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On Numerical Integration of Ordinary Differential Equations

By

Minoru URABE and Takeshi TSUSHIMA

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§ 0. Introduction.

For numerical integration of the ordinary differential equations, there are various formulas, but these formulas are divided into two classes. The one is the class of the formulas for integrating ahead by extrapolation, and the other is the class of the formulas for checking and improving the approximate values found by the former formulas. In this paper, we call the former the extrapolation formulas and the latter the improving formulas. Now, all the extrapolation formulas, except for Runge-Kutta's, are obtained by integrating Newton's interpolation formula over some intervals, and the improving formulas by integrating Newton's or central-difference interpolation formula over some intervals. However, the improving formulas based on central-difference formula are used only when the approximate valus are found sufficiently ahead. Thus, except for Runge-Kutta's formula, all the formulas of both classes used in the first step are based on Newton's interpolation formula.

Thus, in this paper, at first, we integrate Newton's interpolation formula over intervals of arbitrary numbers. Next we consider the general linear combination of the formulas thus obtained and seek for the accurate formulas more convenient for practical use than the customary ones. Namely we seek for the coefficients of the linear combination so that the obtained formulas may not contain the difference of higher orders and moreover not lose their accuracy.

In this paper, we consider the differential equations of the first order and those of the second order. For the equations of the higher order, the similar reasonings will prevail.

§ 1. Integration of Newton's interpolation formula.

Newton's interpolation formula is written as follows:

$$(1.1)\quad f(x)=f_0+\frac{u}{1!}\nabla f_0+\frac{u(u+1)}{2!}\nabla^2 f_0+\cdots+\frac{u(u+1)\cdots(u+p-1)}{p!}\nabla^p f_0+S_{p+1},$$

where $x=x_0+uh$, h being the breadth of an interval. For the remainder S_{p+1}, we have:

$$(1.2)\qquad S_{p+1}=\frac{u(u+1)\cdots(u+p)}{(p+1)!}h^{p+1}f^{(p+1)}(\xi),$$

where ξ is a suitable value of x in the interval containing $x_0, x_{-1}, \cdots, x_{-p}$ and x. Integrating (1.1) m-times over the interval $[x_{-N+1}, x_1]$, we have:

$$\int_{x_{-N+1}}^{x_1}\int_{x_{-N+1}}^{x}\cdots\cdots\int_{x_{-N+1}}^{x} f(x) \underbrace{dx\cdots\cdots dx}_{m\text{-times}} = h^m \sum_{\rho=0}^{p} \alpha_{m,\rho}^{N} \nabla^\rho f_0 + R_{m,p+1}^{N}, \tag{1.3}$$

where

$$\alpha_{m,\rho}^{N} = \int_{-N+1}^{1}\int_{-N+1}^{u}\cdots\cdots\int_{-N+1}^{u} \frac{u(u+1)\cdots(u+\rho-1)}{\rho!} \underbrace{du\cdots\cdots du}_{m\text{-times}}. \tag{1.4}$$

For the remainder $R_{m,p+1}^{N}$, from (1.2), we have the estimation as follows:

$$|R_{m,p+1}^{N}| \leqq h^{m+p+1} \overset{*}{\alpha}{}_{m,p+1}^{N} |f^{(p+1)}|_{\max}, \tag{1.5}$$

where

$$\overset{*}{\alpha}{}_{m,p+1}^{N} = \int_{-N+1}^{1}\int_{-N+1}^{u}\cdots\cdots\int_{-N+1}^{u} \frac{|u(u+1)\cdots(u+p)|}{(p+1)!} \underbrace{du\cdots\cdots du}_{m\text{-times}}. \tag{1.6}$$

Now we put ${}^{(0)}f(x)=f(x)$ and define ${}^{(\nu)}f(x)$ successively by the relations as follows:

$$\frac{d}{dx}\,{}^{(\nu)}f(x) = {}^{(\nu-1)}f(x). \tag{1.7}$$

Then it is easily seen that

$$\int_{x_{-N+1}}^{x_1}\int_{x_{-N+1}}^{x}\cdots\cdots\int_{x_{-N+1}}^{x} f(x) \underbrace{dx\cdots\cdots dx}_{m\text{-times}} = {}^{(m)}f(x_1) - \sum_{\nu=0}^{m-1} \frac{(x_1-x_{-N+1})^\nu}{\nu!}\,{}^{(m-\nu)}f(x_{-N+1}). \tag{1.8}$$

Given the differential equation as follows:

$$y^{(n)} = f(x, y, y', \cdots\cdots, y^{(n-1)}). \tag{E}$$

Then, if, in (1.3), we write $n-m$ instead of m and put $f(x)=y^{(n)}$, then, from (1.8), we have:

$$y_1^{(m)} = \sum_{\nu=0}^{n-m-1} \frac{N^\nu h^\nu}{\nu!} y_{-N+1}^{(m+\nu)} + h^{n-m} \sum_{\rho=0}^{p} \alpha_{n-m,\rho}^{N} \nabla^\rho f_0 + R_{n-m,p+1}^{N} \quad (m=0, 1, \cdots, n-1). \tag{1.9}$$

These are nothing but the extrapolation formulas for the equation (E) with the remainders.

If we integrate (1.1) over the interval $[x_{-N}, x_0]$, then, for (E), similarly we have:

$$y_0^{(m)} = \sum_{\nu=0}^{n-m-1} \frac{N^\nu h^\nu}{\nu!} y_{-N}^{(m+\nu)} + h^{n-m} \sum_{\rho=0}^{p} \beta_{n-m,\rho}^{N} \nabla^\rho f_0 + R'^{N}_{n-m,p+1} \quad (m=0,1,\cdots,n-1), \tag{1.10}$$

where

$$\beta_{m,\rho}^{N} = \int_{-N}^{0}\int_{-N}^{u}\cdots\cdots\int_{-N}^{u} \frac{u(u+1)\cdots(u+\rho-1)}{\rho!} \underbrace{du\cdots\cdots du}_{m\text{-times}}. \tag{1.11}$$

For the remainder $R'^{N}_{m,p+1}$, from (1.2), it is valid that

$$|R'^{N}_{m,p+1}| \leqq h^{m+p+1} \overset{*}{\beta}{}_{m,p+1}^{N} |f^{(p+1)}|_{\max}, \tag{1.12}$$

where

$$\overset{*}{\beta}{}^N_{m,p+1}=\int_{-N}^0\int_{-N}^u\cdots\cdots\int_{-N}^u\frac{|u(u+1)\cdots(u+p)|}{(p+1)!}\underbrace{du\cdots\cdots du}_{m\text{-times}}. \tag{1.13}$$

The formulas (1.10) are nothing but the improving formulas for the equation (E) with the remainders.

§ 2. Calculation of the numbers $\alpha^N_{m,\rho}$, $\beta^N_{m,\rho}$, $\overset{*}{\alpha}{}^N_{m,\rho}$ and $\overset{*}{\beta}{}^N_{m,\rho}$.

From (1.4) and (1.11), it is easily seen that, for $N\geqq 2$,

$$\left\{\begin{aligned}\alpha^N_{1,\rho}&=\alpha^1_{1,\rho}+\beta^{N-1}_{1,\rho},\\ \alpha^N_{2,\rho}&=\alpha^1_{2,\rho}+\beta^{N-1}_{1,\rho}+\beta^{N-1}_{2,\rho}.\end{aligned}\right. \tag{2.1}$$

Making use of these formulas, we calculate $\beta^N_{m,\rho}$ and $\alpha^N_{m,\rho}$. The results are shown in Table 1.

Put

$$U_\rho=\frac{u(u+1)\cdots(u+\rho-1)}{\rho\,!}. \tag{2.2}$$

Then, for $u\geqq 0$, $|U_\rho|=U_\rho$, consequently $\overset{*}{\alpha}{}^1_{1,\rho}=\alpha^1_{1,\rho}$ and $\overset{*}{\alpha}{}^1_{2,\rho}=\alpha^1_{2,\rho}$. Now it is evident that (2.1) holds also for $\overset{*}{\alpha}$ and $\overset{*}{\beta}$. Thus we see that

$$\left\{\begin{aligned}\overset{*}{\alpha}{}^N_{1,\rho}&=\alpha^1_{1,\rho}+\overset{*}{\beta}{}^{N-1}_{1,\rho},\\ \overset{*}{\alpha}{}^N_{2,\rho}&=\alpha^1_{2,\rho}+\overset{*}{\beta}{}^{N-1}_{1,\rho}+\overset{*}{\beta}{}^{N-1}_{2,\rho}.\end{aligned}\right. \tag{2.3}$$

Then, for our purpose, it is sufficient to calculate $\overset{*}{\beta}{}^N_{m,\rho}$.

Now, from (2.2), it is evident that, when $\rho\geqq 1$,

$$\left\{\begin{array}{ll}\text{for } u \text{ such that } -\sigma\leqq u\leqq-(\sigma-1)\ (\sigma\leqq\rho-1), & |U_\rho|=(-1)^\sigma U_\rho;\\ \text{for } u\leqq-(\rho-1), \quad |U_\rho|=(-1)^\rho U_\rho. & \end{array}\right.$$

Then it is readily seen that, when $\rho\geqq 1$,

$$\left\{\begin{array}{ll}\text{for } N\geqq\rho, & \overset{*}{\beta}{}^N_{m,\rho}=(-1)^\rho[\beta^N_{m,\rho}-\beta^{N-1}_{m,\rho}]+\overset{*}{\beta}{}^{N-1}_{m,\rho},\\ \text{for } N\leqq\rho-1, & \overset{*}{\beta}{}^N_{m,\rho}=(-1)^N[\beta^N_{m,\rho}-\beta^{N-1}_{m,\rho}]+\overset{*}{\beta}{}^{N-1}_{m,\rho},\end{array}\right. \qquad (m=1,2)$$

where we agree that $\overset{*}{\beta}{}^0_{m,\rho}=\beta^0_{m,\rho}=0$. Then, adding these formulas, we have:

$$\begin{aligned}\overset{*}{\beta}{}^1_{m,\rho}&=-\beta^1_{m,\rho},\\ \overset{*}{\beta}{}^2_{m,\rho}&=\beta^2_{m,\rho}-2\beta^1_{m,\rho},\\ \overset{*}{\beta}{}^3_{m,\rho}&=-\beta^3_{m,\rho}+2\beta^2_{m,\rho}-2\beta^1_{m,\rho},\\ &\vdots\end{aligned}$$

$$\overset{*}{\beta}{}^{\rho-1}_{m,\rho}=(-1)^{\rho-1}[\beta^{\rho-1}_{m,\rho}-2\beta^{\rho-2}_{m,\rho}+2\beta^{\rho-3}_{m,\rho}-\cdots\cdots+(-1)^{\rho-2}2\beta^{1}_{m,\rho}],$$

$$\overset{*}{\beta}{}^{\rho}_{m,\rho}=(-1)^{\rho}[\beta^{\rho}_{m,\rho}-2\beta^{\rho-1}_{m,\rho}+2\beta^{\rho-2}_{m,\rho}-\cdots\cdots+(-1)^{\rho-1}2\beta^{1}_{m,\rho}],$$

$$\overset{*}{\beta}{}^{\rho+1}_{m,\rho}=(-1)^{\rho}[\beta^{\rho+1}_{m,\rho}-2\beta^{\rho-1}_{m,\rho}+2\beta^{\rho-2}_{m,\rho}-\cdots\cdots+(-1)^{\rho-1}2\beta^{1}_{m,\rho}],$$

$$\vdots$$

$$\overset{*}{\beta}{}^{N}_{m,\rho}=(-1)^{\rho}[\beta^{N}_{m,\rho}-2\beta^{\rho-1}_{m,\rho}+2\beta^{\rho-2}_{m,\rho}-\cdots\cdots+(-1)^{\rho-1}2\beta^{1}_{m,\rho}],$$

for $\rho\geqq 1$. When $\rho=0$, $|U_0|=U_0=1$, consequently

$$\overset{*}{\beta}{}^{N}_{m,0}=\beta^{N}_{m,0}.$$

Thus, by means of Table 1, we can calculate the values of $\overset{*}{\beta}{}^{N}_{m,\rho}$. Then, by (2.3), we can calculate the values of $\overset{*}{\alpha}{}^{N}_{m,\rho}$. These results are shown in Table 2.

Remark For the discussion of the paragraph 3rd and downwards, we have not necessity for the values of $\alpha^{N}_{m,\rho}$ and $\overset{*}{\alpha}{}^{N}_{m,\rho}$ for $N\geqq 2$. However, here, for utility of the future, we have calculated these values also.

§ 3. Extrapolation formula for the equation of the first order.

