2 - \omega a_{m_0+\beta}^2)\overset{\beta}{u} = 0,$$

hence we have

$$a_\beta^2 - \omega a_{m_0+\beta}^2 = 0. \quad (5.24)$$

By the assumption there does not exist κ in K such that $\omega=\kappa^2$, so we have from (5.24)

$$a_\beta = a_{m_0+\beta} = 0, \quad \beta = 0, 1, 2, \dots, m_0-1. \quad (5.25)$$

Therefore, $\overset{\alpha}{u}$ and $\overset{m_0+\alpha}{u}$ are linearly independent. And from (5.20), it is easily verified that

$$p_0 = \sum_{\alpha=0}^{m_0-1} \overset{\alpha}{u}.$$

Now, we shall investigate the ideals of $\mathfrak{o}$. To do this we consider a linear mapping $x \rightarrow \overset{\lambda}{Q}_h(x)$, where

$$\overset{\lambda}{Q}_h(x) = \omega^\lambda x - p_h x p_h^{-1}, \quad h = 1, 2, \dots, n; \ \lambda = 1, 2, \dots, m-1. \tag{5.26}$$

Then we have

$$\overset{\lambda}{Q}_h(p_A) = (\omega^\lambda - 1) p_A \quad \text{for} \quad p_A \in \mathfrak{c} \tag{5.27}$$

and, since it follows by (5.8) that $\lambda_0 \equiv \sum_{i<h_0} \lambda_i - \sum_{j>h_0} \lambda_j \not\equiv 0 \ (mod.\, m)$ for some h_0 and $p_A \notin \mathfrak{c}$, we have for the λ_0,

$$\overset{\lambda_0}{Q}_{h_0}(p_A) = 0 \quad \text{for} \quad p_A \notin \mathfrak{c}. \tag{5.28}$$

So if we write

$$Q(x) = \prod_{\lambda, h} \overset{\lambda}{Q}_h(x), \quad h = 1, 2, \dots, n; \ \lambda = 1, 2, \dots, m-1, \tag{5.29}$$

where $\prod_{\lambda, h} \overset{\lambda}{Q}_h$ means the operator applying $\overset{\lambda}{Q}_h$ successively on x in any definite order, then the mapping $x \rightarrow Q(x)$ is a linear mapping such that

$$Q(p_A) = \prod_{\lambda=1}^{m-1} (\omega^\lambda - 1)^n \cdot p_A \neq 0 \quad \text{for} \quad p_A \in \mathfrak{c}, \tag{5.30}$$

and

$$Q(p_A) = 0 \quad \text{for} \quad p_A \notin \mathfrak{c}. \tag{5.31}$$

Let $\mathfrak{a}$ be any ideal $\neq \{0\}$, and let $x = \sum_A c_A p_A$ be any element $\neq 0$ in $\mathfrak{a}$, i.e., any element such that $c_{A_0} \neq 0$ for some A_0. Then $x' \equiv p_{A_0}^{-1} \cdot x = \sum_A c'_A \cdot p_A$ belongs to $\mathfrak{a}$ and $c_0' \neq 0$. From the definition of $Q(x)$, we have $Q(x') \in \mathfrak{a}$. But it follows from (5.30) and (5.31) that $Q(x') \in \mathfrak{c}$, and from $c_0' \neq 0$ and (5.30) that $Q(x') \neq 0$. Therefore we have

$$\mathfrak{a} \cap \mathfrak{c} \neq \{0\}. \tag{5.32}$$

In the case (I) when n is even, since $\mathfrak{c} = \{p_0\}$, we have

$$Q(x') = c_0' Q(p_0) = c_0' \prod_{\lambda=1}^{m-1} (\omega^\lambda - 1)^n \cdot p_0 \tag{5.33}$$

but $Q(x') \in \mathfrak{a}$, therefore $p_0 \in \mathfrak{a}$ and hence $\mathfrak{a} = \mathfrak{o}$.

In the case (II) when n is odd, first we shall prove the following lemma.

Lemma. *In the cases* (II_1), (II_2) *and* (II_3), *any ideal* $\mathfrak{q}$ *of the center* $\mathfrak{c}$ *of* $\mathfrak{o}$ *is some direct sum of the simple ideals* $\{z\}$ *of* $\mathfrak{c}$; $z \in \mathfrak{a} \cap \mathfrak{c}$ *if and only if* $\mathfrak{a} z \neq 0$. *In the case* (II_4), $\mathfrak{q}$ *is some direct sum of the simple ideals* $\mathfrak{q}_\alpha \equiv \{\overset{\alpha}{u}, \overset{m_0+\alpha}{u}\}$ *of* $\mathfrak{c}$; $\mathfrak{q}_\alpha \subset \mathfrak{a} \cap \mathfrak{c}$ *if and only if* $\overset{\alpha}{u} \mathfrak{a} \neq 0$.

Proof. First we shall consider the cases (II_1), (II_2) and (II_3). Let $\sum_{\lambda=0}^{r-1} a_\lambda z$ be any element of $\mathfrak{q}$, then $(\sum_{\lambda=0}^{r-1} a_\lambda z) \cdot \overset{\mu}{z}$ belongs to $\mathfrak{q}$, and we have by (5.18)

$$\left(\sum_{\lambda=0}^{m-1} a_\lambda \overset{\lambda}{z}\right)\cdot \overset{\mu}{z} = a_\mu \overset{\mu}{z}\,, \tag{5.34}$$

and therefore, if $a_\mu \neq 0$, then $\overset{\mu}{z}$ belongs to $\mathfrak{q}$. Hence $\mathfrak{q}$ is some direct sum of $\{\overset{\lambda}{z}\}$, $(\lambda=0, 1, 2, \dots, m-1)$. If $\overset{\lambda}{z} \in \mathfrak{a}\frown\mathfrak{c}$, then, obviously $\mathfrak{a}\overset{\lambda}{z}\neq 0$. Conversely, suppose that $\mathfrak{a}\overset{\lambda}{z}\neq 0$, then there exists an element $y\in\mathfrak{a}$ such that $y\overset{\lambda}{z}=x=\sum_A c_A p_A \neq 0$. Assume that $c_{A_0}\neq 0$, then we have $0\neq Q(x')=Q(p_{A_0}^{-1}y\cdot\overset{\lambda}{z})=Q(p_{A_0}^{-1}y)\cdot\overset{\lambda}{z}$ where $x'\equiv p_{A_0}^{-1}x=\sum_A c_A'\cdot p_A$ and $c_0'=c_{A0}\neq 0$. And since $y\in\mathfrak{a}$, we have $Q(p_{A_0}^{-1}y)\in\mathfrak{a}$. Hence we see that

$$0 \neq Q(p_{A_0}^{-1}y)\overset{\lambda}{z}\in\mathfrak{a}\,. \tag{5.35}$$

On the other hand, from (5.30) and (5.31) it follows that

$$0\neq Q(p_{A_0}^{-1}\cdot y)\in\mathfrak{c}\,,$$

and therefore we have by (5.18)

$$Q(p_{A_0}^{-1}\cdot y)\overset{\lambda}{z} = c\cdot\overset{\lambda}{z} \quad (c\neq 0)\,. \tag{5.36}$$

