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On Extensions of a Metric

By

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It is well known that a bounded continuous real valued function defined on a closed subset of a space can be extended over the whole space, as in Tietze's theorem. The present paper will be concerned with this function as Euclidean distance. Let F be a subset of a space R and ρ a metric for F. A metric for R satisfying

$$\sigma(x,y)=\rho(x,y) \qquad \text{for} \qquad x, y \text{ in } F$$

is called an extension of ρ over R. It is said to be bounded if $\sup_{x,y \in R} \sigma(x,y) < +\infty$. We will prove the following two theorems:

THEOREM 1. *Let R be a perfectly separable regular space, F a subset of R which is isometric with Euclidean n-cube Q^n. Then, there is a bounded extension of the metric over R.*

THEOREM 2. *Let R be a locally compact perfectly separable regular space, F a closed subset of R which is isometric with Euclidean n-space E^n. Then, there is an extension of the metric over R.*

The method employed in this paper is essentially due to Urysohn* and the general plan is to construct a topological mapping of R into the product space of Q^n or E^n by Q^ω, where Q^ω is the Hilbert fundamental parallelotope, under which the metric for F is preserved invariant.

It will be denoted by $\Pi^i(\alpha)$, $\Gamma^i(\alpha)$. $\Sigma^i(\alpha)$, $\Delta^i(\alpha)$ and $X^i(\alpha)$ for a number α, that is, the sets of points of E^n with the i-th coordinate $x^i=, \geqslant, >, \leqslant$ and $<\alpha$ respectively. In the proof of Theorem 1, we shall also make use of the same notations as above for the subsets of Q^n. Furthermore, since F is isometric with Q^n or E^n, it may be assumed that they are all subsets of F.

R is metrizable, so that we may treate it as a metric space with the distance $d(x,y)$ and formulate in terms of metric the topology of R.

THE PROOF OF THEOREM 1. We begin by constructing in R n systems of open sets $\mathfrak{U}^i=\{U^i(m/2^l); m=1,2,\cdots,2^l; l=1,2,\cdots\}$ $(i=1,2,\cdots,n)$ having the following properties:

* P. Urysohn, „*Über die Metrisation der kompakten topologischen Räume*", Math. Ann. vol. 92 (1924), pp. 275~293.

(i) $U^i(m/2^l) \supset X^i(m/2^l)$

(ii) $U^i(m/2^l) \cap \Gamma^i(m/2^l) = \phi$

(iii) $\overline{U^i(m/2^l)} \cap \Gamma^i(m/2^l) = \Pi^i(m/2^l)$

(iv) $\overline{U^i(m/2^l)} \subset U^i((m+1)/2^l)$.

We shall carry out the construction by induction on l. Let $U^i(1) = R - \Gamma^i(1)$, then the conditions (i)~(iii) are satisfied, while the condition (iv) has no meaning as yet. Suppose that this has been done for $l=k-1$, we then proceed to do it for $l=k$. There is an open set V such that $V \supset \overline{U^i(m/2^{k-1})}$, $\bar{V} \cap \Gamma^i((2m+1)/2^k) = \phi$ and $\bar{V} \subset U^i((m+1)/2^{k-1})$, $1 \leqslant m \leqslant 2^{k-1}-1$, because R is normal and F closed. Let $a_j = (2m+1)/2^k - 1/2^{k+j}$ $(j=0,1,2,\cdots)$ and $\Omega^i{}_j = \Delta^i(a_{j+1}) - X^i(a_j)$. For every j, we obtain an open neighborhood W_j of $\Omega^i{}_j$ such that $\bar{W}_j \subset U^i((m+1)/2^{k-1})$, $\bar{W}_j \cap \Gamma^i((2m+1)/2^k) = \phi$ and $W_j \subset$ the $1/2^j$-neighborhood of $\Omega^i{}_j$. Let $V \cup (\bigcup_{j=1}^{\infty} W_j) = U^i((2m+1)/2^k)$. For $U^i(1/2^k)$ we have to neglect to construct V as above. Then the system of open sets $U^i(m/2^k)$ satisfies (i)~(iv).

Now we define $f^i(x) = \inf r$ for $x \in R - U^i(r)$, then it is a continuous function defined on R. It is clear that $f^i(x) = r$ if $x \in \Pi^i(r)$, hence $f^i(x) = x^i$ for $x = (x^1, x^2, \cdots, x^n) \in F$.