For the differential equation of the first order, from (1.9) and (1.10), we have:

$$(3.1)\quad \begin{cases} \text{(i)} \quad y_{r+1}=y_r+h\sum_{\rho=0}^{p}\alpha^{1}_{1,\rho}\nabla^{\rho}f_r+R^{1}_{1,p+1}, \\ \text{(ii)} \quad y_r=y_{r-s}+h\sum_{\rho=0}^{p}\beta^{s}_{1,\rho}\nabla^{\rho}f_r+R'^{s}_{1,p+1}. \quad (s=1,2,\cdots,N) \end{cases}$$

Multiplying l_s on both sides of (ii) and adding them to (i) we have:

$$(3.2)\qquad y_{r+1}=\sum_{s=0}^{N}l_s y_{r-s}+h\sum_{\rho=0}^{p}a_{1,\rho}\nabla^{\rho}f_r+R_{1,p+1},$$

where

$$(3.3)\quad \begin{cases} l_0=1-\sum_{s=1}^{N}l_s, \\ a_{1,\rho}=\alpha^{1}_{1,\rho}+\sum_{s=1}^{N}l_s\beta^{s}_{1,\rho}. \end{cases}$$

Here the remainder $R_{1,p+1}$ is estimated as follows:

$$(3.4)\qquad |R_{1,p+1}|\leqq h^{p+2}A_1|f^{(p+1)}|_{\max},$$

where

$$(3.5)\qquad A_1=\overset{*}{\alpha}{}^{1}_{1,p+1}+\sum_{s=1}^{N}|l_s|\overset{*}{\beta}{}^{s}_{1,p+1}.$$

The formula (3.2) from which the remainder is removed is nothing but the extrapolation formula. Let the approximate values of y_i calculated by this formula be $\bar{y}_i$. Then it follows that

(3.6) $$\bar{y}_{r+1}=\sum_{s=0}^{N} l_s\bar{y}_{r-s}+h\sum_{\rho=0}^{p} a_{1,\rho}\nabla^{\rho}\bar{f}_r,$$

where $\bar{f}_r=f(x,\bar{y}_r)$. Put

(3.7) $$\alpha_{1,\sigma}=(-1)^{\sigma}\sum_{\rho=\sigma}^{p} a_{1,\rho}\binom{\rho}{\sigma},$$

then (3.6) is written as follows:

(3.8) $$\bar{y}_{r+1}=\sum_{s=0}^{N} l_s\bar{y}_{r-s}+h\sum_{\sigma=0}^{p}\alpha_{1,\sigma}\bar{f}_{r-\sigma}.$$

Then, for the errors $\varepsilon_i=\bar{y}_i-y_i$, from (3.2) and (3.8), we have the following estimation:

(3.9) $$|\varepsilon_{r+1}|\leqq\sum_{s=0}^{N}|l_s||\varepsilon_{r-s}|+hK\sum_{\sigma=0}^{p}|\alpha_{1,\sigma}||\varepsilon_{r-\sigma}|+h^{p+2}A_1L,$$

where $K=|\partial f(x,y)/\partial y|_{\max}$ and $L=|f^{(p+1)}|_{\max}$. Put $\max\limits_{s,\sigma\geqq 0}(|\varepsilon_{r-s}|, |\varepsilon_{r-\sigma}|)=|\varepsilon|$. Then, from (3.9), it follows that

(3.10) $$|\varepsilon_{r+1}|\leqq\Big(\sum_{s=0}^{N}|l_s|+hK\sum_{\sigma=0}^{p}|\alpha_{1,\sigma}|\Big)|\varepsilon|+h^{p+2}A_1L.$$

Consequently, in order that the extrapolation formula be accurate, the quantity $(\sum_{s=0}^{N}|l_s|+hK\sum_{\sigma=0}^{p}|\alpha_{1,\sigma}|)$ should be as small as possible. However, since $h\ll 1$, the quantity $\sum_{s=0}^{N}|l_s|$ should be as small as possible.

Now, for practical computation, it is desirable that the differences of the higher orders do not appear in the formula, in other words, that the coefficients $a_{1,0}$, $a_{1,1}$, ……, $a_{1,p}$ vanish as many as possible, counting from the end.

Thus it is seen that, in order that the extrapolation formula be accurate and moreover be convenient for practical use, it is necessary that the coefficients l_s satisfy the conditions as follows:

(3.11) $$\begin{cases} \text{(i)} & a_{1,\sigma}=\alpha^1_{1,\sigma}+\sum_{s=1}^{N} l_s\beta^s_{1,\sigma}=0 \quad \text{for} \quad \sigma=p,p-1,\cdots, \\ \text{(ii)} & \sum_{s=0}^{N}|l_s|=\min. \end{cases}$$

Now, from (3.3),

(3.11) (iii) $$\sum_{s=0}^{N} l_s=1.$$

The coefficients l_s satisfying (3.11) (i) and (iii) are expressed as follows:

$$l_s=\sum_j c_{sj}\tau_j+c_s,$$

where τ_j's are the parameters. If we consider the τ_j-space E, $l_s=0$ represents a hyperplane in E. Then the space E is divided into several domains by the hyperplanes $l_s=0$ $(s=0,1,2,\cdots,N)$ and in each domain the signs of l_s's are fixed. Consequently, in each domain, $\sum_{s=0}^{N}|l_s|$ is expressed as follows:

$$\sum_{s=0}^{N}|l_s|=\sum_j c_j\tau_j+c\,.$$

If, in the domain, $\sum_{s=0}^{N}|l_s|$ is minimum at the point P, then $\left[\frac{\partial}{\partial\tau_j}\sum_{s=0}^{N}|l_s|\right]_P=c_j=0$, namely $\sum_{s=0}^{N}|l_s|$ is constant. From this fact, it is seen that the minimum value of $\sum_{s=0}^{N}|l_s|$ can be attained at the vertices of the domain. Thus we see that, in order to obtain the desired coefficients l_s's, it is enough to take l_s's such that, of all the vertices, they give the vertex at which $\sum_{s=0}^{N}|l_s|$ is the smallest. Now, from (3.11) (iii), $\sum_{s=0}^{N}|l_s|\geqq 1$, consequently the minimum value of $\sum_{s=0}^{N}|l_s|$ is not less than unity. The procedure of computation for l_s's can be seen from the following example.

Example. The case where $\sigma=5$ and $N=5$.

In this case, the conditions are written as follows:

$$\begin{cases} 475-27l_1-16l_2-27l_3-475l_5=0\,, \\ l_0+l_1+l_2+l_3+l_4+l_5=1\,. \end{cases}$$

The vertices are given by $(l_0, l_1, l_2, l_3, l_4, l_5)$ where any four of l_s's are zeros. Calculating $\sum_{s=0}^{N}|l_s|$ at each vertex, we have:

l_0	l_1	l_2	l_3	l_4	l_5	$\Sigma\,\lvert l_s\rvert$
0	0	0	0	0	1	1
0	0	0	$\frac{475}{27}$	$-\frac{448}{27}$	0	$\frac{923}{27}$
0	0	$\frac{475}{16}$	0	$-\frac{459}{16}$	0	$\frac{934}{16}$
0	0	$-\frac{448}{11}$	$\frac{459}{11}$	0	0	$\frac{907}{11}$
0	$\frac{475}{27}$	0	0	$-\frac{448}{27}$	0	$\frac{923}{27}$
0	$\frac{459}{11}$	$-\frac{448}{11}$	0	0	0	$\frac{907}{11}$
$-\frac{448}{27}$	0	0	$\frac{475}{27}$	0	0	$\frac{923}{27}$
$-\frac{459}{16}$	0	$\frac{475}{16}$	0	0	0	$\frac{934}{16}$
$-\frac{448}{27}$	$\frac{475}{27}$	0	0	0	0	$\frac{923}{27}$

Thus it is seen that the set of the coefficients l_s's which gives the minimum value of $\Sigma | l_s |$ is $(0, 0, 0, 0, 0, 1)$.

Of the extrapolation formulas obtained in the above way, those in which $\Sigma | l_s | -1 < 0.5$, are tabulated in Table 3.

Now, as seen from (3.10), it is supplementarily desirable that $\sum_{\sigma=0}^{p} | \alpha_{1,\sigma} |$ is as small as possible. Therefore we have shown these quantities also in the table.

§ 4. Improving formula for the equation of the first order.

For the differential equation of the first order, from (1.10), we have:

$$y_{r+1} = y_{r+1-s} + h \sum_{s=0}^{p} \beta_{1,\rho}^{s} \nabla^{\rho} f_{r+1} + R_{1,p+1}'^{s}. \quad (s=1, 2, \cdots, N) \tag{4.1}$$

Multiplying l_s on both sides of (4.1) and adding them, we have:

$$\sum_{s=1}^{N} l_s \cdot y_{r+1} = \sum_{s=1}^{N} l_s y_{r+1-s} + h \sum_{\rho=0}^{p} b_{1,\rho} \nabla^{\rho} f_{r+1} + R'_{1,p+1}, \tag{4.2}$$

where

$$b_{1,\rho} = \sum_{s=1}^{N} l_s \beta_{1,\rho}^{s}. \tag{4.3}$$

The remainder $R'_{1,p+1}$ is estimated as follows:

$$| R'_{1,p+1} | \leqq h^{p+2} B_1 L, \tag{4.4}$$

where

$$B_1 = \sum_{s=1}^{N} | l_s | \overset{*}{\beta}{}_{1,p+1}^{s}. \tag{4.5}$$

If we normalize l_s's so that

$$\sum_{s=1}^{N} l_s = 1, \tag{4.6}$$

then, from (4.2), we have:

$$y_{r+1} = \sum_{s=1}^{N} l_s y_{r+1-s} + h \sum_{\rho=0}^{p} b_{1,\rho} \nabla^{\rho} f_{r+1} + R'_{1,p+1}. \tag{4.7}$$

Removing the remainder $R'_{1,p+1}$, we obtain the improving formula.

Let the errors of the approximate values of y_i improved by this formula be ε_i. Put

$$\beta_{1,\sigma} = (-1)^{\sigma} \sum_{\rho=\sigma}^{p} b_{1,\rho} \binom{\rho}{\sigma}, \tag{4.8}$$

then, as in § 3, from (4.7), we have:

$$|\varepsilon_{r+1}|\leqq\sum_{s=1}^{N}|l_s||\varepsilon_{r+1-s}|+hK\sum_{\sigma=0}^{p}|\beta_{1,\sigma}||\varepsilon_{r+1-\sigma}|+h^{p+2}B_1L.$$

Consequently it follows that

$$(4.9)\qquad (1-hK|\beta_{1,0}|)|\varepsilon_{r+1}|\leqq\sum_{s=1}^{N}|l_s||\varepsilon_{r+1-s}|+hK\sum_{\sigma=1}^{p}|\beta_{1,\sigma}||\varepsilon_{r+1-\sigma}|+h^{p+2}B_1L.$$

Put $\max_{s,\sigma\geqq 1}(|\varepsilon_{r+1-s}|,|\varepsilon_{r+1-\sigma}|)=|\varepsilon|$, then it follows that

$$|\varepsilon_{r+1}|\leqq\frac{1}{1-hK|\beta_{1,0}|}\left(\sum_{s=1}^{N}|l_s|+hK\sum_{\sigma=1}^{p}|\beta_{1,\sigma}|\right)|\varepsilon|+\frac{h^{p+2}B_1L}{1-hK|\beta_{1,0}|}.$$

Since $h\ll 1$, $1/(1-hK|\beta_{1,0}|)=1+hK|\beta_{1,0}|$, consequently we have:

$$(4.10)\qquad |\varepsilon_{r+1}|\leqq\left[\sum_{s=1}^{N}|l_s|+hK\left\{\sum_{s=1}^{N}|l_s|\cdot|\beta_{1,0}|+\sum_{\sigma=1}^{p}|\beta_{1,\sigma}|\right\}\right]|\varepsilon|+h^{p+2}B_1L.$$

Then, as in § 3, the desired formulas are determined by the conditions as follows:

$$(4.11)\qquad \begin{cases}\text{(i)} & \sum_{s=1}^{N}l_s=1,\\ \text{(ii)} & \sum_{s=1}^{N}|l_s|=\text{min.},\\ \text{(iii)} & b_{1,\sigma}=\sum_{s=1}^{N}l_s\beta_{1,\sigma}^{s}=0 \quad \text{for} \quad \sigma=p,p-1,\cdots.\end{cases}$$

The procedure of computation for l_s's satisfying (4.11) is the same as in § 3.

Of the improving formulas obtained in this way, those in which $\sum_{s=1}^{N}|l_s|-1<0.3$, are tabulated in Table 4.

For the formula such that $\sum_{s=1}^{N}|l_s|-1\ll 1$, from (4.10), we have:

$$(4.12)\qquad |\varepsilon_{r+1}|\leqq\left[\sum_{s=1}^{N}|l_s|+hK\sum_{\sigma=0}^{p}|\beta_{1,\sigma}|\right]|\varepsilon|+h^{p+2}B_1L,$$

because $h\ll 1$. Then, as in § 3, it is supplementarily desirable that $\sum_{\sigma=0}^{p}|\beta_{1,\sigma}|$ is as small as possible, consequently we have shown these quantities also in the table.

Now, the improved value of y_{r+1} is found by the method of iteration. In this process, $|\beta_{1,0}|$ expresses the rapidity of convergence of iteration process. Hence, we have shown these quantities also in the table.

§ 5. Extrapolation formulas for the equation of the second order.