Thus from (5.35) and (5.36) it follows that $\overset{\lambda}{z}$ belongs to $\mathfrak{a}$, i.e., $\overset{\lambda}{z}\in\mathfrak{a}\frown\mathfrak{c}$. Next we shall consider the case (II$_4$). Let $u\equiv\sum_{\alpha=0}^{m_0-1} a_\alpha\overset{\alpha}{u}+\sum_{\alpha=0}^{m_0-1} a_{m_0+\alpha}\overset{m_0+\alpha}{u}$ be any element of $\mathfrak{q}$, then $u\cdot\overset{\beta}{u}$ and $u\cdot\overset{m_0+\beta}{u}$ belong to $\mathfrak{q}$, and we have by (5.21)

$$\left.\begin{aligned} u\cdot\overset{\beta}{u} &= a_\beta\overset{\beta}{u}+a_{m_0+\beta}\overset{m_0+\beta}{u}\in\mathfrak{q}\,,\\ u\cdot\overset{m_0+\beta}{u} &= \omega a_{m_0+\beta}\overset{\beta}{u}+a_\beta\overset{m_0+\beta}{u}\in\mathfrak{q}\,,\end{aligned}\right\}, \tag{5.37}$$

and therefore we get

$$\text{and}\quad\left.\begin{aligned} &(a_\beta^2-\omega a_{m_0+\beta}^2)\overset{\beta}{u}\in\mathfrak{q}\\ &(a_\beta^2-\omega a_{m_0+\beta}^2)\overset{m_0+\beta}{u}\in\mathfrak{q}\,.\end{aligned}\right\} \tag{5.38}$$

If a_β and $a_{m_0+\beta}$ are not both zero, then from the assumption that there exist no element κ in K such that $\omega=\kappa^2$, we have

$$a_\beta^2-\omega a_{m_0+\beta}^2\neq 0\,,$$

and hence, from (5.38), $\overset{\beta}{u}$ and $\overset{m_0+\beta}{u}$ must belong to $\mathfrak{q}$. That is, $\mathfrak{q}$ contains $\mathfrak{q}_\beta\equiv\{\overset{\beta}{u}, \overset{m_0+\beta}{u}\}$ which is a simple ideal of $\mathfrak{c}$. And, since $\mathfrak{q}_\alpha\mathfrak{q}_\beta=0$ $(\alpha\neq\beta)$, $\mathfrak{q}$ is some direct sum of $\mathfrak{q}_\alpha$. Now if $\mathfrak{q}_\alpha\subset\mathfrak{a}\frown\mathfrak{c}$, then, obviously, $\mathfrak{a}\overset{\alpha}{u}\neq 0$. Conversely suppose that $\mathfrak{a}\overset{\alpha}{u}\neq 0$, then there exists an element $y\in\mathfrak{a}$ such that $0\neq x=\sum_A c_A p_A=y\overset{\alpha}{u}$. Assume that $c_{A_0}\neq 0$, then we have $0\neq Q(x')=Q(p_{A_0}^{-1}y\cdot\overset{\alpha}{u})=Q(p_{A_0}^{-1}y)\cdot\overset{\alpha}{u}$, where $x' \equiv p_{A_0}^{-1} x = \sum_A c_A' \cdot p_A$ and $c_0' = c_{A_0} \neq 0$. And, since $y \in \mathfrak{a}$, we have $Q(p_{A_0}^{-1} \cdot y) \in \mathfrak{a}$. Hence we see that

$$0 \neq Q(p_{A_0}^{-1} y) \overset{\alpha}{u} \in \mathfrak{a} . \tag{5.39}$$

On the other hand, we know that

$$0 \neq Q(p_{A_0}^{-1} \cdot y) \in \mathfrak{c} ,$$

and therefore we get by (5.21) and (5.39),

$$0 \neq a_\alpha \overset{\alpha}{u} + a_{m_0+\alpha} \overset{m_0+\alpha}{u} \in \mathfrak{a} , \tag{5.40}$$

and furthermore, multiplying $\overset{m_0+\alpha}{u}$ on (5.40), we have by (5.21)

$$\omega a_{m_0+\alpha} \overset{\alpha}{u} + a_\alpha \overset{m_0+\alpha}{u} \in \mathfrak{a} . \tag{5.41}$$

Since $a_\alpha^2 - \omega a_{m_0+\alpha}^2 \neq 0$ in this case, it follows from (5.40) and (5.41) that $\overset{\alpha}{u}$ and $\overset{m_0+\alpha}{u}$ belong to $\mathfrak{a}$, and therefore $\mathfrak{q}_\alpha \subset \mathfrak{a} \cap \mathfrak{c}$. q.e.d.

Since, in the cases (II_1), (II_2) and (II_3), the unit element p_0 of $\mathfrak{o}$ is written as the sum $\sum_{\lambda=0}^{m-1} \overset{\lambda}{z}$ of the basic elements $\overset{\lambda}{z}$, $\mathfrak{o}$ is decomposed into the direct sum of the simple ideals $\mathfrak{o}\overset{\lambda}{z}$ of $\mathfrak{o}$:

$$\mathfrak{o} = \sum_{\lambda=0}^{m-1} \oplus \mathfrak{o} \overset{\lambda}{z} . \tag{5.42}$$