Furthermore, let O_j be the $1/2^j$-neighborhood of F, then there is a continuous function h^j defined on R such that

$$h^j(x) = 0 \quad \text{if} \quad x \in F,$$

$$h^j(x) = 1 \quad \text{if} \quad x \notin O_j,$$

and

$$0 \leqslant h^j(x) \leqslant 1 \quad \text{for} \quad x \in R.$$

Here we can avail ourselves of the well known method. Let $\{R_s;\ s=1,2,\cdots\}$ be a countable basis of the space R. Order all pairs R_s, R_t such that $\bar{R}_s \subset R_t$, $R_t \subset R - F$ into a sequence $P_1, P_2, \cdots$. For each such pair $P_j = (R_s, R_t)$, we obtain a continuous function $k^j(x)$ defined on R such that

$$k^j(x) = 0 \quad \text{if} \quad x \in \bar{R}_s,$$

$$k^j(x) = 1 \quad \text{if} \quad x \notin R_t,$$

and

$$0 \leqslant k^j(x) \leqslant 1 \quad \text{for} \quad x \in R.$$

Now we define

$$\sigma(x\ y) = \sqrt{\sum_{i=1}^{n} (f^i(x) - f^i(y))^2 + \sum_{j=1}^{\infty} 2^{-j}\{(h^j(x) - h^j(y))^2 + (k^j(x) - k^j(y))^2\}}$$

for each pair of points $x, y \in R$ and show that this function $\sigma(x, y)$ is a distance function which gives a desired metric.

Obviously σ is a bounded function on $R \times R$. Now if $x = y$, we have $f^i(x) = f^i(y)$

$(i=1,2,\cdots,n)$, $h^j(x)=h^j(y)$ and $k^j(x)=k^j(y)\,(j=1,2,\cdots)$ and thus $\sigma(x,y)=0$. On the other hand we will show that if $x\neq y$, then $\sigma(x,y)>0$. To do this, we have to distinguish three cases: 1) $x,y\notin F$; in this case there exists a pair $P_m=(R_s,R_t)$ such that $x\in R_s, y\in R-R_t$. Whence $k^m(x)=0$, $k^m(y)=1$ and thus $\sigma(x,y)\geqslant\sqrt{1/2^m}>0$. 2) $x\in F$, $y\notin F$; in this case, since $d(y,F)=\xi>0$ and there exists an integer l such that $\xi>1/2^l$, y does not belong to O_l. Therefore $h^l(x)=0$, $h^l(y)=1$ and thus $\sigma(x,y)\geqslant\sqrt{1/2^l}>0$. 3) $x,y\in F$; then there exists a coordinate such that $x^i\neq y^i$. It follows that $\sigma(x,y)\geqslant|f^i(x)-f^i(y)|=|x^i-y^i|>0$.

Now it is obvious that $\sigma(x,y)=\sigma(y,x)$ and $\sigma(x,y)+\sigma(y,z)\geqslant\sigma(x,z)$.

To complete the proof we have to show that σ is topologically equivalent to d, that is, $p\in R$ is a limit point of a point set M if and only if there are points of M distinct from p but arbitrarily near p. Let p be a limit point of M and let $\eta>0$ be arbitrary. Taking N so large that $\sum_{j=N+1}^{\infty}1/2^j<\eta^2/2$. Since f^i, h^j and k^j are continuous functions, we can find a neighborhood U of p throughout which the oscillation of $\sum_{i=1}^{n}(f^i(x)-f^i(y))^2+\sum_{j=1}^{N}2^{-j}\{(h^j(x)-h^j(y))^2+(k^j(x)-k^j(y))^2\}$ is less than $\eta^2/2$. Then if $q\in U\cap M$, we have $\sigma(p,q)<\eta$.

On the other hand suppose that a point p is not a limit point of a set M. If $p\notin F$, then there exists a pair $P_m=(R_s,R_t)$ such that $p\in R_s$ and $R_t\cap(F\cup M)=\phi$. Thus $k^m(p)=0$ and $k^m(q)=1$ for every $q\in M$. This gives $\sigma(p,M)\geqslant\sqrt{1/2^m}>0$.