For the differential equation of the second order, from (1.9) and (1.10), we have:

$$(5.1)\qquad \begin{cases}\text{(i)} & y_{r+1}=y_r+hy'_r+h^2\sum_{\rho=0}^{p}\alpha_{2,\rho}^{1}\nabla^{\rho}f_r+R_{2,p+1}^{1},\\ \text{(ii)} & y_r=y_{r-s}+shy'_{r-s}+h^2\sum_{\rho=0}^{p}\beta_{2,\rho}^{s}\nabla^{\rho}f_r+R_{2,p+1}^{\prime s}.\quad (s=1,2,\cdots,N)\end{cases}$$

Multiplying l_s on both sides of (ii) and adding them to (i), we have:

$$y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h\left(y'_r+\sum_{s=1}^{N} s l_s y'_{r-s}\right)+h^2\sum_{\rho=0}^{p} a_{2,\rho}\nabla^\rho f_r+R_{2,p+1}, \tag{5.2}$$

where

$$\begin{cases} l_0=1-\sum_{s=1}^{N} l_s, \\ a_{2,\rho}=\alpha^1_{2,\rho}+\sum_{s=1}^{N} l_s\beta^s_{2,\rho}. \end{cases} \tag{5.3}$$

The remainder $R_{2,p+1}$ is estimated as follows:

$$|R_{2,p+1}|\leqq h^{p+3}A_2|f^{(p+1)}|_{\max}, \tag{5.4}$$

where

$$A_2=\overset{*}{\alpha}{}^1_{2,p+1}+\sum_{s=1}^{N}|l_s|\,\overset{*}{\beta}{}^s_{2,p+1}.$$

For y'_{r+1}, we can make use of the extrapolation formula for the equation of the first order. Thus, as the extrapolation formulas for the equation of the second order, we have the following simultaneous formulas:

$$\begin{cases} y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h\left(y'_r+\sum_{s=1}^{N} s l_s y'_{r-s}\right)+h^2\sum_{\rho=0}^{p} a_{2,\rho}\nabla^\rho f_r, \\ y'_{r+1}=\sum_{s=0}^{N'} l'_s y'_{r-s}+h\sum_{\rho=0}^{q} a_{1,\rho}\nabla^\rho f_r. \end{cases} \tag{5.5}$$

Put

$$\alpha_{2,\sigma}=(-1)^\sigma\sum_{\rho=\sigma}^{p} a_{2,\rho}\binom{\rho}{\sigma}, \tag{5.6}$$

and let the errors of the approximate values of y_i and y'_i calculated by means of (5.5) be ε_i and ε'_i respectively. Then, putting $K=|\partial f(x,y,y')/\partial y|_{\max}$ and $K'=|\partial f(x,y,y')/\partial y'|_{\max}$, from (5.2) and (3.2), we have:

$$\begin{cases} |\varepsilon_{r+1}|\leqq\sum_{s=0}^{N}|l_s||\varepsilon_{r-s}|+h\left(|\varepsilon'_r|+\sum_{s=1}^{N} s|l_s||\varepsilon'_{r-s}|\right)+h^2\sum_{\sigma=0}^{p}|\alpha_{2,\sigma}|(K|\varepsilon_{r-\sigma}|+K'|\varepsilon'_{r-\sigma}|) \\ \qquad\qquad +h^{p+3}A_2L_2, \\ |\varepsilon'_{r+1}|\leqq\sum_{s=0}^{N'}|l'_s||\varepsilon'_{r-s}|+h\sum_{\sigma=0}^{q}|\alpha_{1,\sigma}|(K|\varepsilon_{r-\sigma}|+K'|\varepsilon'_{r-\sigma}|)+h^{q+2}A_1L_1, \end{cases} \tag{5.7}$$

where $L_1=|f^{(q+1)}|_{\max}$ and $L_2=|f^{(p+1)}|_{\max}$. Put $\max_{s,\sigma\geq 0}(|\varepsilon_{r-s}|, |\varepsilon_{r-\sigma}|)=|\varepsilon|$ and $\max_{s,\sigma\geq 0}(|\varepsilon'_{r-s}|, |\varepsilon'_{r-\sigma}|)=|\varepsilon'|$. Then, from (5.7), it follows that

(5.8) $$\begin{cases} |\varepsilon_{r+1}| \leqq \left(\sum_{s=0}^{N}|l_s| + h^2K\sum_{\sigma=0}^{p}|\alpha_{2,\sigma}|\right)|\varepsilon| + \left\{h\left(1+\sum_{s=1}^{N}s|l_s|\right) + h^2K'\sum_{\sigma=0}^{p}|\alpha_{2,\sigma}|\right\}|\varepsilon'| \\ \qquad\qquad + h^{p+3}A_2L_2, \\ |\varepsilon'_{r+1}| \leqq hK\sum_{\sigma=0}^{q}|\alpha_{1,\sigma}|\cdot|\varepsilon| + \left(\sum_{s=0}^{N'}|l'_s| + hK'\sum_{\sigma=0}^{q}|\alpha_{1,\sigma}|\right)|\varepsilon'| + h^{q+2}A_1L_1. \end{cases}$$

Consequently, in order that the extrapolation formulas be accurate, the coefficients of $|\varepsilon|$ and $|\varepsilon'|$ in the right-hand sides of (5.8) should be as small as possible. We assume that $|\varepsilon|$ and $|\varepsilon'|$ are of the same magnitude. Then, for the second of (5.8), our demand is satisfied, because it is satisfied for the extrapolation formula for the equation of the first order. For the first of (5.8), our demand claimes that the quantities $\sum_{s=0}^{N}|l_s|$ should be as small as possible, for, since $h \ll 1$, the other terms in the coefficients are so small that we can neglect.

Now for practical computation, it is desirable that the differences of the higher orders do not appear in the formulas.

Thus, quite similarly as in § 3, we can obtain the accurate formulas convenient for practical use.

Of the extrapolation formulas obtained in this way, those in which $\sum_{s=0}^{N}|l_s|-1<0.2$, are tabulated in Table 5.

As seen from (5.8), it is supplementarily desirable that the quantities $\left(1+\sum_{s=1}^{N}s\,|l_s|\right)$ and $\sum_{\sigma=0}^{p}|\alpha_{2,\sigma}|$ are as small as possible. Therefore we have shown these quantities also in the table.

Remark. In making use of our formulas, on account of our claim that $|\varepsilon|$ and $|\varepsilon'|$ be of the same magnitude, we must start with the values of y and y' of the same degree of accuracy.

§ 6. Improving formulas for the equation of the second order.

For the differential equation of the second order, from (1.10), we have:

$$y_{r+1}=y_{r+1-s}+shy'_{r+1-s}+h^2\sum_{\rho=0}^{p}\beta^s_{2,\rho}\nabla^\rho f_{r+1}+R'^s_{2,p+1}. \quad (s=1,2,\cdots,N)$$

If we multiply l_s on both sides and add them, then, normalizing l_s's so that $\sum_{s=1}^{N}l_s=1$, we have:

(6.1) $$y_{r+1}=\sum_{s=1}^{N}l_sy_{r+1-s}+h\sum_{s=1}^{N}sl_sy'_{r+1-s}+h^2\sum_{\rho=0}^{p}b_{2,\rho}\nabla^\rho f_{r+1}+R'_{2,p+1},$$

where

(6.2) $$b_{2,\rho}=\sum_{s=1}^{N}l_s\beta^s_{2,\rho}.$$

The remainder $R'_{2,p+1}$ is estimated as follows:

$$|R'_{2,p+1}| \leqq h^{p+3}B_2L_2, \tag{6.3}$$

where

$$B_2 = \sum_{s=1}^{N} |l_s| \overset{*}{\beta}{}^{s}_{2,p+1}. \tag{6.4}$$

For y'_{r+1}, we can make use of the improving formula for the equation of the first order. Thus, as the improving formulas for the equation of the second order, we have the following simultaneous formulas:

$$\left\{\begin{aligned} y_{r+1} &= \sum_{s=1}^{N} l_s y_{r+1-s} + h\sum_{s=1}^{N} s l_s y'_{r+1-s} + h^2 \sum_{\rho=0}^{p} b_{2,\rho} \nabla^\rho f_{r+1}, \\ y'_{r+1} &= \sum_{s=1}^{N'} l'_s y'_{r+1-s} + h \sum_{\rho=0}^{q} b_{1,\rho} \nabla^\rho f_{r+1}. \end{aligned}\right. \tag{6.5}$$

Put

$$\beta_{2,\sigma} = (-1)^\sigma \sum_{\rho=\sigma}^{p} b_{2,\rho} \binom{\rho}{\sigma}, \tag{6.6}$$

and let the errors of the approximate values of y_i and y'_i improved by means of (6.5) be ε_i and ε'_i respectively. Then, as in § 5, we have:

$$\left\{\begin{aligned} |\varepsilon_{r+1}| &\leqq \sum_{s=1}^{N} |l_s|\,|\varepsilon_{r+1-s}| + h\sum_{s=1}^{N} s\,|l_s|\,|\varepsilon'_{r+1-s}| + h^2 \sum_{\sigma=0}^{p} |\beta_{2,\sigma}| (K|\varepsilon_{r+1-\sigma}| + K'|\varepsilon'_{r+1-\sigma}|) \\ &\qquad + h^{p+3}B_2L_2, \\ |\varepsilon'_{r+1}| &\leqq \sum_{s=1}^{N'} |l'_s|\,|\varepsilon'_{r+1-s}| + h\sum_{\sigma=0}^{q} |\beta_{1,\sigma}| (K|\varepsilon_{r+1-\sigma}| + K'|\varepsilon'_{r+1-\sigma}|) + h^{q+2}B_1L_1. \end{aligned}\right.$$

Consequently it follows that

$$\left\{\begin{aligned} &(1-h^2K|\beta_{2,0}|)|\varepsilon_{r+1}| - h^2K'|\beta_{2,0}|\,|\varepsilon'_{r+1}| \leqq \sum_{s=1}^{N} |l_s|\,|\varepsilon_{r+1-s}| + h\sum_{s=1}^{N} s\,|l_s|\,|\varepsilon'_{r+1-s}| \\ &\qquad + h^2K\sum_{\sigma=1}^{p} |\beta_{2,\sigma}|\,|\varepsilon_{r+1-\sigma}| + h^2K'\sum_{\sigma=1}^{p} |\beta_{2,\sigma}|\,|\varepsilon'_{r+1-\sigma}| + h^{p+3}B_2L_2, \\ &-hK|\beta_{1,0}|\,|\varepsilon_{r+1}| + (1-hK'|\beta_{1,0}|)\,|\varepsilon'_{r+1}| \leqq hK\sum_{\sigma=1}^{q} |\beta_{1,\sigma}|\,|\varepsilon_{r+1-\sigma}| + \sum_{s=1}^{N'} |l'_s| \times \\ &\qquad \times |\varepsilon'_{r+1-s}| + hK'\sum_{\sigma=1}^{q} |\beta_{1,\sigma}|\,|\varepsilon'_{r+1-\sigma}| + h^{q+2}B_1L_1. \end{aligned}\right. \tag{6.7}$$

Put $\max\limits_{s,\sigma\geqq 1}(|\varepsilon_{r+1-s}|, |\varepsilon_{r+1-\sigma}|) = |\varepsilon|$ and $\max\limits_{s,\sigma\geqq 1}(|\varepsilon'_{r+1-s}|, |\varepsilon'_{r+1-\sigma}|) = |\varepsilon'|$. If we solve (6.7) with regard to $|\varepsilon_{r+1}|$ and $|\varepsilon'_{r+1}|$, then, neglecting the terms of the higher orders with regard to h, we have:

$$(6.8)\quad\begin{cases} |\varepsilon_{r+1}|\leqq\Big[\sum_{s=1}^{N}|l_s|+h^2K\Big(|\beta_{2,0}|\sum_{s=1}^{N}|l_s|+\sum_{\sigma=1}^{p}|\beta_{2,\sigma}|\Big)\Big]|\varepsilon| \\ \qquad\qquad +\Big[h\sum_{s=1}^{N}s|l_s|+h^2K'\Big(|\beta_{2,0}|\sum_{s=1}^{N'}|l'_s|+\sum_{\sigma=1}^{p}|\beta_{2,\sigma}|\Big)\Big]|\varepsilon'|+h^{p+3}B_2L_2, \\ |\varepsilon'_{r+1}|\leqq hK\Big(|\beta_{1,0}|\sum_{s=1}^{N}|l_s|+\sum_{\sigma=1}^{q}|\beta_{1,\sigma}|\Big)|\varepsilon| \\ \qquad\qquad +\Big[\sum_{s=1}^{N'}|l'_s|+hK'\Big(|\beta_{1,0}|\sum_{s=1}^{N'}|l'_s|+\sum_{\sigma=1}^{q}|\beta_{1,\sigma}|\Big)\Big]|\varepsilon'|+h^{q+2}B_1L_1. \end{cases}$$

As in § 5, we assume that $|\varepsilon|$ and $|\varepsilon'|$ are of the same magnitude. Then, as in § 5, it is readily seen that, in order that the improving formulas (6.5) be accurate, it is necessary that the quantity $\sum_{s=1}^{N}|l_s|$ is as small as possible.