As we shall see from the above Lemma, any ideal $\mathfrak{a}$ of $\mathfrak{o}$ contains all basic elements $\overset{\alpha}{z}$ such that $\mathfrak{a}\overset{\alpha}{z} \neq 0$. If we write $\mathfrak{a}' = \sum_{\mathfrak{a}\overset{\alpha}{z} \neq 0} \oplus \mathfrak{o}\overset{\alpha}{z}$, then $\mathfrak{a}' \subset \mathfrak{a}$. Suppose that $\mathfrak{a}' \neq \mathfrak{a}$, then any element $\sum_{\lambda=0}^{m-1} (x_\lambda \overset{\lambda}{z} \,;\, x_\lambda \in \mathfrak{o})$ of $\mathfrak{a} - \mathfrak{a}'$ contains the term $x_\alpha \overset{\alpha}{z}$ such that $x_\alpha \overset{\alpha}{z} \neq 0$, $\mathfrak{a}\overset{\alpha}{z} = 0$ for some α. However, from that $\mathfrak{a}\overset{\alpha}{z} = 0$ and $\sum_{\lambda=0}^{m-1} x_\lambda \overset{\lambda}{z} \in \mathfrak{a}$, it follows that $\sum_{\lambda=0}^{m-1} x_\lambda \overset{\lambda}{z} \cdot \overset{\alpha}{z} = 0$, and therefore $x_\alpha \overset{\alpha}{z} = 0$ by (5.18). This contradicts the above assumption: $x_\alpha \overset{\alpha}{z} \neq 0$. Thus we have $\mathfrak{a} = \sum_{\mathfrak{a}\overset{\alpha}{z} \neq 0} \oplus \mathfrak{o}\overset{\alpha}{z}$.

Also, in the case (II_4), similarly as the above cases (II_1), (II_2) and (II_3) we have

$$\mathfrak{o} = \sum_{\alpha=1}^{m_0-1} \oplus \mathfrak{o}\mathfrak{q}_\alpha , \tag{5.43}$$

and it is easily seen that

$$\mathfrak{o}\mathfrak{q}_\alpha = \mathfrak{o}\overset{\alpha}{u} .$$

If we write $\mathfrak{a}' = \sum_{\mathfrak{a}\overset{\alpha}{u} \neq 0} \oplus \mathfrak{o}\mathfrak{q}_\alpha$, then by the above Lemma $\mathfrak{a}' \subset \mathfrak{a}$. Suppose that

$\mathfrak{a}'\neq\mathfrak{a}$, then there exists an element $x=\sum_{\alpha=0}^{m_0-1} x_\alpha \overset{\alpha}{u}+\sum_{\alpha=0}^{m_0-1} x_{m_0+\alpha} \overset{m_0+\alpha}{u}$, $(x_\alpha, x_{m_0+\alpha}\in\mathfrak{o})$ in $\mathfrak{a}-\mathfrak{a}'$ such that $0\neq x_\alpha\overset{\alpha}{u}+x_{m_0+\alpha}\overset{m_0+\alpha}{u}$ and $\mathfrak{a}\overset{\alpha}{u}=0$ for some α. Since $x\in\mathfrak{a}$ and $\mathfrak{a}\overset{\alpha}{u}=0$, we have $x\overset{\alpha}{u}=0$, and therefore $x_\alpha u+x_{m_0+\alpha}\overset{m_0+\alpha}{u}=0$. This contradicts the above assumption $x_\alpha\overset{\alpha}{u}+x_{m_0+\alpha}\overset{m_0+\alpha}{u}\neq 0$. Thus we have $\mathfrak{a}=\sum_{\mathfrak{a}\overset{\alpha}{u}\neq 0}\oplus\mathfrak{o}\mathfrak{q}_\alpha$.

Moreover, we can prove that, in the cases (II_1), (II_2) and (II_3), the simple ideal $\mathfrak{o}\overset{\lambda}{z}$ is isomorphic to $\mathfrak{o}\overset{\mu}{z}$ $(\lambda\neq\mu)$ and also, in the case (II_4), the simple ideal $\mathfrak{o}\overset{\alpha}{\mathfrak{q}}$ is isomorphic to $\mathfrak{o}\overset{\beta}{\mathfrak{q}}$. Let $\mathfrak{o}_{2r}$ be the subalgebra of $\mathfrak{o}$ generated by $p_1, p_2, \dots, p_{2r}$, then $x\overset{\lambda}{z}=0$ $(x\in\mathfrak{o}_{2r})$ implies $x=0$. For, since $x=\sum_{\lambda_1,\dots,\lambda_{2r}} c_{\lambda_1\dots\lambda_{2r}} p_1^{\lambda_1}\dots p_{2r}^{\lambda_{2r}}$ and $\overset{\lambda}{z}=\sum_{\mu=0}^{m-1} a_{\lambda\mu} p_1^u p_2^{-\mu}\dots p_{2r+1}^u$, $a_{\lambda\mu}\neq 0$, we have

$$0=x\overset{\lambda}{z}=\sum_{\mu=0}^{m-1} a_{\lambda\mu}\sum_{\lambda_1,\dots,\lambda_{2r}} c_{\lambda_1\dots\lambda_{2r}}\omega^{\kappa_\mu;\lambda_1,\dots,\lambda_{2r}} p_1^{\lambda_1+} p_2^{\lambda_2-}\dots p_{2r}^{\lambda_{2r}-\mu} p_{2r+1}^u,$$

$(\kappa_{\mu,\lambda_1\dots\lambda_{2r}}$ is some integer),

from which, by the linearly independency of $p_1^{\nu_1}p_2^{\nu_2}\dots p_{2r+1}^{\nu_{2r+1}}$, (taking $\mu=0$),

$$\sum_{\lambda_1\dots\lambda_{2r}} c_{\lambda_1\dots\lambda_{2r}}\omega^{\kappa_0;\lambda_1\dots\lambda_{2r}} p_1^{\lambda_1}p_2^{\lambda_2}\dots p_{2r}^{\lambda_{2r}}=0\,.$$

