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Iteration of Certain Finite Transformation (Continued).

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Chapter II. Transformation of Type *B*.

§ 6. Condition I.

We consider the transformation of the form as follows:

(6.1) $$T:\ 'x^\nu=\varphi^\nu(x)=x^\nu+a^\nu_{\lambda_1\cdots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}+\cdots,\qquad (N\geqq 3)$$

where at least one of $a^\nu_{\lambda_1\cdots\lambda_N}$'s does not vanish.

Quite similarly as in § 1, we have

Theorem 1′. *For the transformation* (6.1), *we put as follows:*

$$a^\nu_{\lambda_1\cdots\lambda_N}=R^\nu_{\lambda_1\cdots\lambda_N}e^{'\Omega^\nu_{\lambda_1\cdots\lambda_N}},\quad '\Omega^\nu_{\lambda_1\cdots\lambda_N}+\omega_{\lambda_1}+\cdots+\omega_{\lambda_N}-\omega_\nu=\Omega^\nu_{\lambda_1\cdots\lambda_N}.$$

We consider the function as follows:

$$R=R(r,\omega\,;\,\rho,\theta)$$
$$=\sum_{\lambda_1,\cdots,\lambda_N,\nu}R^\nu_{\lambda_1\cdots\lambda_N}\sigma_{\lambda_1}\cdots\sigma_{\lambda_N}\rho_\nu\cos(\Omega^\nu_{\lambda_1\cdots\lambda_N}+\phi_{\lambda_1}+\cdots+\phi_{\lambda_N}-\theta_\nu),$$

where

$\sigma_\lambda=r_\lambda$ *or* ρ_λ,

$\phi_\lambda=0$ *or* θ_λ *according as* $\sigma_\lambda=r_\lambda$ *or* $\sigma_\lambda=\rho_\lambda$,

$r_\lambda,\rho_\lambda\geqq 0$ *and* $\sum_\lambda r_\lambda^2=\sum_\lambda\rho_\lambda^2=1$.

We assume that there exists a set of (r_ν,ω_ν) *such that* $R<0$ *for all* (ρ_ν,θ_ν) *except for* $(\rho_\nu=r_\nu,\ \theta_\nu=\pi)$, *and that* $\left[\dfrac{d^P}{d\varepsilon^P}R(r,\omega\,;\,r_\nu+\varepsilon\eta_\nu,\ \pi+\varepsilon\varepsilon^\nu)\right]_{\varepsilon=0}\neq 0$ *for any* $(\eta_\nu,\varepsilon_\nu)$ *such that* $|\eta_\nu|,|\varepsilon_\nu|\leqq 1$ *except for* $\eta_\nu=\varepsilon_\nu=0$, *where* $P=N$ *or* $N+1$ *according as N is even or odd. Then, in the space* E_{2n} *of the complex numbers* x^ν*'s, there exists a small hypersphere passing through the origin with the center* $\alpha^\nu=rr_\nu e^{i\omega_\nu}$ *and with the radius r, such that all the points of that hypersphere converge to the origin remaining in it when T is infinitely iterated on these points.*

When $\left[\frac{d^P R}{d\varepsilon^P}\right]_{\varepsilon=0}=0$, the similar result as in the remark after Theorem 1 is concluded.

In this paragraph, as Julia did in the case of one variable,[1] instead of the hypersphere in E_{2n} of x^ν's, we consider the hypersphere in the space $\tilde{E}_{2n}$ of $\tilde{x}^\nu$'s such that $\tilde{x}^\nu=(x^\nu)^{N-1}$. Put

$$(6.2)\qquad \begin{cases} \tilde{\alpha}^\nu = r\, r_\nu\, e^{i\omega_\nu}, & \text{where } r_\nu \geqq 0,\ \sum_\nu r_\nu^2=1; \\ \tilde{x}^\nu-\tilde{\alpha}^\nu = r\,\rho_\nu\, e^{i\theta_\nu}, & \text{where } \rho_\nu \geqq 0,\ \sum_\nu \rho_\nu^2=1. \end{cases}$$

Then it follows that

$$(6.3)\qquad \tilde{x}^\nu=(x^\nu)^{N-1}=r\,X_\nu\, e^{i\tau_\nu},$$

where

$$(6.4)\qquad X_\nu \cos\tau_\nu = r_\nu \cos\omega_\nu+\rho_\nu\cos\theta_\nu,\quad X_\nu\sin\tau_\nu=r_\nu\sin\omega_\nu+\rho_\nu\sin\theta_\nu .$$