Thus, quite similarly as in § 5, we can obtain the accurate formulas convenient for practical use.

Of the improving formulas obtained in this way, those in which $\sum_{s=0}^{N}|l_s|-1<0.1$, are tabulated in Table 6.

For the formulas such that $\sum_{s=1}^{N}|l_s|-1\ll 1$ and $\sum_{s=1}^{N'}|l'_s|-1\ll 1$, from (6.8), we have:

$$(6.9)\quad\begin{cases} |\varepsilon_{r+1}|\leqq\Big(\sum_{s=1}^{N}|l_s|+h^2K\sum_{\sigma=0}^{p}|\beta_{2,\sigma}|\Big)|\varepsilon|+\Big(h\sum_{s=1}^{N}s|l_s|+h^2K'\sum_{\sigma=0}^{p}|\beta_{2,\sigma}|\Big)|\varepsilon'|+h^{p+3}B_2L_2, \\ |\varepsilon'_{r+1}|\leqq hK\sum_{\sigma=0}^{q}|\beta_{1,\sigma}|\cdot|\varepsilon|+\Big(\sum_{s=1}^{N'}|l'_s|+hK'\sum_{\sigma=0}^{q}|\beta_{1,\sigma}|\Big)|\varepsilon'|+h^{q+2}B_1L_1. \end{cases}$$

Then, as in § 5, it is supplementarily desirable that the quantities $\sum_{s=1}^{N}s|l_s|$ and $\sum_{\sigma=0}^{p}|\beta_{2,\sigma}|$ are as small as possible. Therefore we have shown these quantities also in Table 6.

Now the values of y_{r+1} and y'_{r+1} are found by the method of iteration from (6.5). Let the m-th values of them in the iteration process be $y_{r+1}^{(m)}$ and $y_{r+1}^{\prime(m)}$ respectively. Then, from (6.5), it is easily seen that

$$\begin{cases} |y_{r+1}^{(m+1)}-y_{r+1}^{(m)}|\leqq h^2|\beta_{2,0}|(K|y_{r+1}^{(m)}-y_{r+1}^{(m-1)}|+K'|y_{r+1}^{\prime(m)}-y_{r+1}^{\prime(m-1)}|), \\ |y_{r+1}^{\prime(m+1)}-y_{r+1}^{\prime(m)}|\leqq h|\beta_{1,0}|(K|y_{r+1}^{(m)}-y_{r+1}^{(m-1)}|+K'|y_{r+1}^{\prime(m)}-y_{r+1}^{\prime(m-1)}|). \end{cases}$$

Consequently, it follows that

$$(K|y_{r+1}^{(m+1)}-y_{r+1}^{(m)}|+K'|y_{r+1}^{\prime(m+1)}-y_{r+1}^{\prime(m)}|)\leqq h(hK|\beta_{2,0}|+K'|\beta_{1,0}|)\times$$
$$\times(K|y_{r+1}^{(m)}-y_{r+1}^{(m-1)}|+K'|y_{r+1}^{\prime(m)}-y_{r+1}^{\prime(m-1)}|).$$

Thus the quantity $h(hK|\beta_{2,0}|+K'|\beta_{1,0}|)$ expresses the rapidity of convergence of iteration process. Hence, we have shown the quantity $|\beta_{2,0}|$ also in Table 6.

As in § 5, in making use of our formulas, on account of our claim that $|\varepsilon|$ and $|\varepsilon'|$ be of the same magnitude, we must start with the values of y and y' of the accuracy of the same degree.

§ 7. Example.

As an example, we shall find correctly to five decimal places the solution of the equation

$$y''=-y'^2/y, \tag{7.1}$$

with the initial condition that $y=1$, $y'=1$ when $x=0$.

In order to obtain the solution correct to five decimal places, we adopt the formulas for $p=3$ and $q=4$. Of such formulas, for the sake of simplicity and accuracy, we avail ourselves of the formulas as follows:

$$(7.2)\quad \begin{cases} \text{for extrapolation,} \\ y'_{r+1}=\dfrac{1}{112}(39y'_{r-1}+96y'_{r-4}-23y'_{r-5})+\dfrac{h}{28}(105f_r-111\nabla f_r+87\nabla^2 f_r), \\ y_{r+1}=\dfrac{1}{351}(313y_r+38y_{r-3})+\dfrac{h}{351}(351y'_r+114y'_{r-3}) \\ \qquad\qquad +\dfrac{h^2}{312}(308f_r-252\nabla f_r+229\nabla^2 f_r); \\ \text{for improving,} \\ y'_{r+1}=\dfrac{1}{531}(250y'_r+300y'_{r-1}-25y'_{r-3}+6y'_{r-4})+\dfrac{h}{177}(260f_{r+1}-200\nabla f_{r+1}), \\ y_{r+1}=\dfrac{1}{23}(16y_r+7y_{r-1})+\dfrac{h}{23}(16y'_r+14y'_{r-1})+\dfrac{h^2}{23}(22f_{r+1}-24\nabla f_{r+1}+4\nabla^2 f_{r+1}). \end{cases}$$

In order to find the new values of y and y' by means of these formulas, it is necessary to have the six beforehand known values of y and y'. Thus, the starting values must be computed for the six values of x. For this purpose, at first, differentiating successively the equation (7.1), we find the values of the derivatives for $x=0$. They are as follows:

$$y''=-1,\quad y'''=3,\quad y^{(3)}=-15,\quad y^{(5)}=105,\quad y^{(6)}=-945,$$

$$y^{(7)}=10{,}395,\quad y^{(8)}=-135{,}135,\quad y^{(9)}=2{,}027{,}025,\quad y^{(10)}=-34{,}459{,}425.$$

By means of Taylor's series determined by these values, we compute the starting values of y and y' correct to five decimal places for $x=-0.10$, -0.05, 0, 0.05, 0.10 and 0.15.

Next, by means of (7.2) with $h=0.05$, we compute the values of y and y' to five decimal places for x=0.20, 0.25, 0.30, 0.35 and 0.40.

Lastly, starting with the values found in the above way for x=−0.1, 0, 0.1, 0.2, 0.3 and 0.4, by means of (7.2) with h=0.1, we compute successively the values of y and y' to five decimal places. These values are tabulated in the following table as y_1 and y'_1.

Now, the equation (7.1) is easily integrated and the solution satisfying the given initial condition becomes

$$y=\sqrt{2x+1}\ .$$

Consequently it follows that $y'=1/\sqrt{2x+1}$. For comparison, the true values of y and y' computed from these functions are also tabulated in the table.

Solution of the equation $y''=-y'^2/y$ with the initial conaition that $y(0)=y'(0)=1$.

x	y					y'				
	y_1		true values	y_2		y_1'		true values	y_2'	
	values	errors		values	errors	values	errors		values	errors
−0.10	0.89443		0.89443	0.89443		1.11803		1.11803	1.11803	
−0.05	0.94868		0.94868	0.94868		1.05409		1.05409	1.05409	
0	1.00000		1.00000	1.00000		1.00000		1.00000	1.00000	
0.05	1.04881		1.04881	1.04881		0.95346		0.95346	0.95346	
0.10	1.09544		1.09544	1.09544		0.91287		0.91287	0.91287	
0.15	1.14018		1.14018	1.14018		0.87706		0.87706	0.87706	
0.20	1.18321	−1	1.18322	1.18322	0	0.84515	0	0.84515	0.84516	+1
0.25	1.22474	0	1.22474	1.22475	+1	0.81650	0	0.81650	0.81650	0
0.30	1.26490	−1	1.26491	1.26491	0	0.79057	0	0.79057	0.79057	0
0.35	1.30384	*	1.30384	——		0.76697	*	0.76696	——	
0.40	1.34164	0	1.34164	1.34164	0	0.74536	0	0.74536	0.74538	+2
0.5	1.41420	−1	1.41421	1.41422	+1	0.70711	0	0.70711	0.70714	+3
0.6	1.48323	−1	1.48324	1.48324	0	0.67421	+1	0.67420	0.67423	+3
0.7	1.54918	−1	1.54919	1.54919	0	0.64550	0	0.64550	0.64553	+3
0.8	1.61244	−1	1.61245	1.61245	0	0.62018	+1	0.62017	0.62020	+3
0.9	1.67331	−1	1.67332	1.67332	0	0.59762	+1	0.59761	0.59764	+3
1.0	1.73203	−2	1.73205	1.73205	0	0.57736	+1	0.57735	0.57737	+2
1.1	1.78884	−1	1.78885	1.78886	+1	0.55902	0	0.55902	0.55904	+2
1.2	1.84389	−2	1.84391	1.84391	0	0.54233	0	0.54233	0.54235	+2
1.3	1.89735	−2	1.89737	1.89738	+1	0.52705	0	0.52705	0.52707	+2
1.4	1.94935	−1	1.94936	1.94938	+2	0.51299	0	0.51299	0.51301	+2
1.5	1.99999	−1	2.00000	2.00002	+2	0.50000	0	0.50000	0.50002	+2
1.6	2.04938	−1	2.04939	2.04941	+2	0.48795	0	0.48795	0.48797	+2
1.7	2.09761	−1	2.09762	2.09764	+2	0.47673	0	0.47673	0.47675	+2
1.8	2.14476	0	2.14476	2.14479	+3	0.46625	0	0.46625	0.46627	+2
1.9	2.19088	−1	2.19089	2.19092	+3	0.45643	−1	0.45644	0.45645	+1
2.0	2.23605	−2	2.23607	2.23609	+2	0.44721	0	0.44721	0.44723	+2

For comparison, we have computed the values of y and y' by means of the customary difference formulas as follows:

$$
(7.3)\quad \begin{cases}
\text{for extrapolation,} \\
y'_{r+1}=y'_r+\dfrac{h}{720}(720f_r+360\nabla f_r+300\nabla^2 f_r+270\nabla^3 f_r+251\nabla^4 f_r), \\
y_{r+1}=y_r+hy'_r+\dfrac{h^2}{360}(180f_r+60\nabla f_r+45\nabla^2 f_r+38\nabla^3 f_r); \\
\text{for improving,} \\
y'_{r+1}=y'_r+\dfrac{h}{720}(720f_{r+1}-360\nabla f_{r+1}-60\nabla^2 f_{r+1}-30\nabla^3 f_{r+1}-19\nabla^4 f_{r+1}), \\
y_{r+1}=y_r+hy'_r+\dfrac{h^2}{360}(180f_{r+1}-120\nabla f_{r+1}-15\nabla^2 f_{r+1}-7\nabla^3 f_{r+1}).
\end{cases}
$$

These values are tabulated in the table as y_2 and y'_2.

The formulas (7.2) do not contain the differences of 3rd and 4th orders contained in (7.3). In this respect, (7.2) is simpler than (7.3), but the terms of y and y' in the right-hand sides of (7.2) are more complicated than those of (7.3). As seen from our example, in practical computation, the extrapolation process does not make much difference in labour whether by means of (7.2) or (7.3). However, in the improving process, the computation by means of (7.2) is much simpler than that by means of (7.3), especially when the iteration is carried out two and more times. In accuracy, as seen from the table, (7.2) is pretty better than (7.3).

Faculty of Science, Hiroshima University
and Hiroshima Women's College.

Table 1. *The numbers* $\alpha_{m,\rho}^{N}$ *and* $\beta_{m,\rho}^{N}$.