And moreover the basic elements $p_1^{\lambda_1}p_2^{\lambda_2}\dots p_{2r}^{\lambda_{2r}}$ are linearly independent, therefore we have $c_{\lambda_1\dots\lambda_{2r}}=0$ i.e., $x=0$. Hence, if we make $x\overset{\lambda}{z}$ $(x\in\mathfrak{o}_{2r})$ correspond to $x\overset{\mu}{z}$, then we get a one to one correspondence from $\mathfrak{o}_{2r}\overset{\lambda}{z}$ to $\mathfrak{o}_{2r}\overset{\mu}{z}$. By this correspondence, it follows from $\overset{\lambda}{z}\overset{\mu}{z}=\delta_{\lambda\mu}\overset{\lambda}{z}$ that $\mathfrak{o}_{2r}\overset{\lambda}{z}$ is isomorphic to $\mathfrak{o}_{2r}\overset{\mu}{z}$. By the fact that $x\overset{\mu}{z}=0 (x\in\mathfrak{o}_{2r})$ implies $x=0$ we know that each of $\mathfrak{a}_{2r}z(\lambda=0,1,2,\dots,m-1)$ contains m^{n-1} linearly independent elements respectively. And since $\mathfrak{o}_{2r}\overset{\lambda}{z}\subset\mathfrak{o}\overset{\lambda}{z}$, $\mathfrak{o}=\sum_{\lambda=0}^{m-1}\oplus\mathfrak{o}\overset{\lambda}{z}$ and $\mathfrak{o}$ contains m^n linearly independent elements, we have $\mathfrak{o}_{2r}\overset{\lambda}{z}=\mathfrak{o}\overset{\lambda}{z}$. Thus the simple ideal $\mathfrak{o}\overset{\lambda}{z}$ is isomorphic to $\mathfrak{o}\overset{\mu}{z}$, and therefore these simple ideals contain m^{n-1} linearly independent elements respectively. Next, in the case (II_4), similarly as the above case, we can see that $x\overset{\alpha}{u}+x'\overset{m_0+\alpha}{u}$ $(x, x'\in\mathfrak{o}_{2r})$ implies $x, x'\in\mathfrak{o}_{2r}$. By the same consideration as above, we obtain that these simple ideals $\mathfrak{o}\overset{\alpha}{\mathfrak{q}}$ are mutually isomorphic and have $2m^{n-1}$ linearly independent elements respectively.

Collecting the above results, we obtain the theorem.

Theorem 6. *If n is even, then the center $\mathfrak{c}$ of $\mathfrak{o}$ is $\{p_0\}$ and the ideals of $\mathfrak{o}$ are only $\{0\}$ and the whole algebra $\mathfrak{o}$. If n is odd, then the center $\mathfrak{c}$ of*

$\mathfrak{o}$ *is* $\{\overset{0}{p}, \overset{1}{p}, \overset{2}{p}, \ldots, \overset{m-1}{p}\}$, *where* $\overset{\mu}{p}=p_1^{\mu}p_2^{-\mu}p_3^{\mu}\ldots p_{n-1}^{-\mu}p_n^{\mu}$. *And, in the cases* (II_1), (II_2) *and* (II_3), $\mathfrak{o}$ *is decomposed into the direct sum* $\sum_{\lambda=0}^{m-1}\oplus\mathfrak{o}\overset{\lambda}{z}$ *of the simple ideals of* $\mathfrak{o}$, *where*

$$\overset{\lambda}{z}=\frac{1}{m}\sum_{\mu=0}^{m-1}\omega^{\lambda\mu-\frac{1}{4}(n-1)\cdot(\mu-1)}\cdot\overset{\mu}{p} \quad \text{for the cases } (II_1) \text{ and } (II_2),$$

$$\overset{\lambda}{z}=\frac{1}{m}\sum_{\mu=0}^{m-1}\omega^{\lambda\mu-\frac{1}{4}(n-1)\mu^2}\cdot\overset{\mu}{p} \quad \text{for the case } (II_3);$$

moreover, as for any ideal $\mathfrak{a}$ *of* $\mathfrak{o}$, $\mathfrak{a}\frown\mathfrak{c}$ *is an ideal* $\sum_{\mathfrak{a}\overset{\lambda}{z}\neq 0}\oplus\{\overset{\lambda}{z}\}$ *of* $\mathfrak{c}$, *and* $\mathfrak{a}=\sum_{\mathfrak{c}\overset{\alpha}{z}\neq 0}\oplus\mathfrak{o}\overset{\alpha}{z}$. *Similarly, in the case* (II_4), *we have* $\mathfrak{o}=\sum_{\alpha=0}^{m_0-1}\oplus\mathfrak{o}\mathfrak{q}_{\alpha}$, $(\mathfrak{o}\mathfrak{q}_{\alpha}=\mathfrak{q}\overset{\alpha}{u})$, $\mathfrak{a}\frown\mathfrak{c}=\sum_{\mathfrak{c}\cdot\overset{\alpha}{z}\neq 0}\oplus\mathfrak{q}_{\alpha}$ *and* $\mathfrak{a}=\sum_{\mathfrak{a}\overset{\alpha}{z}\neq 0}\oplus\mathfrak{o}\mathfrak{q}_{\alpha}$, *where* $\mathfrak{q}_{\alpha}=\{\overset{\alpha}{u}, \overset{m_0+\alpha}{u}\}$,

$$\overset{\alpha}{u}=\frac{2}{m}\sum_{\substack{\mu=0\\(\text{even})}}^{m-2}\omega^{\alpha\mu-\frac{1}{4}(n-1)\mu^2}\cdot\overset{\mu}{p}$$

and

$$\overset{m_0+\alpha}{u}=\frac{2}{m}\sum_{\substack{\nu=1\\(\text{odd})}}^{m-1}\omega^{\alpha\nu-\frac{1}{4}(n-1)\nu^2+\frac{1}{2}}\cdot\overset{\nu}{p}.$$

Moreover, $\mathfrak{o}\overset{\lambda}{z}$ $(\lambda=0, 1, \ldots, m-1)$ *are mutually isomorphic and have* m^{n-1} *linearly independent elements respectively, and* $\mathfrak{o}\overset{\alpha}{\mathfrak{q}}$ $(\alpha=0, 1, \ldots, m_0-1)$ *are mutually isomorphic and have* $2m^{n-1}$ *linearly independent elements respectively.*

§ 6, Matric Representation of the G. C. Algebra Associated with the Linearization of $\sum_{i=1}^{n}(x^i)^m$.

We can investigate the representation of the G. C. algebra $\mathfrak{o}$ by means of the structure of $\mathfrak{o}$ discussed in § 5 and the general theory of the representations of algebra. But, in this section, we shall consider directly the actual representation of the G. C. algebra $\mathfrak{o}$. We must suppose that K contains a primitive m-th root ω of unity, and the square root $\omega^{\frac{1}{2}}$ of ω.