From (6.1), we have the transformation $\tilde{T}$ as follows:

$$(6.5)\qquad '\tilde{x}^\nu=\tilde{x}^\nu+M\,a^\nu_{\lambda_1\cdots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}(x^\nu)^{M-1}+M\,a^\nu_{\lambda_1\cdots\lambda_{N+1}}x^{\lambda_1}\cdots x^{\lambda_{N+1}}(x^\nu)^{M-1}+\cdots,$$

where $M=N-1$. Then, in the space $\tilde{E}_{2n}$, the distance d from the center $\tilde{\alpha}^\nu$ to the transform $'\tilde{x}^\nu$ of $\tilde{x}^\nu$ is calculated as follows:

$$(6.6)\qquad d^2=\sum_\nu('\tilde{x}^\nu-\tilde{\alpha}^\nu)(\overline{'\tilde{x}^\nu}-\bar{\tilde{\alpha}}^\nu)=\sum_\nu(\tilde{x}^\nu-\tilde{\alpha}^\nu)(\bar{\tilde{x}}^\nu-\bar{\tilde{\alpha}}^\nu)$$
$$+M\{\sum_\nu a^\nu_{\lambda_1\cdots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}(x^\nu)^{M-1}(\bar{\tilde{x}}^\nu-\bar{\tilde{\alpha}}^\nu)+\sum_\nu \bar{a}^\nu_{\lambda_1\cdots\lambda_N}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_N}(\bar{x}^\nu)^{M-1}(\tilde{x}^\nu-\tilde{\alpha}^\nu)\}$$
$$+M\{\sum_\nu a^\nu_{\lambda_1\cdots\lambda_{N+1}}x^{\lambda_1}\cdots x^{\lambda_{N+1}}(x^\nu)^{M-1}(\bar{\tilde{x}}^\nu-\bar{\tilde{\alpha}}^\nu)$$
$$+\sum_\nu \bar{a}^\nu_{\lambda_1\cdots\lambda_{N+1}}\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_{N+1}}(\bar{x}^\nu)^{M-1}(\tilde{x}^\nu-\tilde{\alpha}^\nu)\}$$

....

$$=r^2+2M\,r^3R+[r]_{3+\frac{1}{M}},$$

where R is a real part of $\frac{1}{r^3}\sum_\nu a^\nu_{\lambda_1\cdots\lambda_N}x^{\lambda_1}\cdots x^{\lambda_N}(x^\nu)^{M-1}(\bar{\tilde{x}}^\nu-\bar{\tilde{\alpha}}^\nu)$. Substituting (6.2) and (6.3) into R, we have:

$$(6.7)\qquad R=\sum_{\lambda_1,\cdots,\lambda_N,\nu} R^\nu_{\lambda_1\cdots\lambda_N}(X_{\lambda_1}\cdots X_{\lambda_N}X_\nu^{M-1})^{1/M}\rho_\nu\times$$
$$\times\cos\left['\Omega^\nu_{\lambda_1\cdots\lambda_N}+\frac{1}{M}\left\{\tau_{\lambda_1}+\cdots+\tau_{\lambda_N}+(M-1)\tau_\nu\right\}-\theta_\nu\right].$$

1) G. Julia, Jour. Math. Pures Appl. (1918).

It seems almost impossible to elliminate τ in simple form in the general case; therefore we use reluctantly this formula itself as the formula of condition. For the origin ($\rho^\nu = r^\nu$, $\theta_\nu = \omega_\nu + \pi$), since $X_\nu = 0$, it is evident that $R=0$. As in § 1, we assume that, except for the origin, $R<0$ for all ρ_ν and θ_ν.