$m=1$

N	ρ	0	1	2	3	4	5	6
1	$\alpha_{1,\rho}^{1}$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{9}{24}$	$\frac{251}{720}$	$\frac{475}{1440}$	$\frac{19087}{60480}$
	$\beta_{1,\rho}^{1}$	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{27}{1440}$	$-\frac{863}{60480}$
2	$\alpha_{1,\rho}^{2}$	2	0	$\frac{4}{12}$	$\frac{8}{24}$	$\frac{232}{720}$	$\frac{448}{1440}$	$\frac{18224}{60480}$
	$\beta_{1,\rho}^{2}$	2	$-\frac{4}{2}$	$\frac{4}{12}$	0	$-\frac{8}{720}$	$-\frac{16}{1440}$	$-\frac{592}{60480}$
3	$\alpha_{1,\rho}^{3}$	3	$-\frac{3}{2}$	$\frac{9}{12}$	$\frac{9}{24}$	$\frac{243}{720}$	$\frac{459}{1440}$	$\frac{18495}{60480}$
	$\beta_{1,\rho}^{3}$	3	$-\frac{9}{2}$	$\frac{27}{12}$	$-\frac{9}{24}$	$-\frac{27}{720}$	$-\frac{27}{1440}$	$-\frac{783}{60480}$
4	$\alpha_{1,\rho}^{4}$	4	$-\frac{8}{2}$	$\frac{32}{12}$	0	$\frac{224}{720}$	$\frac{448}{1440}$	$\frac{18304}{60480}$
	$\beta_{1,\rho}^{4}$	4	$-\frac{16}{2}$	$\frac{80}{12}$	$-\frac{64}{24}$	$\frac{224}{720}$	0	$-\frac{512}{60480}$
5	$\alpha_{1,\rho}^{5}$	5	$-\frac{15}{2}$	$\frac{85}{12}$	$-\frac{55}{24}$	$\frac{475}{720}$	$\frac{475}{1440}$	$\frac{18575}{60480}$
	$\beta_{1,\rho}^{5}$	5	$-\frac{25}{2}$	$\frac{175}{12}$	$-\frac{225}{24}$	$\frac{2125}{720}$	$-\frac{475}{1440}$	$-\frac{1375}{60480}$
6	$\alpha_{1,\rho}^{6}$	6	$-\frac{24}{2}$	$\frac{180}{12}$	$-\frac{216}{24}$	$\frac{2376}{720}$	0	$\frac{17712}{60480}$

$m=2$

N	ρ	0	1	2	3	4	5	6
1	$\alpha_{2,\rho}^{1}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{3}{24}$	$\frac{38}{360}$	$\frac{135}{1440}$	$\frac{863}{10080}$	$\frac{9625}{120960}$
	$\beta_{2,\rho}^{1}$	$\frac{1}{2}$	$-\frac{2}{6}$	$-\frac{1}{24}$	$-\frac{7}{360}$	$-\frac{17}{1440}$	$-\frac{82}{10080}$	$-\frac{731}{120960}$
2	$\alpha_{2,\rho}^{2}$	$\frac{4}{2}$	$-\frac{4}{6}$	0	$\frac{16}{360}$	$\frac{80}{1440}$	$\frac{592}{10080}$	$\frac{7168}{120960}$
	$\beta_{2,\rho}^{2}$	$\frac{4}{2}$	$-\frac{16}{6}$	$\frac{16}{24}$	$\frac{16}{360}$	$\frac{16}{1440}$	$\frac{32}{10080}$	$\frac{64}{120960}$
3	$\alpha_{2,\rho}^{3}$	$\frac{9}{2}$	$-\frac{27}{6}$	$\frac{27}{24}$	$\frac{54}{360}$	$\frac{135}{1440}$	$\frac{783}{10080}$	$\frac{8505}{120960}$
	$\beta_{2,\rho}^{3}$	$\frac{9}{2}$	$-\frac{54}{6}$	$\frac{135}{24}$	$-\frac{351}{360}$	$-\frac{81}{1440}$	$-\frac{162}{10080}$	$-\frac{891}{120960}$
4	$\alpha_{2,\rho}^{4}$	$\frac{16}{2}$	$-\frac{80}{6}$	$\frac{192}{24}$	$-\frac{448}{360}$	0	$\frac{512}{10080}$	$\frac{7168}{120960}$
	$\beta_{2,\rho}^{4}$	$\frac{16}{2}$	$-\frac{128}{6}$	$\frac{512}{24}$	$-\frac{3328}{360}$	$\frac{1792}{1440}$	$\frac{512}{10080}$	$\frac{1024}{120960}$
5	$\alpha_{2,\rho}^{5}$	$\frac{25}{2}$	$-\frac{175}{6}$	$\frac{675}{24}$	$-\frac{4250}{360}$	$\frac{2375}{1440}$	$\frac{1375}{10080}$	$\frac{9625}{120960}$
	$\beta_{2,\rho}^{5}$	$\frac{25}{2}$	$-\frac{250}{6}$	$\frac{1375}{24}$	$-\frac{14375}{360}$	$\frac{19375}{1440}$	$-\frac{15250}{10080}$	$-\frac{6875}{120960}$
6	$\alpha_{2,\rho}^{6}$	$\frac{36}{2}$	$-\frac{324}{6}$	$\frac{1728}{24}$	$-\frac{17712}{360}$	$\frac{23760}{1440}$	$-\frac{17712}{10080}$	0

Table 2. *The numbers* $\overset{*}{\alpha}{}^{N}_{m,\rho}$ *and* $\overset{*}{\beta}{}^{N}_{m,\rho}$.

$m=1$

N	ρ	0	1	2	3	4	5	6
1	$\overset{*}{\alpha}{}^{1}_{1,\rho}$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{9}{24}$	$\frac{251}{720}$	$\frac{475}{1440}$	$\frac{19087}{60480}$
	$\overset{*}{\beta}{}^{1}_{1,\rho}$	1	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{19}{720}$	$\frac{27}{1440}$	$\frac{863}{60480}$
2	$\overset{*}{\alpha}{}^{2}_{1,\rho}$	2	$\frac{2}{2}$	$\frac{6}{12}$	$\frac{10}{24}$	$\frac{270}{720}$	$\frac{502}{1440}$	$\frac{19950}{60480}$
	$\overset{*}{\beta}{}^{2}_{1,\rho}$	2	$\frac{4}{2}$	$\frac{6}{12}$	$\frac{2}{24}$	$\frac{30}{720}$	$\frac{38}{1440}$	$\frac{1134}{60480}$
3	$\overset{*}{\alpha}{}^{3}_{1,\rho}$	3	$\frac{5}{2}$	$\frac{11}{12}$	$\frac{11}{24}$	$\frac{281}{720}$	$\frac{513}{1440}$	$\frac{20221}{60480}$
	$\overset{*}{\beta}{}^{3}_{1,\rho}$	3	$\frac{9}{2}$	$\frac{29}{12}$	$\frac{11}{24}$	$\frac{49}{720}$	$\frac{49}{1440}$	$\frac{1325}{60480}$
4	$\overset{*}{\alpha}{}^{4}_{1,\rho}$	4	$\frac{10}{2}$	$\frac{34}{12}$	$\frac{20}{24}$	$\frac{300}{720}$	$\frac{524}{1440}$	$\frac{20412}{60480}$
	$\overset{*}{\beta}{}^{4}_{1,\rho}$	4	$\frac{16}{2}$	$\frac{82}{12}$	$\frac{66}{24}$	$\frac{300}{720}$	$\frac{76}{1440}$	$\frac{1596}{60480}$
5	$\overset{*}{\alpha}{}^{5}_{1,\rho}$	5	$\frac{17}{2}$	$\frac{87}{12}$	$\frac{75}{24}$	$\frac{551}{720}$	$\frac{551}{1440}$	$\frac{20683}{60480}$
	$\overset{*}{\beta}{}^{5}_{1,\rho}$	5	$\frac{25}{2}$	$\frac{177}{12}$	$\frac{227}{24}$	$\frac{2201}{720}$	$\frac{551}{1440}$	$\frac{2459}{60480}$
6	$\overset{*}{\alpha}{}^{6}_{1,\rho}$	6	$\frac{26}{2}$	$\frac{182}{12}$	$\frac{236}{24}$	$\frac{2452}{720}$	$\frac{1026}{1440}$	$\frac{21546}{60480}$

$m=2$

N	ρ	0	1	2	3	4	5	6
1	$\overset{*}{\alpha}{}^{1}_{2,\rho}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{3}{24}$	$\frac{38}{360}$	$\frac{135}{1440}$	$\frac{863}{10080}$	$\frac{9625}{120960}$
	$\overset{*}{\beta}{}^{1}_{2,\rho}$	$\frac{1}{2}$	$\frac{2}{6}$	$\frac{1}{24}$	$\frac{7}{360}$	$\frac{17}{1440}$	$\frac{82}{10080}$	$\frac{731}{120960}$
2	$\overset{*}{\alpha}{}^{2}_{2,\rho}$	$\frac{4}{2}$	$\frac{6}{6}$	$\frac{6}{24}$	$\frac{60}{360}$	$\frac{190}{1440}$	$\frac{1134}{10080}$	$\frac{12082}{120960}$
	$\overset{*}{\beta}{}^{2}_{2,\rho}$	$\frac{4}{2}$	$\frac{16}{6}$	$\frac{18}{24}$	$\frac{30}{360}$	$\frac{50}{1440}$	$\frac{196}{10080}$	$\frac{1526}{120960}$
3	$\overset{*}{\alpha}{}^{3}_{2,\rho}$	$\frac{9}{2}$	$\frac{29}{6}$	$\frac{33}{24}$	$\frac{98}{360}$	$\frac{245}{1440}$	$\frac{1325}{10080}$	$\frac{13419}{120960}$
	$\overset{*}{\beta}{}^{3}_{2,\rho}$	$\frac{9}{2}$	$\frac{54}{6}$	$\frac{137}{24}$	$\frac{397}{360}$	$\frac{147}{1440}$	$\frac{390}{10080}$	$\frac{2481}{120960}$
4	$\overset{*}{\alpha}{}^{4}_{2,\rho}$	$\frac{16}{2}$	$\frac{82}{6}$	$\frac{198}{24}$	$\frac{600}{360}$	$\frac{380}{1440}$	$\frac{1596}{10080}$	$\frac{14756}{120960}$
	$\overset{*}{\beta}{}^{4}_{2,\rho}$	$\frac{16}{2}$	$\frac{128}{6}$	$\frac{514}{24}$	$\frac{3374}{360}$	$\frac{2020}{1440}$	$\frac{1064}{10080}$	$\frac{4396}{120960}$
5	$\overset{*}{\alpha}{}^{5}_{2,\rho}$	$\frac{25}{2}$	$\frac{177}{6}$	$\frac{681}{24}$	$\frac{4402}{360}$	$\frac{2755}{1440}$	$\frac{2459}{10080}$	$\frac{17213}{120960}$
	$\overset{*}{\beta}{}^{5}_{2,\rho}$	$\frac{25}{2}$	$\frac{250}{6}$	$\frac{1377}{24}$	$\frac{14421}{360}$	$\frac{19603}{1440}$	$\frac{16826}{10080}$	$\frac{12295}{120960}$
6	$\overset{*}{\alpha}{}^{6}_{2,\rho}$	$\frac{36}{2}$	$\frac{326}{6}$	$\frac{1734}{24}$	$\frac{17864}{360}$	$\frac{24140}{1440}$	$\frac{21546}{10080}$	$\frac{26838}{120960}$

Table 3. *Extrapolation formulas for the equation of the first order.* $y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h\sum_{\rho=0}^{p} a_{1,\rho}\nabla^{\rho} f_r$.

l_0	l_1	l_2	l_3	l_4	l_5	$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$\Sigma\|l_s\|$	$\Sigma\|\alpha_{1,\sigma}\|$	A_1
1	0	0	0	0	0	1440/1440	720/1440	600/1440	540/1440	502/1440	475/1440	1	22.800	0.315592
0	0	0	0	0	1	60/10	−120/10	150/10	−90/10	33/10	0	1	22.800	0.356250
1	0	0	0	0	0	720/720	360/720	300/720	270/720	251/720	*	1	12.244	0.329861
0	0	0	297/269	0	−28/269	1020/269	−852/269	372/269	252/269	0	*	1.208178	10.483	0.407259
0	0	88/79	0	0	−9/79	210/79	−24/79	−69/79	114/79	0	*	1.227848	10.101	0.402848
0	0	0	1755/4009	3024/4009	−770/4009	17520/4009	−2046/4009	14550/4009	0	0	*	1.384136	17.867	0.458060
0	39/112	0	0	96/112	−23/112	105/28	−111/28	87/28	0	0	*	1.410714	8.250	0.460206
1	0	0	0	0	0	24/24	12/24	10/24	9/24	*	*	1	6.667	0.348611
0	0	0	1	0	0	12/3	−12/3	8/3	0	*	*	1	6.667	0.416667
0	55/63	0	0	8/63	0	50/21	−20/21	25/21	0	*	*	1	5.238	0.424559
0	27/28	0	0	0	1/28	15/7	−3/7	6/7	0	*	*	1	4.714	0.483234
0	0	55/64	0	9/64	0	210/64	−150/64	105/64	0	*	*	1	5.156	0.443012
0	0	24/25	0	0	1/25	78/25	−48/25	33/25	0	*	*	1	4.560	0.510889
55/64	0	0	0	9/64	0	75/48	−30/48	65/48	0	*	*	1	5.729	0.407205
24/25	0	0	0	0	1/25	6/5	0	5/5	0	*	*	1	5.200	0.470889
1	0	0	0	0	0	12/12	6/12	5/12	*	*	*	1	3.667	0.375000
0	45/44	0	0	0	−1/44	21/11	3/11	0	*	*	*	1.045455	2.455	0.632576
0	85/81	0	0	−4/81	0	50/27	10/27	0	*	*	*	1.098765	2.593	0.554527
0	8/7	0	−1/7	0	0	12/7	4/7	0	*	*	*	1.285714	2.857	0.488095
27/50	0	25/50	0	0	−2/50	9/5	0	0	*	*	*	1.080000	1.800	0.795000
0	10/14	5/14	0	−1/14	0	15/7	0	0	*	*	*	1.142857	2.143	0.630952

Table 4. *Improving formulas for the equation of the first order.* $y_{r+1}=\sum_{s=1}^{N} l_s y_{r+1-s}+h\sum_{\rho=0}^{p} b_{1,\rho}\nabla^{\rho} f_{r+1}$.