We shall determine the general system of matrices $P_1, P_2, \ldots, P_n$ such that

$$P_iP_j=\omega P_jP_i \quad (i>j), \tag{6.1}$$

$$P_i^m=E, \tag{6.2}$$

where E is the unit matrix of the same order as P_i. If we transform P_1 to the Jordan's canonical form $\overset{0}{P}_1$ by T, then it follows from $P_1^m=E$ that

$$\overset{0}{P}_1 = TP_1T^{-1} = \begin{pmatrix} \omega^{\alpha_1} & & & 0 \\ & \omega^{\alpha_2} & & \\ & & \ddots & \\ 0 & & & \omega^{\alpha_s} \end{pmatrix}, \quad 0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_s \leqq m-1.^{1)} \tag{6.3}$$

If we write

$$\overset{0}{P}_i = TP_iT^{-1}, \tag{6.4}$$

then we have

$$\overset{0}{P}_i\overset{0}{P}_j = \omega\overset{0}{P}_j\overset{0}{P}_i \quad (i>j), \quad \overset{0}{P}{}_i^m = E. \tag{6.5}$$

It follows from $\overset{0}{P}{}_2^m=E$ that $|\overset{0}{P}_2| \neq 0$, and therefore there exists the inverse matrix $\overset{0}{P}{}_2^{-1}$ of $\overset{0}{P}_2$. Hence we have from (6.5)

$$\overset{0}{P}_2\overset{0}{P}_1\overset{0}{P}{}_2^{-1} = \omega\overset{0}{P}_1. \tag{6.6}$$

It follows from (6.3) and (6.6) that if ω^{α_i} is a characteristic value of $\overset{0}{P}_1$, then ω^{α_i+1} also is a characteristic value of $\overset{0}{P}_1$. Therefore $\overset{0}{P}_1$ must be the matrix of order mt such that

$$\overset{0}{P}_1 = \begin{pmatrix} \omega^0E_t & & & 0 \\ & \omega^1E_t & & \\ & & \ddots & \\ 0 & & & \omega^{m-1}E_t \end{pmatrix}, \tag{6.7}$$

where E_t is the unit matrix of order t, $t=1, 2, \ldots$. If we substitute (6.7) into $\overset{0}{P}_2\overset{0}{P}_1=\omega\overset{0}{P}_1\overset{0}{P}_2$, then we have the following conditions for $\overset{0}{P}_2=\|S_{ij}\|$ whose element S_{ij} is a matrix of order t:

$$\left.\begin{array}{llll} (1-\omega)S_{11}=0 & (\omega-\omega)S_{12}=0 & \cdots\cdots & (\omega^{m-1}-\omega)S_{1m}=0 \\ (1-\omega^2)S_{21}=0 & (\omega-\omega^2)S_{22}=0 & \cdots\cdots & (\omega^{m-1}-\omega^2)S_{2m}=0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ (1-\omega^{m-1})S_{m-1\,1}=0 & (\omega-\omega^{m-1})S_{m-1\,2}=0 & \cdots\cdots & (\omega^{m-1}-\omega^{m-1})S_{m-1m}=0 \\ (1-\omega^m)S_{m1}=0 & (\omega-\omega^m)S_{m2}=0 & \cdots\cdots & (\omega^{m-1}-\omega^m)S_{mm}=0 \end{array}\right\} \tag{6.8}$$

Therefore we have

$$S_{ij}=0, \quad \text{except} \quad S_{12}, S_{23}, \ldots, S_{m-1,m}, S_{m,1}. \tag{6.9}$$

Substituting this $\overset{0}{P}_2$ into $\overset{0}{P}{}_2^m=E$, we have

$$\begin{aligned} S_{12}S_{23} \cdots\cdots S_{m1} &= E_t \\ S_{23}S_{34} \cdots\cdots S_{m1}S_{12} &= E_t \\ &\cdots\cdots \\ S_{m1}S_{12} \cdots\cdots S_{m-1m} &= E_t \end{aligned}$$

and it is easily seen that all these relations are equivalent to

1) J. H. M. Wedderburn, Lectures on Matrices, (New York), (1934) p. 119.

$$S_{12}S_{23}\cdots\cdots S_{m-1m}S_{m1}=E_t\,. \tag{6.10}$$

Thus we obtain

$$\overset{\circ}{P}_2=\begin{pmatrix} 0 & S_{12} & 0 & \cdots\cdots & 0 \\ 0 & 0 & S_{23} & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & S_{m-1m} \\ S_{m1} & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{6.11}$$

where $S_{12}S_{23}\ldots S_{m-1m}S_{m1}=E_t$. Since $S_{12}, S_{23}, \ldots, S_{m-1m}, S_{m1}$ are regular matrices by (6.10), there exist the matrices $S_2, S_3, \ldots, S_m$, for an arbitrary regular matrix S_1, satisfying the following relations:

$$S_{12}=S_1^{-1}S_2\,,\ S_{23}=S_2^{-1}S_3\,,\ \ldots,\ S_{m-1m}=S_{m-1}^{-1}S_m\,,\ S_{m1}=S_m^{-1}S_1\,. \tag{6.12}$$

So if we put

$$S=\begin{pmatrix} S_1 & & & 0 \\ & S_2 & & \\ & & \ddots & \\ 0 & & & S_m \end{pmatrix}, \tag{6.13}$$

then we have

$$S\overset{\circ}{P}{}^2S^{-1}=\begin{pmatrix} 0 & E_t & 0 & \cdots\cdots & 0 \\ 0 & 0 & E_t & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & E_t \\ E_t & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{6.14}$$

Since it is obvious that

$$S\overset{\circ}{P}_1S^{-1}=\overset{\circ}{P}_1\,,$$

from (6.3) we have, by means of the matrix $U=ST$,

$$UP_1U^{-1}=\begin{pmatrix} E_t & & & 0 \\ & \omega E_t & & \\ & & \ddots & \\ 0 & & & \omega^{m-1}E_t \end{pmatrix}=\Omega_1\times E_t\,,$$

$$UP_2U^{-1}=\begin{pmatrix} 0 & E_t & 0 & \cdots\cdots & 0 \\ 0 & 0 & E_t & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & E_t \\ E_t & 0 & \cdots & \cdots & 0 \end{pmatrix}=\Omega_2\times E_t\,, \tag{6.15}$$

where $\Omega_1=\begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{m-1} \end{pmatrix}$ and $\Omega_2=\begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$. (6.16)

And also we have

$$\left.\begin{aligned} VP_1V^{-1} &= E_t\times\Omega_1 = \begin{pmatrix}\Omega_1 & & 0\\ & \Omega_1 & \\ & & \ddots \\ 0 & & \Omega_1\end{pmatrix}, \\ VP_2V^{-1} &= E_t\times\Omega_2 = \begin{pmatrix}\Omega_2 & & 0\\ & \Omega_2 & \\ & & \ddots \\ 0 & & \Omega_2\end{pmatrix}. \end{aligned}\right\}$$