Next, as in § 1, we consider the point $P(\tilde{x}^\nu)$ on the hypersurface of the hypersphere which satisfies the relations as follows:

$$\rho_\nu = r_\nu + \varepsilon\,\eta_\nu\,, \qquad \theta_\nu = \omega_\nu + \pi + \varepsilon\,\varepsilon_\nu\,, \tag{6.8}$$

where $|\eta_\nu|$, $|\varepsilon_\nu| \leqq 1$ and $|\varepsilon| < \delta$ for an arbitrary small number δ. For (ρ_ν, θ_ν) of (6.8), (1.13) is valid, consequently

$$\tan \tau_\nu \doteqdot \frac{\eta_\nu \sin \omega_\nu + \varepsilon_\nu r_\nu \cos \omega_\nu}{\eta_\nu \cos \omega_\nu - \varepsilon_\nu r_\nu \sin \omega_\nu}\,, \qquad X_\nu \doteqdot |\varepsilon| \sqrt{\eta_\nu^2 + \varepsilon_\nu^2 r_\nu^2}\,. \tag{6.9}$$

Therefore

$$R = \varepsilon^2 H_2 + \varepsilon^3 H_3 + \varepsilon^4 H_4 + \cdots\,. \tag{6.10}$$

When $H_2 \neq 0$, by our assumption that $R<0$ for all ρ_ν and θ_ν except for the origin, it must be $H_2 < 0$ for all $(\eta_\nu, \varepsilon_\nu)$. Now, by means of (6.3) and (6.9), all the terms of $[r]_{3+\frac{1}{M}}$ in (6.6) are of the order at least $2+\frac{1}{M}$ with regard to ε. Thus (6.6) is written as follows:

$$d^2 \doteqdot r^2 + r^3 \varepsilon^2 \left\{ 2M H_2 + [r\,\varepsilon]_{\frac{1}{M}} \right\}.$$

Then, for any sufficiently small ε, it follows that $d<r$ except for $\varepsilon=0$. Thus, corresponding to Theorem 1, we have

Theorem 4. *For the transformation*

$$T: \quad {}'x^\nu = \varphi^\nu(x) = x^\nu + a^\nu_{\lambda_1 \cdots \lambda_N} x^{\lambda_1} \cdots x^{\lambda_N} + \cdots\,, \quad (N \geqq 3)\,,$$

we put

$$\begin{cases} a^\nu_{\lambda_1\cdots\lambda_N} = R^\nu_{\lambda_1\cdots\lambda_N} e^{i'\Omega^\nu_{\lambda_1\cdots\lambda_N}}\,, \\ (x^\nu)^{N-1} = r\,X_\nu\, e^{i\tau_\nu}\,, \end{cases}$$

where

$$X_\nu \cos \tau_\nu = r_\nu \cos \omega_\nu + \rho_\nu \cos \theta_\nu\,, \qquad X_\nu \sin \tau_\nu = r_\nu \sin \omega_\nu + \rho_\nu \sin \theta_\nu\,,$$

and $r_\nu, \rho_\nu \geqq 0$, $\sum_\nu r_\nu^2 = \sum_\nu \rho_\nu^2 = 1$. *For the following function*

$$R = R(r, \omega; \rho, \theta) = \sum_{\lambda,\nu} R^\nu_{\lambda_1\cdots\lambda_N} (X_{\lambda_1} \cdots X_{\lambda_N} X_\nu^{N-2})^{\frac{1}{N-1}} \rho_\nu \times$$

$$\times \cos\left[{}'\Omega^\nu_{\lambda_1\cdots\lambda_N} + \frac{1}{N-1}\left\{ \tau_{\lambda_1} + \cdots + \tau_{\lambda_N} + (N-2)\,\tau_\nu \right\} - \theta_\nu \right],$$

we assume that there exists a set of (r_ν, ω_ν) *such that* $R<0$ *for all* (ρ_ν, θ_ν) *except for* $(\rho_\nu = r_\nu, \theta_\nu = \omega_\nu + \pi)$, *and that* $\left[\frac{d^2}{d\varepsilon^2} R(r, \omega;\, r_\nu + \varepsilon\eta_\nu, \omega_\nu + \pi + \varepsilon\varepsilon_\nu)\right]_{\varepsilon=0} \neq 0$ *for any* $(\eta_\nu, \varepsilon_\nu)$ *such that* $|\eta_\nu|, |\varepsilon_\nu| \leq 1$ *except for* $\eta_\nu = \varepsilon_\nu = 0$. *Then, in the space* E_{2n} *of the complex numbers* x^ν*'s, there exists a small domain D passing through the origin such that all the points of that domain converge to the origin remaining in it when T is infinitely iterated on these points. The boundary of D is determined by the equation as follows:*

$$\sum_\nu \{(x^\nu)^{N-1} - \alpha^\nu)\}\{(\bar{x}^\nu)^{N-1} - \bar{\alpha}^\nu\} = r^2,$$

where $\alpha^\nu = r r_\nu e^{i\omega_\nu}$. *The domain D is composed of* $(N-1)^n$ *sub-domains, the boundary of which is determined by each branch of the roots.*