l_1	l_2	l_3	l_4	l_5	$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$b_{1,4}$	$b_{1,5}$	$\Sigma\|l_s\|$	$\Sigma\|\beta_{1,\sigma}\|$	B_1	$\|\beta_{1,0}\|$
1	0	0	0	0	$\frac{1440}{1440}$	$-\frac{720}{1440}$	$-\frac{120}{1440}$	$-\frac{60}{1440}$	$-\frac{38}{1440}$	$-\frac{27}{1440}$	1	2.349	0.014269	0.330
0	0	0	1	0	$\frac{180}{45}$	$-\frac{360}{45}$	$\frac{300}{45}$	$-\frac{120}{45}$	$\frac{14}{45}$	0	1	4.000	0.026389	0.311
0	$\frac{1900}{2511}$	0	$\frac{675}{2511}$	$-\frac{64}{2511}$	$\frac{2060}{837}$	$-\frac{2800}{837}$	$\frac{1400}{837}$	$-\frac{400}{837}$	0	0	1.050976	2.461	0.022318	0.311
1	0	0	0	0	$\frac{720}{720}$	$-\frac{360}{720}$	$-\frac{60}{720}$	$-\frac{30}{720}$	$-\frac{19}{720}$	*	1	1.786	0.018750	0.349
0	0	$\frac{2125}{2152}$	0	$\frac{27}{2152}$	$\frac{6510}{2152}$	$-\frac{9900}{2152}$	$\frac{5175}{2152}$	$-\frac{1050}{2152}$	0	*	1	3.025	0.038402	0.342
0	$\frac{2125}{2133}$	0	0	$\frac{8}{2133}$	$\frac{1430}{711}$	$-\frac{1450}{711}$	$\frac{275}{711}$	$-\frac{25}{711}$	0	*	1	2.011	0.027725	0.323
$\frac{2125}{2144}$	0	0	0	$\frac{19}{2144}$	$\frac{1665}{1608}$	$-\frac{975}{1608}$	$\frac{75}{1608}$	$-\frac{200}{1608}$	0	*	1	1.688	0.021975	0.351
0	0	$\frac{224}{251}$	$\frac{27}{251}$	0	$\frac{780}{251}$	$-\frac{1224}{251}$	$\frac{684}{251}$	$-\frac{156}{251}$	0	*	1	3.108	0.036045	0.335
0	$\frac{28}{29}$	0	$\frac{1}{29}$	0	$\frac{180}{87}$	$-\frac{192}{87}$	$\frac{48}{87}$	$-\frac{8}{87}$	0	*	1	2.069	0.02[illegible]299	0.322
$\frac{224}{243}$	0	0	$\frac{19}{243}$	0	$\frac{100}{81}$	$-\frac{88}{81}$	$\frac{36}{81}$	$-\frac{20}{81}$	0	*	1	1.827	0.021411	0.346
0	$\frac{10700}{10539}$	0	$-\frac{225}{10539}$	$\frac{64}{10539}$	$\frac{6940}{3513}$	$-\frac{6800}{3513}$	$\frac{1000}{3513}$	0	0	*	1.042699	1.976	0.030242	0.325
0	$\frac{350}{326}$	$-\frac{25}{326}$	0	$\frac{1}{326}$	$\frac{630}{326}$	$-\frac{600}{326}$	$\frac{75}{326}$	0	0	*	1.153374	1.933	0.032115	0.322
$\frac{535}{504}$	0	0	$-\frac{40}{504}$	$\frac{4}{504}$	$\frac{35}{42}$	$-\frac{5}{42}$	$-\frac{15}{42}$	0	0	*	1.158730	1.548	0.030925	0.357
$\frac{250}{531}$	$\frac{300}{531}$	0	$-\frac{25}{531}$	$\frac{6}{531}$	$\frac{260}{177}$	$-\frac{200}{177}$	0	0	0	*	1.094162	1.469	0.030545	0.339
1	0	0	0	0	$\frac{24}{24}$	$-\frac{12}{24}$	$-\frac{2}{24}$	$-\frac{1}{24}$	*	*	1	1.417	0.026389	0.375
0	1	0	0	0	$\frac{6}{3}$	$-\frac{6}{3}$	$\frac{1}{3}$	0	*	*	1	2.000	0.041667	0.333
$\frac{225}{324}$	$\frac{100}{324}$	0	0	$-\frac{1}{324}$	$\frac{35}{27}$	$-\frac{25}{27}$	0	0	*	*	1.006173	1.296	0.040621	0.370
$\frac{64}{99}$	$\frac{36}{99}$	0	$-\frac{1}{99}$	0	$\frac{44}{33}$	$-\frac{32}{33}$	0	0	*	*	1.020202	1.333	0.036420	0.364
$\frac{9}{17}$	$\frac{9}{17}$	$-\frac{1}{17}$	0	0	$\frac{24}{17}$	$-\frac{18}{17}$	0	0	*	*	1.117647	1.412	0.040033	0.353

Table 5. *Extrapolation formulas for the equation of the second order.* (i)

$$y_{r+1}=\sum_{s=0}^{N} l_s y_{r-s}+h\left(y'_r+\sum_s s l_s y'_{r-s}\right)+h^2\sum_{\rho=0}^{p} a_{2,\rho}\Gamma^{\rho} f_r .$$

$p=5$

l_0	l_1	l_2	l_3	l_4	l_5	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$\Sigma\|l_s\|$	$1+\Sigma s\|l_s\|$	$\Sigma\|\alpha_{2,\sigma}\|$	A_2
1	0	0	0	0	0	$\frac{5040}{10080}$	$\frac{1680}{10080}$	$\frac{1260}{10080}$	$\frac{1064}{10080}$	$\frac{945}{10080}$	$\frac{863}{10080}$	1	1	6.417	0.079572
$\frac{14387}{15250}$	0	0	0	0	$\frac{863}{15250}$	$\frac{2121120}{1756800}$	$-\frac{384960}{1756800}$	$\frac{591540}{1756800}$	$-\frac{378436}{1756800}$	$\frac{150235}{1756800}$	0	1	1.282951	1.761	0.085324
0	$\frac{14387}{15168}$	0	0	0	$\frac{781}{15168}$	$\frac{552150}{341280}$	$-\frac{783210}{341280}$	$\frac{1035930}{341280}$	$-\frac{671950}{341280}$	$\frac{264609}{341280}$	0	1	2.205960	5.824	0.090538
0	0	$\frac{14387}{15282}$	0	0	$\frac{895}{15282}$	$\frac{7616400}{2445120}$	$-\frac{11697600}{2445120}$	$\frac{10044420}{2445120}$	$-\frac{5357652}{2445120}$	$\frac{2181543}{2445120}$	0	1	3.175697	6.724	0.097402
0	0	0	$\frac{14387}{15088}$	0	$\frac{701}{15088}$	$\frac{3647160}{678960}$	$-\frac{7027950}{678960}$	$\frac{5533845}{678960}$	$-\frac{1819171}{678960}$	$\frac{451669}{678960}$	0	1	4.092922	6.517	0.103852
0	0	0	0	$\frac{14387}{15762}$	$\frac{1375}{15762}$	$\frac{201836880}{22697280}$	$-\frac{520685760}{22697280}$	$\frac{558243300}{22697280}$	$-\frac{268186420}{22697280}$	$\frac{54549999}{22697280}$	0	1	5.087235	9.716	0.121611

Table 5. *Extrapolation formulas for the equation of the second order.* (ii)

$p=4$

l_0	l_1	l_2	l_3	l_4	l_5	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$\Sigma\|l_s\|$	$1+\Sigma s\|l_s\|$	$\Sigma\|\alpha_{2,\sigma}\|$	A_2
1	0	0	0	0	0	$\frac{720}{1440}$	$\frac{240}{1440}$	$\frac{180}{1440}$	$\frac{152}{1440}$	$\frac{135}{1440}$	1	1	3.331	0.085615
0	$\frac{1927}{1809}$	0	0	$-\frac{118}{1809}$	0	$\frac{12320}{24120}$	$\frac{29020}{24120}$	$-\frac{31620}{24120}$	$\frac{16591}{24120}$	0	1.130459	2.326147	3.176	0.101166
0	0	$\frac{1927}{1776}$	0	$-\frac{151}{1776}$	0	$\frac{79515}{39960}$	$-\frac{36480}{39960}$	$-\frac{38580}{39960}$	$\frac{37553}{39960}$	0	1.170045	3.510135	3.869	0.115687
0	0	0	$\frac{1927}{1873}$	$-\frac{54}{1873}$	0	$\frac{3303360}{674280}$	$-\frac{5716380}{674280}$	$\frac{3571740}{674280}$	$-\frac{425491}{674280}$	0	1.057662	4.201815	5.346	0.128464
0	$\frac{9755}{9696}$	0	0	0	$-\frac{59}{9696}$	$\frac{101115}{109080}$	$\frac{9255}{109080}$	$-\frac{28965}{109080}$	$\frac{35884}{109080}$	0	1.012170	2.036510	2.666	0.103957
0	0	$\frac{19510}{19359}$	0	0	$-\frac{151}{19359}$	$\frac{624160}{258120}$	$-\frac{566780}{258120}$	$\frac{90340}{258120}$	$\frac{119201}{258120}$	0	1.015600	3.054600	3.342	0.118231
0	0	0	$\frac{9755}{9728}$	0	$-\frac{27}{9728}$	$\frac{2179080}{437760}$	$-\frac{3827190}{437760}$	$\frac{2454345}{437760}$	$-\frac{333277}{437760}$	0	1.005551	4.022204	5.351	0.129046
$\frac{1927}{1792}$	0	0	0	$-\frac{135}{1792}$	0	$-\frac{1035}{10080}$	$\frac{17880}{10080}$	$-\frac{14940}{10080}$	$\frac{8084}{10080}$	0	1.150670	1.301339	3.932	0.093567
$\frac{3902}{3875}$	0	0	0	0	$-\frac{27}{3875}$	$\frac{4608}{11160}$	$\frac{5100}{11160}$	$-\frac{3060}{11160}$	$\frac{4283}{11160}$	0	1.013935	1.034839	3.300	0.097246
0	$\frac{425491}{834840}$	0	$\frac{447957}{834840}$	$-\frac{38608}{834840}$	0	$\frac{779036}{278280}$	$-\frac{1070222}{278280}$	$\frac{594249}{278290}$	0	0	1.092492	3.304386	3.650	0.115403
0	0	$\frac{425491}{976160}$	$\frac{600848}{976160}$	$-\frac{50179}{976160}$	0	$\frac{5462169}{1464240}$	$-\frac{7963644}{1464240}$	$\frac{4072448}{1464240}$	0	0	1.102809	3.923949	3.978	0.123331
0	$\frac{333277}{474400}$	0	$\frac{143536}{474400}$	0	$-\frac{2413}{474400}$	$\frac{1529382}{711600}$	$-\frac{1834962}{711600}$	$\frac{1071838}{711600}$	0	0	1.010173	2.635645	3.017	0.111527
0	0	$\frac{5332432}{8500680}$	$\frac{3218427}{8500680}$	0	$-\frac{50179}{8500680}$	$\frac{9590296}{2833560}$	$-\frac{13226030}{2833560}$	$\frac{6615451}{2833560}$	0	0	1.011806	3.419929	3.388	0.122314
0	$\frac{36560501}{38409120}$	0	0	$\frac{2296576}{38409120}$	$-\frac{447957}{38409120}$	$\frac{4188163}{3200760}$	$-\frac{3009505}{3200760}$	$\frac{2217267}{3200760}$	0	0	1.023326	2.249354	2.199	0.119138
0	0	$\frac{36560501}{39178080}$	0	$\frac{3218427}{39178080}$	$-\frac{600848}{39178080}$	$\frac{18491143}{6529680}$	$-\frac{22431572}{6529680}$	$\frac{10584520}{6529680}$	0	0	1.030673	3.271652	2.832	0.138032
$\frac{425491}{898560}$	0	0	$\frac{517376}{898560}$	$-\frac{44307}{898560}$	0	$\frac{33653}{12480}$	$-\frac{49464}{12480}$	$\frac{28852}{12480}$	0	0	1.098618	2.924586	4.017	0.113097
$\frac{5332432}{7965000}$	0	0	$\frac{2676875}{7965000}$	0	$-\frac{44307}{7965000}$	$\frac{68776}{35400}$	$-\frac{92970}{35400}$	$\frac{60065}{35400}$	0	0	1.011125	2.036053	3.477	0.107904
$\frac{36560501}{38720000}$	0	0	0	$\frac{2676875}{38720000}$	$-\frac{517379}{38720000}$	$\frac{514617}{580800}$	$-\frac{436440}{580800}$	$\frac{484580}{580800}$	0	0	1.026724	1.343347	2.720	0.115217

Table 5. *Extrapolation formulas for the equation of the second order.* (iii)