Next if we write

$$\bar{P}_i = UP_iU^{-1}, \quad (i=1, 2, \dots, n), \tag{6.17}$$

then we have

$$\bar{P}_i\bar{P}_j = \omega\bar{P}_j\bar{P}_i \quad (i>j), \quad \bar{P}_i^m = E, \tag{6.18}$$

where $\bar{P}_1 = \Omega_1\times E_t = \begin{pmatrix}E_t & & 0\\ & \omega E_t & \\ & & \ddots \\ 0 & & \omega^{m-1}E_t\end{pmatrix}$ and $\bar{P}_2 = \Omega_2\times E_t = \begin{pmatrix}0 & E_t & 0 & \cdots\cdots & 0\\ 0 & 0 & E_t & 0 \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & E_t\\ E_t & 0 & \cdots & \cdots & 0\end{pmatrix}$.

By the same method as we obtained (6.11), it follows from $\bar{P}_i\bar{P}_1 = \omega\bar{P}_1\bar{P}_i$, $\bar{P}_i^m = E$ $(i=3, 4, \dots, n)$ that

$$\bar{P}_i = \begin{pmatrix} 0 & \overset{i}{S}_{12} & 0 & \cdots\cdots & 0\\ 0 & 0 & \overset{i}{S}_{23} & 0\cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots\cdots & 0 & \overset{i}{S}_{m-1m}\\ \overset{i}{S}_{m1} & 0 & \cdots\cdots & 0 & 0\end{pmatrix}, \quad (i=3, 4, \dots, n), \tag{6.19}$$

where $\overset{i}{S}_{12}\overset{i}{S}_{23}\overset{i}{S}_{34}\dots\overset{i}{S}_{m-1m}\overset{i}{S}_{m1} = E_t$. And each $\bar{P}_i\,(i=3, 4, \dots, n)$ satisfies $\bar{P}_i\bar{P}_2 = \omega\bar{P}_2\bar{P}_i$. By substituting the actual forms of $\bar{P}_i$ and $\bar{P}_2$ in (6.15) and (6.19) into the above relation $\bar{P}_i\bar{P}_2 = \omega\bar{P}_2\bar{P}_i$, we have

$$\overset{i}{S}_{12} = \omega\overset{i}{S}_{23}, \overset{i}{S}_{23} = \omega\overset{i}{S}_{34}, \dots, \overset{i}{S}_{m-1m} = \omega\overset{i}{S}_{m1}, \overset{i}{S}_{m1} = \omega\overset{i}{S}_{12}. \tag{6.20}$$

If we write $\overset{i}{S} = \overset{i}{S}_{12}$, then we have

$$\overset{i}{S}_{23} = \omega^{-1}\overset{i}{S}, \overset{i}{S}_{34} = \omega^{-2}\overset{i}{S}, \dots, \overset{i}{S}_{m1} = \omega^{-(m-1)}\overset{i}{S}. \tag{6.21}$$

Moreover it follows from $\overset{i}{S}_{12}\overset{i}{S}_{23}\dots\overset{i}{S}_{m1} = E_t$ that $(\overset{i}{S})^m = \omega^{\frac{m(m-1)}{2}}E_t$. Therefore we have

$$\bar{P}_i = \begin{pmatrix} 0 & \overset{i}{S} & 0 & \cdots\cdots & 0\\ 0 & 0 & \omega^{-1}\overset{i}{S} & 0 \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & \omega^{-(m-2)}\overset{i}{S}\\ \omega^{-(m-1)}\overset{i}{S} & 0 & \cdots & \cdots & 0\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0\\ 0 & 0 & \omega^{-1} & 0 \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & \cdots & \cdots & \omega^{-(m-2)}\\ \omega^{-(m-1)} & 0 & \cdots & \cdots & 0\end{pmatrix}\times\overset{i}{S}$$

$$(i=3, 4, \dots, n), \tag{6.22}$$

where

$$(\overset{t}{S})^m = \omega^{\frac{m(m-1)}{2}} E_t\,. \tag{6.23}$$

Since $\Omega_1^{-1}\Omega_2 = \begin{pmatrix} 1 & & & & 0 \\ & \omega^{1} & & & \\ & & \omega^{-2} & & \\ & & & \ddots & \\ 0 & & & & \omega^{-(m-1)} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & & 1 \\ 1 & 0 & \cdots\cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & \omega^{-1} & 0 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots\cdots & & & \omega^{-(m-2)} \\ \omega^{-(m-1)} & 0 & \cdots\cdots & & 0 \end{pmatrix}$,

if we write $Q_i = \omega^{-\frac{m-1}{2}} \overset{i}{S}$ and $\Omega_3 = \omega^{\frac{m-1}{2}} \Omega_1^{-1}\Omega_2$, then the relation (6.22) is expressed by

$$\bar{P}_i = \Omega_3 \times Q_i\,, \quad (i = 3, 4, \dots, n), \tag{6.24}$$

where, from (6.23),

$$(Q_i)^m = E_t\,. \tag{6.25}$$

And, from $\bar{P}_i\bar{P}_j = \omega\bar{P}_j\bar{P}_i$ $(i > j\,;\, i, j = 3, 4, \dots, n)$, we get

$$Q_iQ_j = \omega Q_jQ_i \quad (i > j\,;\, i, j = 3, 4, \dots, n)\,. \tag{6.26}$$

The above relations (6.26) and (6.25) for Q_k $(k=3, 4, \dots, n)$ have the same forms as the relations (6.1) and (6.2) for P_i $(i=1, 2, \dots, n)$. Hence, by repeating the above method, we find that the matrices Q_k $(k=3, 4, \dots, n)$ satisfying (6.25) and (6.26) must be the matrices of order $t=mt'$ such that

$$\left.\begin{aligned} Q_3 &= U'^{-1}(\Omega_1 \times E_{t'})U'\,, \\ Q_4 &= U'^{-1}(\Omega_2 \times E_{t'})U'\,, \\ Q_i &= U'^{-1}(\Omega_3 \times R_i)\, U'\,, \\ & (i = 5, 6, \dots, n)\,, \end{aligned}\right\} \tag{6.27}$$

where R_i $(i=5, 6, \dots, n)$ are the matrices of order t' such that

$$R_iR_j = \omega R_jR_i\,, (R_i)^m = E_{t'}\,, \quad (i > j\,;\, i, j = 5, 6, \dots, n) \tag{6.28}$$