Furthermore let us show that if $p=(p^1,p^2,\cdots,p^n)\in F$ and $p\notin\bar{M}$, then $\sigma(p,\bar{M})>0$. Let

$$G^i(\alpha^i,\beta^i)=\begin{cases} U^i(\alpha^i)-\overline{U^i(\beta^i)} & \text{if} \quad 0\leqslant\beta^i<p^i<\alpha^i\leqslant 1\\ U^i(\alpha^i) & \text{if} \quad 0=\beta^i=p^i<\alpha^i\leqslant 1\\ R-\overline{U^i(\beta^i)} & \text{if} \quad 0\leqslant\beta^i<p^i=\alpha^i=1,\end{cases}$$

then $O_j\cap\{\bigcap_{i=1}^{n}G^i(\alpha^i,\beta^i)\}$ for j, α^i and β^i forms a complete system of neighborhoods of p. For if not, then there exists a neighborhood of p, $U(p)$, such that

$$U(p)\not\supset O_j\cap\{\bigcap_{i=1}^{n}G^i(\alpha^i,\beta^i)\}\qquad\text{for every } j,\ \alpha^i \text{ and } \beta^i.$$

Since $U(p)\cap F$ is open relative to F, we can select a cube $Q=\{(x^1,x^2,\cdots,x^n);\ \xi^i\leqslant x^i\leqslant\eta^i\}$ where $\xi^i\leqslant p^i\leqslant\eta^i$, which is contained in $U(p)\cap F$. For each integer j, there exists a point a_j such that $a_j\notin U(p)$ and $a_j\in O_j\cap\{\bigcap_{i=1}^{n}G^i(\xi^i,\eta^i)\}$. Since $a_j\in O_j$ and $d(a_j,F)<1/2^j$, F contains a point b_j such that $d(a_j,b_j)<1/2^j$. Because of compactness of F we can suppose them so chosen that the sequence $\{b_j\}$ converges to a point $b\in F$ and $\lim_{j\to\infty}d(a_j,b_j)=0$. Hence $\{a_j\}$ converges to b and $b\notin U(p)$ because $a_j\notin U(p)$. Since $Q\subset U(p)$, $b\notin Q$. On the other hand $b\in\overline{\bigcap_{i=1}^{n}G^i(\xi^i,\eta^i)}\cap F=Q$, this contradiction proves $O_j\cap\{\bigcap_{i=1}^{n}G^i(\alpha^i,\beta^i)\}$ is a complete system of neighborhoods of p. Now if $p \notin \bar{M}$, then there exists an open set $O_j \cap (\bigcap_{i=1}^{n} G^i(\xi^i, \eta^i)) = U$ such that $U \cap \bar{M} = \phi$. For every $q \in \bar{M}$, $q \notin O_j$ or $q \notin G^i(\xi^i, \eta^i)$ for some i. Hence $\sigma(p, \bar{M}) \geqslant \min_{i=1,2,\cdots,n} (\sqrt{1/2^j}, \gamma^i) > 0$, where

$$\gamma^i = \begin{cases} \min(|\xi^i - p^i|, |\eta^i - p^i|) & \text{if} \quad 0 \leqslant \xi^i < p^i < \eta^i \leqslant 1 \\ |\xi^i - p^i| & \text{if} \quad 0 \leqslant \xi^i < p^i = \eta^i = 1 \\ |\eta^i - p^i| & \text{if} \quad 0 = \xi^i = p^i < \eta^i \leqslant 1 . \end{cases}$$

Finally we will show that σ is agreeable with the metric for F. If $x=(x^1, x^2, \cdots, x^n)$ and $y=(y^1, y^2, \cdots, y^n)$ are two elements of F, then $f^i(x)=x^i$, $f^i(y)=y^i$ $(i=1, 2, \cdots, n)$ and $h^j(x)=h^j(y)=k^j(x)=k^j(y)=0$ for all j. Thus $\sigma(x, y)=\sqrt{\sum_{i=1}^{n}(f^i(x)-f^i(y))^2}$ $=\sqrt{\sum_{i=1}^{n}(x^i-y^i)^2}$.

THE PROOF OF THEOREM 2. Since R is normal and F closed, we can construct n systems of open subsets $'\mathfrak{U}^i=\{U^i(m/2^l);\ m=0, -1, -2, \cdots;\ l=0, 1, 2, \cdots\}$ $(i=1, 2, \cdots, n)$:

(i) $U^i(m/2^l) \supset X^i(m/2^l)$

(ii) $U^i(m/2^l) \cap \Gamma^i(m/2^l)=\phi$

(iii) $\overline{U^i(m/2^l)} \cap \Gamma^i(m/2^l)=\Pi^i(m/2^l)$

(iv) $\overline{U^i(m/2^l)} \subset U^i((m+1)/2^l)$

(v) $U^i(m) \subset$ the $1/|m-1|$ -neighborhood of $\Delta^i(m)$.