$p=3$

l_0	l_1	l_2	l_3	l_4	l_5	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$\Sigma\|l_s\|$	$1+\Sigma s\|l_s\|$	$\Sigma\|\alpha_{2,\sigma}\|$	A_2
1	0	0	0	0	0	$\frac{180}{360}$	$\frac{60}{360}$	$\frac{45}{360}$	$\frac{38}{360}$	1	1	2.178	0.093750
0	$\frac{313}{344}$	0	$\frac{31}{344}$	0	0	$\frac{1404}{1032}$	$-\frac{978}{1032}$	$\frac{613}{1032}$	0	1	2.180233	1.841	0.113691
0	0	$\frac{313}{367}$	$\frac{54}{367}$	0	0	$\frac{25260}{8808}$	$-\frac{30228}{8808}$	$\frac{13399}{8808}$	0	1	3.147139	2.868	0.138384
0	$\frac{3290}{3321}$	0	0	$\frac{31}{3321}$	0	$\frac{9476}{8856}$	$-\frac{3212}{8856}$	$\frac{2505}{8856}$	0	1	2.028004	1.476	0.118540
0	0	$\frac{1645}{1672}$	0	$\frac{27}{1672}$	0	$\frac{13026}{5016}$	$-\frac{14052}{5016}$	$\frac{5645}{5016}$	0	1	3.032297	2.597	0.150564
0	$\frac{14337}{14368}$	0	0	0	$\frac{31}{14368}$	$\frac{11055}{10776}$	$-\frac{2757}{10776}$	$\frac{2231}{10776}$	0	1	2.008630	1.342	0.134902
0	0	$\frac{531}{533}$	0	0	$\frac{2}{533}$	$\frac{32484}{12792}$	$-\frac{33852}{12792}$	$\frac{12845}{12792}$	0	1	3.011257	2.539	0.179423
0	0	0	$\frac{14337}{14024}$	0	$-\frac{313}{14024}$	$\frac{202848}{42072}$	$-\frac{340962}{42072}$	$\frac{193399}{42072}$	0	1.044638	4.178551	7.000	0.501943
$\frac{313}{351}$	0	0	$\frac{38}{351}$	0	0	$\frac{308}{312}$	$-\frac{252}{312}$	$\frac{229}{312}$	0	1	1.324786	2.308	0.104802
$\frac{1645}{1664}$	0	0	0	$\frac{19}{1664}$	0	$\frac{369}{624}$	$-\frac{48}{624}$	$\frac{230}{624}$	0	1	1.045673	1.912	0.109767
$\frac{14337}{14375}$	0	0	0	0	$\frac{38}{14375}$	$\frac{7356}{13800}$	$\frac{780}{13800}$	$\frac{3815}{13800}$	0	1	1.013217	1.752	0.129736
0	$\frac{292825}{275184}$	0	$-\frac{22545}{275184}$	$\frac{4904}{275184}$	0	$\frac{36964}{45864}$	$\frac{7757}{45864}$	0	0	1.163854	2.381170	1.144	0.139674
0	$\frac{193399}{185088}$	0	$-\frac{8924}{185088}$	0	$\frac{613}{185088}$	$\frac{39187}{46272}$	$\frac{5289}{46272}$	0	0	1.096430	2.206107	1.075	0.156094
0	$\frac{2849935}{2801088}$	0	0	$-\frac{71392}{2801088}$	$\frac{22545}{2801088}$	$\frac{634047}{700272}$	$\frac{25131}{700272}$	0	0	1.050974	2.159631	0.977	0.251082
$\frac{1547192}{1458000}$	0	0	$-\frac{95375}{1458000}$	0	$\frac{6183}{1458000}$	$\frac{838}{3240}$	$\frac{1875}{3240}$	0	0	1.130830	1.217449	1.416	0.158158
$\frac{569987}{556800}$	0	0	0	$-\frac{19075}{556800}$	$\frac{5888}{556800}$	$\frac{997}{2784}$	$\frac{1272}{2784}$	0	0	1.068517	1.189907	1.272	0.285906
0	$\frac{226751}{239136}$	$\frac{28208}{239136}$	$-\frac{16786}{239136}$	0	$\frac{963}{239136}$	$\frac{56465}{59784}$	0	0	0	1.140389	2.414843	0.944	0.171026
0	$\frac{294350}{296928}$	$\frac{8377}{296928}$	0	$-\frac{8353}{296928}$	$\frac{2594}{296928}$	$\frac{46279}{49488}$	0	0	0	1.056532	2.204487	0.935	0.265010
0	$\frac{762961}{759264}$	0	$\frac{16754}{759264}$	$-\frac{28208}{759264}$	$\frac{7757}{759264}$	$\frac{176951}{189816}$	0	0	0	1.074304	2.270757	0.931	0.299060
$\frac{5887}{7680}$	0	$\frac{2120}{7680}$	0	$-\frac{455}{7680}$	$\frac{128}{7680}$	$\frac{151}{192}$	0	0	0	1.118490	1.872396	0.786	0.413329

Table 5. *Extrapolation formulas for the equation of the second order.* (iv)

$p=2$

l_0	l_1	l_2	l_3	l_4	l_5	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$\Sigma\|l_s\|$	$1+\Sigma s\|l_s\|$	$\Sigma\|\alpha_{2,\sigma}\|$	A_2
1	0	0	0	0	0	$\frac{12}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	1	1	1.292	0.105556
0	$\frac{69}{68}$	0	$-\frac{1}{68}$	0	0	$\frac{48}{51}$	$-\frac{2}{51}$	0	1.029412	2.058824	0.941	0.141503
0	$\frac{515}{513}$	0	0	$-\frac{2}{513}$	0	$\frac{332}{342}$	$-\frac{29}{342}$	0	1.007797	2.019493	0.971	0.161615
0	0	$\frac{515}{496}$	0	$-\frac{19}{496}$	0	$\frac{1689}{744}$	$-\frac{1328}{744}$	0	1.076613	3.229839	2.270	0.551098
0	$\frac{689}{688}$	0	0	0	$-\frac{1}{688}$	$\frac{507}{516}$	$-\frac{55}{516}$	0	1.002907	2.008721	0.983	0.183253
0	0	$\frac{1378}{1359}$	0	0	$-\frac{19}{1359}$	$\frac{2132}{906}$	$-\frac{1771}{906}$	0	1.027962	3.097866	2.353	0.750104
$\frac{46}{45}$	0	0	$-\frac{1}{45}$	0	0	$\frac{12}{30}$	$\frac{11}{30}$	0	1.044444	1.066667	1.133	0.130062
$\frac{515}{512}$	0	0	0	$-\frac{3}{512}$	0	$\frac{87}{192}$	$\frac{56}{192}$	0	1.011719	1.023438	1.036	0.160471
$\frac{1378}{1375}$	0	0	0	0	$-\frac{3}{1375}$	$\frac{156}{330}$	$\frac{85}{330}$	0	1.004364	1.010909	0.988	0.192956
0	$\frac{1063}{1020}$	$-\frac{32}{1020}$	$-\frac{11}{1020}$	0	0	$\frac{232}{255}$	0	0	1.084314	2.137255	0.910	0.140327
0	$\frac{5312}{5040}$	$-\frac{261}{5040}$	0	$-\frac{11}{5040}$	0	$\frac{761}{840}$	0	0	1.107937	2.166270	0.906	0.150820
0	$\frac{9769}{9540}$	0	$-\frac{261}{9540}$	$\frac{32}{9540}$	0	$\frac{728}{795}$	0	0	1.054717	2.119497	0.916	0.187074
0	$\frac{1449}{1368}$	$-\frac{80}{1368}$	0	0	$-\frac{1}{1368}$	$\frac{103}{114}$	0	0	1.118421	2.179825	0.904	0.160307
0	$\frac{2417}{2364}$	0	$-\frac{55}{2364}$	0	$\frac{2}{2364}$	$\frac{542}{591}$	0	0	1.046531	2.096447	0.917	0.184983
0	$\frac{46771}{46152}$	0	0	$-\frac{880}{46152}$	$\frac{261}{46152}$	$\frac{3557}{3846}$	0	0	1.038135	2.117958	0.925	0.530504
$\frac{32}{324}$	$\frac{297}{324}$	0	$-\frac{5}{324}$	0	0	$\frac{8}{9}$	0	0	1 030864	1.962963	0.889	0.140398
$\frac{1063}{1296}$	0	$\frac{297}{1296}$	$-\frac{64}{1296}$	0	0	$\frac{53}{72}$	0	0	1.098765	1.606481	0.736	0.179111
$\frac{261}{1152}$	$\frac{896}{1152}$	0	0	$-\frac{5}{1152}$	0	$\frac{41}{48}$	0	0	1.008681	1.795139	0.854	0.161357
$\frac{83}{96}$	0	$\frac{14}{96}$	0	$-\frac{1}{96}$	0	$\frac{17}{24}$	0	0	1.020833	1.333333	0.708	0.215336
$\frac{9769}{10368}$	0	0	$\frac{896}{10368}$	$-\frac{297}{10368}$	0	$\frac{95}{144}$	0	0	1.057292	1.373843	0.660	0.469332
$\frac{176}{600}$	$\frac{425}{600}$	0	0	0	$-\frac{1}{600}$	$\frac{5}{6}$	0	0	1.003333	1.716667	0.833	0.186092
$\frac{15939}{18000}$	0	$\frac{2125}{18000}$	0	0	$-\frac{64}{18000}$	$\frac{83}{120}$	0	0	1.007111	1.253889	0.692	0.257823
$\frac{38672}{40500}$	0	0	$\frac{2125}{40500}$	0	$-\frac{297}{40500}$	$\frac{29}{45}$	0	0	1.014667	1.194074	0.644	0.457178
$\frac{46771}{48000}$	0	0	0	$\frac{2125}{48000}$	$-\frac{896}{48000}$	$\frac{149}{240}$	0	0	1.037333	1.270417	0.621	1.268227

Table 6. *Improving formulas for the equation of the second order.* (i) $y_{r+1}=\sum_{s=1}^{N} l_s y_{r+1-s}+h\sum_{s=1}^{N} s l_s y'_{r+1-s}+h^2\sum_{\rho=0}^{p} b_{2,\rho}\nabla^\rho f_{r+1}$.

$p=5$

l_1	l_2	l_3	l_4	l_5	$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$\Sigma' \lvert l_s \rvert$	$\Sigma s \lvert l_s \rvert$	$\Sigma \lvert \beta_{2,\sigma} \rvert$	B_2	$\lvert \beta_{2,0} \rvert$
1	0	0	0	0	$\frac{5040.}{10080}$	$-\frac{3360}{10080}$	$-\frac{420}{10080}$	$-\frac{196}{10080}$	$-\frac{119}{10080}$	$-\frac{82}{10080}$	1	1	1.109	0.006043	0.086
$\frac{16}{57}$	$\frac{41}{57}$	0	0	0	$\frac{4050}{2565}$	$-\frac{5160}{2565}$	$\frac{1200}{2565}$	$\frac{68}{2565}$	$\frac{12}{2565}$	0	1	1.719298	1.669	0.010771	0.066
$\frac{256}{297}$	0	0	$\frac{41}{297}$	0	$\frac{152}{99}$	$-\frac{320}{99}$	$\frac{288}{99}$	$-\frac{128}{99}$	$\frac{16}{99}$	0	1	1.414141	1.535	0.010226	0.081
$\frac{7625}{7584}$	0	0	0	$-\frac{41}{7584}$	$\frac{29700}{68256}$	$-\frac{7500}{68256}$	$-\frac{24000}{68256}$	$\frac{13400}{68256}$	$-\frac{5775}{98256}$	0	1.010812	1.032437	1.145	0.006626	0.085
0	$\frac{7625}{7641}$	0	0	$\frac{16}{7641}$	$\frac{5150}{2547}$	$-\frac{7000}{2547}$	$\frac{2000}{2547}$	$-\frac{100}{2547}$	$\frac{100}{2547}$	0	1	2.006282	6.773	0.012802	0.059
0	0	$\frac{256}{337}$	$\frac{81}{337}$	0	$\frac{9000}{1685}$	$-\frac{20160}{1685}$	$\frac{15840}{1685}$	$-\frac{4992}{1685}$	$\frac{432}{1685}$	0	1	3.240356	5.341	0.024316	0.071
0	0	$\frac{7625}{7544}$	0	$-\frac{81}{7544}$	$\frac{133200}{30176}$	$-\frac{261000}{30176}$	$\frac{153000}{30176}$	$-\frac{16800}{30176}$	$-\frac{6075}{30176}$	0	1.021474	3.085896	4.817	0.021823	0.077
0	0	0	$\frac{7625}{7881}$	$\frac{256}{7881}$	$\frac{577800}{70929}$	$-\frac{1560000}{70929}$	$\frac{1596000}{70929}$	$-\frac{726400}{70929}$	$\frac{116400}{70929}$	0	1	4.032483	8.146	0 038464	0.054
0	$\frac{81}{97}$	$\frac{16}{97}$	0	0	$\frac{234}{97}$	$-\frac{360}{97}$	$\frac{144}{97}$	$-\frac{12}{97}$	0	0	1	2.164948	2.412	0 013918	0.062
$\frac{64}{243}$	$\frac{180}{243}$	0	$-\frac{1}{243}$	0	$\frac{384}{243}$	$-\frac{480}{243}$	$\frac{96}{243}$	$\frac{16}{243}$	0	0	1.008230	1.761317	1.712	0.011086	0.066
$\frac{4500}{14121}$	$\frac{9625}{14121}$	0	0	$-\frac{4}{14121}$	$\frac{21450}{14121}$	$-\frac{27000}{14121}$	$\frac{6000}{14121}$	$\frac{500}{14121}$	0	0	1.000566	1.683308	1.590	0.010554	0.067
$\frac{38800}{40581}$	0	0	$\frac{1925}{40581}$	$-\frac{144}{40581}$	$\frac{33000}{40581}$	$-\frac{48000}{40581}$	$\frac{31200}{40581}$	$-\frac{12800}{40581}$	0	0	1.007097	1.163599	1.168	0.007863	0.084
0	$\frac{48500}{47439}$	0	$-\frac{1125}{47439}$	$\frac{64}{47439}$	$\frac{88800}{47439}$	$-\frac{108000}{47439}$	$\frac{12000}{47439}$	$\frac{10000}{47439}$	0	0	1.047429	2.146335	2.293	0.013897	0.059
0	0	$\frac{776000}{861941}$	$-\frac{91125}{861941}$	$-\frac{5184}{861941}$	$\frac{4156200}{861941}$	$-\frac{8712000}{861941}$	$\frac{6012000}{861941}$	$-\frac{1392000}{861941}$	0	0	1.012029	3.153835	4.822	0.022919	0.074
$\frac{192}{1117}$	$\frac{864}{1117}$	$\frac{64}{1117}$	$-\frac{3}{1117}$	0	$\frac{2088}{1117}$	$-\frac{2880}{1117}$	$\frac{864}{1117}$	0	0	0	1.005372	1.901522	1.869	0.012070	0.064
$\frac{3375}{13622}$	$\frac{9750}{13622}$	$\frac{500}{13622}$	0	$-\frac{3}{13622}$	$\frac{11700}{6811}$	$-\frac{15750}{6811}$	$\frac{4500}{6811}$	0	0	0	1.000440	1.790486	1.718	0.011302	0.066
$\frac{10000}{26109}$	$\frac{16000}{26109}$	0	$\frac{125}{26109}$	$-\frac{16}{26109}$	$\frac{12600}{8703}$	$-\frac{16000}{8703}$	$\frac{4000}{8703}$	0	0	0	1.001226	1.630855	1.448	0.010282	0.069
0	$\frac{348000}{384817}$	$\frac{40000}{384817}$	$-\frac{3375}{384817}$	$\frac{192}{384817}$	$\frac{851400}{384817}$	$-\frac{1224000}{384817}$	$\frac{396000}{384817}$	0	0	0	1.017541	2.158065	2.212	0.013910	0.061