Hence we have, from (6.15), (6.24) and (6.27),

$$\left.\begin{aligned} P_1 &= U^{-1}(\Omega_1 \times E_t)U = U^{-1}(E_m \times U')^{-1}[\Omega_1 \times E_m \times E_{t'}](E_m \times U')U \\ P_2 &= U^{-1}(\Omega_2 \times E_t)U = U^{-1}(E_m \times U')^{-1}[\Omega_2 \times E_m \times E_{t'}](E_m \times U')U \\ & \qquad\qquad (\text{substituting } E_t = E_m \times E_{t'}) \\ P_3 &= U^{-1}[\Omega_3 \times U'^{-1}(\Omega_1 \times E_{t'})U']U = U^{-1}\cdot(E_m \times U')^{-1}[\Omega_3 \times \Omega_1 \times E_{t'}](E_m \times U')U \\ P_4 &= U^{-1}[\Omega_3 \times U'^{-1}(\Omega^2 \times E_{t'})U']U = U^{-1}(E_m \times U')^{-1}[\Omega_3 \times \Omega_2 \times E_{t'}](E_m \times U')U \\ P_i &= U^{-1}[\Omega_3 \times U'^{-1}(\Omega_3 \times R_i)U']U = U^{-1}(E_m \times U')^{-1}[\Omega_3 \times \Omega_3 \times R_i](E_m \times U')U \end{aligned}\right\}$$

$$(i = 5, 6, \dots, n)\,. \tag{6.29}$$

In the case where $n=2r$, by repeating this process r times, we can obtain all matrices $P_i\,(i=1,2,\dots,n)$, and in the case where $n=2r+1$ by repeating this process r times, we can obtain $P_1, P_2, \dots, P_{2r}$, and then, as we see from (6.27), P_{2r+1} is expressed by:

$$P_{2r+1}=W^{-1}[\Omega_3\times\Omega_3\times\cdots\times\Omega_3\times R]W$$

where R is any matrix of order l such that $(R)^m=E_l$.

Thus we have the following results.

Theorem 7. *Let ω be a primitive m-th root of unity, if K contians $\omega^{\frac{1}{2}}$, then the general system of matrices $P_i\,(i=1,2,\dots,n)$ such that*

$$P_iP_j=\omega P_jP_i\ (i>j)\quad and\quad P_i^m=E$$

are written, by means of Ω_1, Ω_2 in (6.16) and $\Omega_3=\omega^{\frac{m-1}{2}}\Omega_1^{-1}\Omega_2$ as follows:

(I) *If n is even: $n=2r$, then*

$$\left.\begin{aligned} P_{2s-1}&=W^{-1}[\underbrace{\underbrace{\Omega_3\times\cdots\times\Omega_3}_{s}\times\Omega_1\times E_m\times E_m\times\cdots\times E_m}_{r}\times E_l]W\\ P_{2s}&=W^{-1}[\underbrace{\underbrace{\Omega_3\times\cdots\times\Omega_3}_{s}\times\Omega_2\times E_m\times E_m\times\cdots\times E_m}_{r}\times E_l]W\\ &(s=1,2,\dots,r)\end{aligned}\right\}\quad(6.30)$$

where W is an arbitrary regular matrix of oder $m^r l$, $l=1,2,\dots\dots$.

(II) *If n is odd: $n=2r+1$, then*

$$\left.\begin{aligned} P_{2s-1}&=W^{-1}[\underbrace{\underbrace{\Omega_3\times\cdots\times\Omega_3}_{s}\times\Omega_1\times E_m\times\cdots\times E_m}_{r}\times E_l]W\\ P_{2s}&=W^{-1}[\underbrace{\underbrace{\Omega_3\times\cdots\times\Omega_3}_{s}\times\Omega_2\times E_m\times\cdots\times E_m}_{r}\times E_l]W\\ P_n&=P_{2r+1}=W^{-1}[\underbrace{\Omega_3\times\cdots\times\Omega_3}_{r}\times R]W\\ &(s=1,2,\dots,r)\end{aligned}\right\}\quad(6.31)$$

where R is any matrix of order l such tnat $R^m=E_l$, we may take $R=\begin{pmatrix}\omega^{\alpha_1} & & & 0\\ & \omega^{\alpha_2} & & \\ & & \ddots & \\ 0 & & & \omega^{\alpha_l}\end{pmatrix}$, ($\alpha_1,\dots,\alpha_l$: any integers, $l=1,2,\dots$), and W is an arbitrary regular matrix of order $m^r l$.

Now if we make correspond the matrix $P_1^{\lambda_1}P_2^{\lambda_2}\dots P_n^{\lambda_n}$, P_i being the matrices in (6.30) or (6.31), to the basic element $p_1^{\lambda_1}p_2^{\lambda_2}\dots p_n^{\lambda_n}$ of the G. C. algebra $\mathfrak{o}$, then we obtain the general representation of $\mathfrak{o}$.

(I) The case where n is even: $n=2r$.

If we take $l=1$ in (6.30), then the representing matrices of $\mathfrak{o}$ are the matrices of order m^r. By the Theorem 6, the ideals of $\mathfrak{o}$ are only $\{0\}$ and $\mathfrak{o}$. Hence all representations of $\mathfrak{o}$ are always faithful. And therefore m^{2r} matrices $P_1^{\lambda_1}P_2^{\lambda_2}\dots P_{2r}^{\lambda_{2r}}$ $(\lambda_1, \lambda_2, \dots, \lambda_{2r}=0, 1, 2, \dots, m-1)$ are linearly independent, and the complete matric algebra in m^r dimensions has the $(m^r)^2$ basic matrices, and therefore these $P_1^{\lambda_1}P_2^{\lambda_2}\dots P_{2r}^{\lambda_{2r}}$, being the representing matrix P_A of the basic element p_A in $\mathfrak{o}$, constitute a basis of this complete matric algebra. Hence this representation $\mathfrak{M}$ of $\mathfrak{o}$, generated by $P_1, \dots, P_n$ (for $l=1$) over the field K, yields the complete matric algebra in m^r dimensions. Consequently, as a matter of course, this representation $\mathfrak{M}$ is irreducible.

(II) The case where n is odd: $n=2r+1$.