Similarly we have $'\mathfrak{V}^i=\{V^i(m/2^l);\ m=0, 1, 2, \cdots;\ l=0, 1, 2, \cdots\}$ $(i=1, 2, \cdots, n)$:

(i) $V^i(m/2^l) \supset \Sigma^i(m/2^l)$

(ii) $V^i(m/2^l) \cap \Delta^i(m/2^l)=\phi$

(iii) $\overline{V^i(m/2^l)} \cap \Delta^i(m/2^l)=\Pi^i(m/2^l)$

(iv) $\overline{V^i((m+1)/2^l)} \subset V^i(m/2^l)$

(v) $V^i(m) \subset$ the $1/(m+1)$ -neighborhood of $\Gamma^i(m)$

(vi) $\overline{U^i(0)} \cap \overline{V^i(0)}=\Pi^i(0)$.

Before taking up the construction of $''\mathfrak{U}^i=\{U^i(m/2^l);\ m=1, 2, \cdots;\ l=0, 1, 2, \cdots\}$ and $''\mathfrak{V}^i=\{V^i(m/2^l);\ m=-1, -2, \cdots;\ l=0, 1, 2, \cdots\}$ $(i=1, 2, \cdots, n)$, we shall show here a property of F. Let $K(a^1, a^2, \cdots, a^n)$ be the set of points of which coordinates satisfy the condition: $a^i \leqslant x^i \leqslant a^i+2$ $(i=1, 2, \cdots, n)$. It can easily be verified that there is an open neighborhood[1] $V(\Omega^i(k, k+2))$ of $\Omega^i(k, k+2)=\Delta^i(k+2)-X^i(k)$ $(k=0, \pm 1, \pm 2, \cdots)$ such that

(§) $V(\Omega^i(k, k+2)) \cap V(\Omega^i(j, j+2))=\phi$ if $|j-k| \geqslant 3$.

Since R is locally compact by our assumption and $K(a^1, a^2, \cdots, a^n)$ is compact,

1) This means an open set which contains $\Omega^i(k, k+2)$.

there is a compact neighborhood[2] of $K(a^1, a^2, \cdots, a^n)$, we denote it $W(K(a^1, a^2, \cdots, a^n))$, which is contained in $\bigcap_{i=1}^{n} V(\Omega^i(a^i, a^i+2))$. Let $W^i(a^i)=\bigcup_{\substack{b^j=0,\pm1,\pm2,\cdots(i\neq j)\\ b^i=a^i}} W(K(b^1, b^2, \cdots, b^n))$. Then

$\overline{\bigcap_{i=1}^{n} W^i(a^i)}$ is compact, since

$$\begin{aligned}\bigcap_{i=1}^{n} W^i(a^i) &= \bigcap_{i=1}^{n} \{(\bigcup_{\substack{|a^j-b^j|\geq 3 \text{ for some } j(\neq i)\\ a^i=b^i}} W(K(b^1, b^2, \cdots, b^n))) \cap (\bigcup_{\substack{|a^j-b^j|\leq 2 \text{ for each } j(\neq i)\\ a^i=b^i}} W(K(b^1, b^2, \cdots, b^n)))\}\\ &= \bigcap_{i=1}^{n} (\bigcup_{\substack{|a^j-b^j|\leq 2 \text{ for each } j(\neq i)\\ a^i=b^i}} W(K(b^1, b^2, \cdots, b^n))) \qquad \text{by } (\S)\end{aligned}$$

and $\bigcup_{\substack{|a^j-b^j|\leq 2 \text{ for each } j(\neq i)\\ a^i=b^i}} W(K(b^1, b^2, \cdots b^n))$ is a sum of a finite number of compact neighborhoods. It is easily seen that $\bigcap_{i=1}^{n} W^i(a^i)$ contains $K(a^1, a^2, \cdots, a^n)$.

Now we obtain the following system of open sets $''\mathfrak{U}^i=\{U^i(m/2^l);\ m=1, 2, \cdots;\ l=0, 1, 2, \cdots\}$ having the properties:

(i) $U^i(m/2^l)\supset X^i(m/2^l)$

(ii) $U^i(m/2^l)\cap\Gamma^i(m/2^l)=\phi$

(iii) $\overline{U^i(m/2^l)}\cap\Gamma^i(m/2^l)=\Pi^i(m/2^l)$

(iv) $\overline{U^i(m/2^l)}\subset U^i((m+1)/2^l)$

(v) $U^i(m+1)\subset W^i(m-1)\cup(R-\overline{V^i(m-1)})$

In order to construct this system, we shall rely on complete induction. For $l=0$ and $m\geqslant 2$, since $\Omega^i{}_j=\Delta^i(a_{j+1})-X^i(a_j)$, where $a_j=m-(1/2^{j-1})\ (j=1, 2, \cdots)$, is closed and contained in $W^i(m-2)$, there exists an open neighborhood of $\Omega^i{}_j$ (we denote it by $V(\Omega^i{}_j)$) such that

$$(\#)\quad \begin{aligned}&\overline{V(\Omega^i{}_j)}\subset R-\overline{V^i(m)}\\ \text{and}\quad &V(\Omega^i{}_j)\subset W^i(m+2)\cap\{\text{the } 1/2^j\text{-neighborhood of } \Omega^i{}_j\}.\end{aligned}$$

Let $\{\bigcup_{j=1}^{\infty} V(\Omega^i{}_j)\}\cap(R-\overline{V^i(m-2)})=U^i(m)$. For $l=0$ and $m=1$ there exists an open set V such that $\overline{U^i(0)}\subset V\subset\overline{V}\subset U^i(2)\cap(R-\Gamma^i(1))$. Let $U^i(1)=(\bigcup_{j=1}^{\infty} V(\Omega^i{}_j))\cup V$, where $\{V(\Omega^i{}_j)\}$ satisfies the conditions (#). Then $\{U^i(m)\}$ satisfies (i)~(v). Suppose we have demonstrated (i)~(v) for all values of $l<k$, and let us consider the case $l=k>1$. If we denote

$$U^i(m/2^k)=\begin{cases}(\bigcup_{j=1}^{\infty} V(\Omega^i{}_j))\cup(R-\overline{V^i(m/2^k-2)}) & \text{if } m/2^k\geqslant 2\\ (\bigcup_{j=1}^{\infty} V(\Omega^i{}_j))\cup V & \text{if } m/2^k<2\end{cases}$$

2) That is, an open set of which closure is compact.

where $V(\Omega^i{}_j)$ satisfies the conditions (#) and V is obtained in such a manner as in the case $l=0$, $m=1$. Thus we have the desired sets $U^i(m/2^l)$ for $m=1, 2, \cdots$; $l=0, 1, 2, \cdots$.

Similarly, we have $''\mathcal{V}^i=\{V^i(m/2^l)\,;\ m=-1, -2, \cdots;\ l=0, 1, 2, \cdots\}$:

(i) $V^i(m/2^l) \supset \Sigma^i(m/2^l)$

(ii) $V^i(m/2^l) \cap \Delta^i(m/2^l)=\phi$

(iii) $\overline{V^i(m/2^l)} \cap \Delta^i(m/2^l)=\Pi^i(m/2^l)$

(iv) $\overline{V^i((m+1)/2^l)} \subset V^i(m/2^l)$

(v) $V^i(m-1) \subset W^i(m-1) \cup (R-\overline{U^i(m+1)})$.

Let $\mathcal{U}^i={}'\mathcal{U}^i \cup {}''\mathcal{U}^i$ and $\mathcal{V}^i={}'\mathcal{V}^i \cap {}''\mathcal{V}^i$. These two systems of open sets define 2n continuous functions on R:

$$f^i(x)=\inf r \quad \text{for} \quad x \in R-U^i(r)$$

$$g^i(x)=\sup r \quad \text{for} \quad x \in R-V^i(r).$$

Obviously, for $x=(x^1, x^2, \cdots, x^n) \in F$ we have $f^i(x)=x^i$ and $g^i(x)=x^i$. Furthermore $|f^i(x)|$, $|g^i(x)|<+\infty$ for every $x \in R$. If $x \in F$ and $x=(x^1, x^2, \cdots, x^n)$, then $f^i(x)=x^i$ and $g^i(x)=x^i$. If $x \notin F$, then there exists an integer l such that $0<1/l<\xi$. By the condition (v) of ${}'\mathcal{U}^i$ and ${}'\mathcal{V}^i$, $x \notin V^i(-l-1)$ and $x \notin V^i(l-1)$. Furthermore, since $x \in R-V^i(l-1) \subset R-\overline{V^i(l)}$ and $x \in R-U^i(-l-1) \subset R-\overline{U^i(-l-2)}$, we have $x \in U^i(l+2)$ and $x \in V^i(-l-4)$ (by the condition (v) of ${}''\mathcal{U}^i$ and ${}''\mathcal{V}^i$). Hence $-l-1 \leqslant f^i(x) \leqslant l+2$, $-l-4 \leqslant g^i(x) \leqslant l-1$.