Table 6. *Improving formulas for the equation of the second order.* (ii)

$p=4$

l_1	l_2	l_3	l_4	l_5	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$b_{2,4}$	$\Sigma\|l_s\|$	$\Sigma s\|l_s\|$	$\Sigma\|\beta_{2,\sigma}\|$	B_2	$\|\beta_{2,0}\|$
1	0	0	0	0	$\frac{720}{1440}$	$-\frac{480}{1440}$	$-\frac{60}{1440}$	$-\frac{28}{1440}$	$-\frac{17}{1440}$	1	1	0.865	0.0081349	0.094
$\frac{16}{33}$	$\frac{17}{33}$	0	0	0	$\frac{378}{297}$	$-\frac{456}{297}$	$\frac{96}{297}$	$\frac{4}{297}$	0	1	1.515152	1.300	0.013961	0.074
$\frac{1792}{1809}$	0	0	$\frac{17}{1809}$	0	$\frac{344}{603}$	$-\frac{320}{603}$	$\frac{96}{603}$	$-\frac{64}{603}$	0	1	1.028192	0.889	0.009050	0.093
$\frac{19375}{19392}$	0	0	0	$\frac{17}{19392}$	$\frac{89100}{174528}$	$-\frac{64500}{174528}$	$\frac{1500}{174528}$	$-\frac{9500}{174528}$	0	1	1.003507	0.820	0.009591	0.095
0	$\frac{112}{111}$	0	$-\frac{1}{111}$	0	$\frac{1944}{999}$	$-\frac{2496}{999}$	$\frac{480}{999}$	$\frac{128}{999}$	0	1.018018	2.054054	2.202	0.020571	0.056
0	$\frac{19375}{19359}$	0	0	$-\frac{16}{19359}$	$\frac{12850}{6453}$	$-\frac{17000}{6453}$	$\frac{4000}{6453}$	$\frac{500}{6453}$	0	1.001653	2.005785	2.146	0.020840	0.054
0	0	$\frac{1792}{1873}$	$\frac{81}{1873}$	0	$\frac{8712}{1873}$	$-\frac{17856}{1873}$	$\frac{11808}{1873}$	$-\frac{2496}{1873}$	0	1	3.043246	4.651	0.041582	0.090
0	0	$\frac{19375}{19456}$	0	$\frac{81}{19456}$	$\frac{88200}{19456}$	$-\frac{177750}{19456}$	$\frac{113625}{19456}$	$-\frac{22125}{19456}$	0	1	3.008326	4.533	0.045479	0.100
$\frac{108}{247}$	$\frac{135}{247}$	$\frac{4}{247}$	0	0	$\frac{342}{247}$	$-\frac{432}{247}$	$\frac{108}{247}$	0	0	1	1.578947	1.385	0.014811	0.073
$\frac{512}{945}$	$\frac{432}{945}$	0	$\frac{1}{945}$	0	$\frac{376}{315}$	$-\frac{448}{315}$	$\frac{96}{315}$	0	0	1	1.460317	1.194	0.013408	0.076
0	$\frac{5616}{6101}$	$\frac{512}{6101}$	$-\frac{27}{6101}$	0	$\frac{13320}{6101}$	$-\frac{19008}{6101}$	$\frac{6048}{6101}$	0	0	1.008851	2.110474	2.183	0.021613	0.059
$\frac{3375}{5751}$	$\frac{2375}{5751}$	0	0	$\frac{1}{5751}$	$\frac{2150}{1917}$	$-\frac{2500}{1917}$	$\frac{500}{1917}$	0	0	1	1.413667	1.122	0.013094	0.078
0	$\frac{7375}{7871}$	$\frac{500}{7871}$	0	$-\frac{4}{7871}$	$\frac{16950}{7871}$	$-\frac{24000}{7871}$	$\frac{7500}{7871}$	0	0	1.001016	2.067082	2.153	0.021525	0.057
$\frac{242000}{240057}$	0	0	$-\frac{2375}{240057}$	$\frac{432}{240057}$	$\frac{35800}{80019}$	$-\frac{16000}{80019}$	$-\frac{12000}{80019}$	0	0	1.019787	1.056666	0.747	0.012249	0.097
0	$\frac{242000}{244863}$	0	$\frac{3375}{244863}$	$-\frac{512}{244863}$	$\frac{168200}{81621}$	$-\frac{232000}{81621}$	$\frac{68000}{81621}$	0	0	1.004182	2.042203	2.061	0.024162	0.051
$\frac{672}{859}$	$\frac{216}{859}$	$-\frac{32}{859}$	$\frac{3}{859}$	0	$\frac{648}{859}$	$-\frac{576}{859}$	0	0	0	1.074505	1.410943	0.752	0.013063	0.084
$\frac{16875}{20884}$	$\frac{4500}{20884}$	$-\frac{500}{20884}$	0	$\frac{9}{20884}$	$\frac{3825}{5221}$	$-\frac{3375}{5221}$	0	0	0	1.047884	1.312967	0.733	0.012409	0.086
$\frac{17000}{19899}$	$\frac{3000}{19899}$	0	$-\frac{125}{19899}$	$\frac{24}{19899}$	$\frac{4600}{6633}$	$-\frac{4000}{6633}$	0	0	0	1.012563	1.186994	0.694	0.012558	0.090
$\frac{17250}{17929}$	0	$\frac{1000}{17929}$	$-\frac{375}{17929}$	$\frac{54}{17929}$	$\frac{10800}{17929}$	$-\frac{9000}{17929}$	0	0	0	1.041832	1.228178	0.602	0.017220	0.100

Table 6. *Improving formulas for the equation of the second order.* (iii)

$p=3$

l_1	l_2	l_3	l_4	l_5	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$\Sigma\|l_s\|$	$\Sigma s\|l_s\|$	$\Sigma\|\beta_{2,\sigma}\|$	B_2	$\|\beta_{2,0}\|$
1	0	0	0	0	$\frac{180}{360}$	$-\frac{120}{360}$	$-\frac{15}{360}$	$-\frac{7}{360}$	1	1	0.700	0.011806	0.106
$\frac{16}{23}$	$\frac{7}{23}$	0	0	0	$\frac{22}{23}$	$-\frac{24}{23}$	$\frac{4}{23}$	0	1	1.304348	0.957	0.018780	0.087
$\frac{351}{344}$	0	$-\frac{7}{344}$	0	0	$\frac{72}{172}$	$-\frac{27}{172}$	$-\frac{27}{172}$	0	1.040698	1.081395	0.732	0.014123	0.105
0	$\frac{351}{367}$	$\frac{16}{367}$	0	0	$\frac{774}{367}$	$-\frac{1080}{367}$	$\frac{324}{367}$	0	1	2.043597	2.109	0.037659	0.049
$\frac{3328}{3321}$	0	0	$-\frac{7}{3321}$	0	$\frac{536}{1107}$	$-\frac{320}{1107}$	$-\frac{96}{1107}$	0	1.004216	1.010539	0.658	0.014787	0.108
0	$\frac{208}{209}$	0	$\frac{1}{209}$	0	$\frac{424}{209}$	$-\frac{576}{209}$	$\frac{160}{209}$	0	1	2.009569	2.029	0.041268	0.038
$\frac{14375}{14368}$	0	0	0	$-\frac{7}{14368}$	$\frac{7100}{14368}$	$-\frac{4500}{14368}$	$-\frac{1000}{14368}$	0	1.000974	1.002923	0.633	0.018444	0.111
0	$\frac{14375}{14391}$	0	0	$\frac{16}{14391}$	$\frac{9650}{4797}$	$-\frac{13000}{4797}$	$-\frac{3500}{4797}$	0	1	2.003335	2.012	0.049819	0.031
0	0	$\frac{14375}{14024}$	0	$-\frac{351}{14024}$	$\frac{60300}{14024}$	$-\frac{114750}{14024}$	$\frac{60750}{14024}$	0	1.050057	3.200228	5.262	0.445356	0.449
$\frac{162}{187}$	$\frac{27}{187}$	$-\frac{2}{187}$	0	0	$\frac{126}{187}$	$-\frac{108}{187}$	0	0	1.021390	1.187166	0.674	0.016332	0.096
$\frac{640}{711}$	$\frac{72}{711}$	0	$-\frac{1}{711}$	0	$\frac{152}{237}$	$-\frac{128}{237}$	0	0	1.002813	1.108298	0.641	0.016116	0.101
$\frac{624}{637}$	0	$\frac{16}{637}$	$-\frac{3}{637}$	0	$\frac{360}{637}$	$-\frac{288}{637}$	0	0	1.009419	1.073783	0.565	0.020735	0.113
$\frac{2625}{2874}$	$\frac{250}{2874}$	0	0	$-\frac{1}{2874}$	$\frac{300}{479}$	$-\frac{250}{479}$	0	0	1.000696	1.089074	0.626	0.018540	0.104
$\frac{30375}{30848}$	0	$\frac{500}{30848}$	0	$-\frac{27}{30848}$	$\frac{4275}{7712}$	$-\frac{3375}{7712}$	0	0	1.001750	1.037668	0.554	0.025194	0.117
$\frac{14500}{14589}$	0	0	$\frac{125}{14589}$	$-\frac{36}{14589}$	$\frac{2600}{4863}$	$-\frac{2000}{4863}$	0	0	1.004935	1.040510	0.535	0.057345	0.123

Table 6. *Improving formulas for the equation of the second order.* (iv)

$p=2$

l_1	l_2	l_3	l_4	l_5	$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$\Sigma\|l_s\|$	$\Sigma s\|l_s\|$	$\Sigma'\|\beta_{2,\sigma}\|$	B_2	$\|\beta_{2,p}\|$
1	0	0	0	0	$\frac{12}{24}$	$-\frac{8}{24}$	$-\frac{1}{24}$	1	1	0.583	0.019444	0.125
$\frac{16}{17}$	$\frac{1}{17}$	0	0	0	$\frac{10}{17}$	$-\frac{8}{17}$	0	1	1.058824	0.588	0.023203	0.118
$\frac{135}{136}$	0	$\frac{1}{136}$	0	0	$\frac{36}{68}$	$-\frac{27}{68}$	0	1	1.014706	0.529	0.027410	0.132
$\frac{512}{513}$	0	0	$\frac{1}{513}$	0	$\frac{88}{171}$	$-\frac{64}{171}$	0	1	1.005848	0.515	0.037676	0.140
0	$\frac{32}{31}$	0	$-\frac{1}{31}$	0	$\frac{56}{31}$	$-\frac{64}{31}$	0	1.064516	2.193548	2.323	0.388351	0.258
$\frac{1375}{1376}$	0	0	0	$\frac{1}{1376}$	$\frac{175}{344}$	$-\frac{125}{344}$	0	1	1.002907	0.509	0.048543	0.145
0	$\frac{1375}{1359}$	0	0	$-\frac{16}{1359}$	$\frac{850}{453}$	$-\frac{1000}{453}$	0	1,023547	2.082414	2.539	0.555936	0.331