It follows from $\Omega_3=\omega^{\frac{m-1}{2}}\Omega_1^{-1}\Omega_2$ that, for the matrices P_i in (6.31),

$$\omega^{-\frac{r(m+1)}{2}}P_1P_2^{-1}P_3P_4^{-1}\dots P_{n-1}^{-1}P_n = W^{-1}[E_{m^r}\times R]W. \tag{6.32}$$

Since $R^m=E_l$, among the matrices R^λ, $\lambda=0, 1, 2, \dots\dots$, there exist at most m linearly independent matrices. And there exist exactly m linealy independent matrices if and only if the minimal polynomial of R is x^m-1, in other words, if and only if R has the characteristic roots $\omega^0, \omega^1, \omega^2, \dots, \omega^{m-1}$. Therefore such matrix R of minimal order is transformable to Ω_1. As we stated in the case (I), the matrices $P_1, \dots, P_{2r}$ for $l=1$ generate the complete matric algebra in dimensions m^r. Hence, as we can see from (6.31), the number of linearly independent matrices among $P_1^{\lambda_1}P_2^{\lambda_2}\dots P_n^{\lambda_n}$, $(\lambda_1, \lambda_2, \dots, \lambda_n =0, 1, \dots, m-1)$ is exactly equal to m^{2r+1} if and only if the minimal polynomial of R is x^m-1. And since the number of the basic elements in $\mathfrak{o}$ is equal to m^{2r+1}, the faithful representation of minimal order of $\mathfrak{o}$ is given by the matric algebra $\mathfrak{M}$ generated by P_i in (6.31) for $l=m$ and $R=\Omega_1$. Then we can see from the form of these matrices P_i that this representation of $\mathfrak{o}$ is the direct sum of m complete matric algebras in m^r dimensions. That is, this representation is completely reducible.

Summarizing these results we obtain the following theorem.

Theorem 8.[1)] *The general representation of the G. C. algebra $\mathfrak{o}$ associated with the linearization of $\sum_{i=1}^{n}(x^i)^m$ is generated by the system of matrices $P_i\,(i=1, 2, \dots, n)$ in Theorem 7. If n is even: $n=2r$, then the faithful*

1) We can obtain the same results, by constructing the regular representation of $\mathfrak{o}$ by means of $p_h p_{(\lambda_1\dots\lambda_n)}=\omega^{\sum_{i\leq h}\lambda_i}p_{(\lambda_1,\dots,\lambda_h+1,\dots\lambda_n)}$ and by decomposing this representation into irreducible parts.

By means of the matrices P_i in Theorem 7, we obtain a linearization of $\sum_{i=1}^{n}\frac{\partial^m}{\partial x^{i\,m}}$:

$$E\cdot\sum_{i=1}^{n}\frac{\partial^m}{\partial x^{i\,m}}=\Big(\sum_{i=1}^{n}P_i\frac{\partial}{\partial x^i}\Big)^m.$$

representation of minimal order yields the complete matric algebra in m^r dimensions, and therefore the representation is irreducible. If n is odd: $n=2r+1$, the faithful representation of minimal order is the direct sum of m complete matric algebras in m^r dimensions, and therefore it is completely reducible.

Moreover, we have the following corollary.[1]

Corollary. *Any automorphism of the G. C. algebra (of course the matric algebra $\mathfrak{M}$) associated with the linearization of $\sum_{i=1}^{n}(x^i)^m$ is inner.*

§ 7. Linear Transformation leaving invariant $\sum_{i=1}^{n}(x^i)^m$, ($m>2$)

In this last section, we shall determine the linear transformation

$$'x^i = \sum_{j=1}^{n} h^i_j x^j \tag{7.1}$$

leaving invariant $\sum_{i=1}^{n}(x^i)^m$, $m>2$; namely,

$$\sum_{i=1}^{n}(x^i)^m = \sum_{i=1}^{n}('x^i)^m . \tag{7.2}$$

Substituting (7.1) into (7.2), we get by comparing the coefficient of $(x^j)^m$,

$$\sum_{i=1}^{n}(h^i_j)^m = 1, \quad j=1,2,\dots,n, \tag{7.3}$$

Similarly, from the coefficients of $(x^j)^{m-2}(x^k)^2$ $(j\neq k)$ and from the coefficients $(x^j)^{m-2}x^kx^l(k\neq l)$, we have

$$\sum_{i=1}^{n}(h^i_j)^{m-2}(h^i_k)^2 = 0 \quad (j\neq k), \tag{7.4}$$

and

$$\sum_{i=1}^{n}(h^i_j)^{m-2}h^i_k h^i_l = 0 \quad (k\neq l). \tag{7.5}$$

respectively. If we write (7.3) and (7.4) together, then we have

$$\sum_{i=1}^{n}(h^i_j)^{m-2}(h^i_k)^2 = \delta_{jk} \quad \text{(kronecker's delta)} \tag{7.6}$$

and therefore it follows from (7.6) that

$$\det |(h^i_j)^{m-2}| \neq 0 . \tag{7.7}$$

From (7.5) and (7.7) we get

$$h^i_k h^i_l = 0 \quad (k\neq l). \tag{7.8}$$

We shall see from (7.7) that there exists no index i such that $h^i_j=0$ for $j=1,2,\dots,n$, and, by means of (7.8), that there exists only one $j(i)$ for each

1) H. Weyl, The classical groups (Princeton), (1939), p. 280.

i such that

$$h^i_{j(i)} \neq 0 \tag{7.9}$$

Moreover, by means of (7.7) again, it is seen that $j(i)$, $i=1, 2, \dots, n$ is a permutation of $1, 2, \dots, n$. From (7.3) and (7.9) we have

$$(h^i_{j(i)})^m = 1, \tag{7.10}$$

and therefore

$$h^i_{j(i)} = \omega^{\lambda_i} \tag{7.11}$$

where ω is a primitive m-th root of uuity and λ_i is any integer. Hence, it is easily seen that there exist $m^n \cdot n!$ linear transformations leaving invariant $\sum_{i=1}^{n} (x^i)^m$, $(m>2)$.

Thus we have the theorem.

Theorem 9. *The set of all linear transformations leaving invariant* $\sum_{i=1}^{n} (x^i)^m$, $(m>2)$ *is a finite group of order* $m^n \cdot n!$. *These linears transformations are written as follows:*

$$'x^i = \omega^{\lambda_i} x^{j(i)}$$

where $j(i)$, $i=1, 2, \dots, n$ *is any permutation of* $1, 2, 3, \dots, n$, ω *is a primitive* m*-th root of unity, and* λ_i *is any integer mod.* m.

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