Let

$$\sigma(x, y)=\sqrt{2^{-1}\sum_{i=1}^{n}\{(f^i(x)-f^i(y))^2+(g^i(x)-g^i(y))^2\} + \sum_{j=1}^{\infty}2^{-j}\{(h^j(x)-h^j(y))^2+(k^j(x)-k^j(y))^2\}}$$

for each pair of points $x, y \in R$, where $h^j(x)$ and $k^j(x)$ are defined as in Theorem 1. By using the condition (v) of ${}''\mathcal{U}^i$ and ${}''\mathcal{V}^i$ we will verfy that $O_j \cap \bigcap_{i=1}^{n}((U^i(\xi^i) - \overline{U^i(\eta^i)}) \cap (V^i(\eta^i)-\overline{V^i(\xi^i)}))\}=O_j \cap \{\bigcap_{i=1}^{n}(U^i(\xi^i)-V^i(\eta^i))\}$ for j, ξ^i and η^i ($\xi^i>p^i >\eta^i$) ($i=1, 2, \cdots, n$) is a complete system of neighborhoods of $p=(p^1, p^2, \cdots, p^n) \in F$. Suppose, on the contrary, that it is not a complete system of neighborhoods of $p \in F$. Then there exists a neighborhood of p, $W(p)$, such that

$$(*) \qquad W(p) \not\supset O_j \cap \{\bigcap_{i=1}^{n}(U^i(\xi^i)-V^i(\eta^i))\} \text{ for every } j,\ \xi^i \text{ and } \eta^i.$$

Then we can find an open n-cube $Q=\{(x^1, x^2, \cdots, x^n): \tau^i{}_1<x^i<\tau^i{}_2,\ \tau^i{}_1<p^i<\tau^i{}_2\}$ such that $\bar{Q} \subset W(p) \cap F$. We may suppose $m^i \leqslant \tau^i{}_1<p^i<\tau^i{}_2 \leqslant m^i+2$ where m^i is an integer. It is obviouse that

$$Q \subset \bigcap_{i=1}^{n} (V^i(\tau^i_1) \cap U^i(\tau^i_2)) \subset \bigcap_{i=1}^{n} (V^i(m^i) \cap U^i(m^i+2)) \subset \bigcap_{i=1}^{n} W^i(m^i)$$

(by (v) of $''\mathcal{U}^i$ and $''\mathcal{V}^i$)

and $\overline{\bigcap_{i=1}^{n}(V^i(\tau^i_1) \cap U^i(\tau^i_2))}$ is compact. Since $\bar{Q}$ is compact and $\bar{Q} \cap (R-W(p))=\phi$, we have $d(\bar{Q}, R-W(p))=\eta>0$. For each j, there exists a point q_j which belongs to $\{\bigcap_{i=1}^{n}(V^i(\tau^i_1) \cap U^i(\tau^i_2))\} \cap O_j \cap (R-W(p))$ by (*). Obviously $d(q_j, \dot{Q}) \geqslant \eta$ and $d(q_j, F) < 1/2^j$. Since $\overline{\bigcap_{i=1}^{n}(V^i(\tau^i_1) \cap U^i(\tau^i_2))}$ is compact, we may suppose that $\lim_{j\to\infty} q_j=q$, where $q \in F$, because $d(q, F)=0$ and F is closed. Furthermore $d(q, \bar{Q}) \geqslant \eta$, thus $q \notin \bar{Q}$. On the other hand $q \in \overline{\bigcap_{i=1}^{n}\{ V^i(\tau^i_1) \cap U^i(\tau^i_2)\}} \cap F=\bar{Q}$. This contradiction proves $O_j \cap \{\bigcap_{i=1}^{n}(U^i(\xi^i)-V^i(\eta^i))\}$ is a complete system of neighborhoods of p.

By the same argument as given for the proof of Theorem 1 we can show that this function $\sigma(x, y)$ is a distance function effecting the desired metrization.

REMARK. We may replace the words "Q^n(in Theorem 1)" and "E^n(in Theorem 2)" by "a bounded closed subset of E^n" and "an unbounded closed subset of E^n", respectively.

EXAMPLE. Let R be the (x, y)-plane, F the set of points consisting of $(0, 0)$ and the curve M, $y=x\sin(1/x)$, $0<x\leqslant 1$ in R. The arc F has infinite length for Euclidean distance d, while there is a bounded metric for R topologically equivalent to d which makes finite the length of F.

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