When $H_2=0$, since $R<0$ for any sufficiently small ε, it must be $H_3=0$. When $H_4 \neq 0$, by our assumption, it must be $H_4<0$. In (6.6),

$$\begin{aligned}[r]_{3+\frac{1}{M}} &= r^{3+\frac{1}{M}} \varepsilon^{2+\frac{1}{M}} (K_{10} + K_{11}\varepsilon + \cdots\cdots) \\ &\quad + r^{3+\frac{2}{M}} \varepsilon^{2+\frac{2}{M}} (K_{20} + K_{21}\varepsilon + \cdots\cdots) \\ &\quad + \cdots\cdots,\end{aligned}$$

consequently (6.6) becomes

$$d^2 = r^2 + r^3 \varepsilon^{2+\frac{1}{M}} \left[2MH_4 \varepsilon^{2-\frac{1}{M}} + r^{\frac{1}{M}} (K_{10} + K_{11}\varepsilon + \cdots)\right].$$

Therefore, when $[r]_{3+\frac{1}{M}}$ does not contain the terms of the order less than 4 with regard to ε, as in § 1, it follows that $d<r$ except for $\varepsilon \neq 0$. When $[r]_{3+\frac{1}{M}}$ contains the terms of the order less than 4, the constant relation between d and r does not hold in general. When $H_4=0$, similar reasonings are continually applied.

Example.

$$T: \begin{cases} 'x^1 = x^1 - c(x^1)^3 - \frac{1}{2} c^2 (x^1)^5 + a^1_{\lambda_1 \cdots \lambda_7} x^{\lambda_1} \cdots x^{\lambda_7} + \cdots, \\ 'x^\omega = x^\omega - 2c(x^1)^2 x^\omega - 2c^2 (x^1)^4 x^\omega + a^\omega_{\lambda_1 \cdots \lambda_7} x^{\lambda_1} \cdots x^{\lambda_7} + \cdots, \quad (\omega \neq 1) \end{cases}$$

where $c>0$.

$M=N-1=2$. Put $\tilde{x}^\nu = (x^\nu)^2$. Then we have:

$$\tilde{T}:\begin{cases} '\tilde{x}^1=\tilde{x}^1-2c\,(\tilde{x}^1)^2+\left(2a^1_{\lambda_1\cdots\lambda_7}\,x^{\lambda_1}\cdots x^{\lambda_7}\,x^1+c^3(x^1)^8\right)+\cdots\,, \\ '\tilde{x}^\omega=\tilde{x}^\omega-4c\tilde{x}^1\,\tilde{x}^\omega+\left(2a^\omega_{\lambda_1\cdots\lambda_7}\,x^{\lambda_1}\cdots x^{\lambda_7}\,x^\omega+8c^3(x^1)^6\,(x^\omega)^2\right)+\cdots\,. \end{cases}$$

$\tilde{T}$ is of the form of Ex. 2 in § 1. Therefore, adopting r_ν and ω_ν such that $\omega_\nu=0$ and $r_1=1$, $r_2=\cdots=r_n=0$, it follows that

$$R=-c\{2+\rho_1\,(3-\rho_1^2)\cos\theta_1\}\,,$$

consequently it follows that $R<0$ except for the origin.

Next, for the neighborhood of the origin, namely for ρ_ν and θ_ν given by (6.8)[1], by Ex. 2 in § 1, we have:

$$\left[\frac{d^2 R}{d\,\varepsilon^2}\right]_{\varepsilon=0}=-2c\,\varepsilon_1^2\,.$$

Therefore, for $\varepsilon_1\neq 0$, $\left[\frac{d^2 R}{d\,\varepsilon^2}\right]_{\varepsilon=0}\neq 0$. For $\varepsilon_1=0$, R becomes as follows:

$$R=-c\,\varepsilon^4\,\eta_1^2\,(3+\varepsilon^2\,\eta_1)\,,$$

therefore, for $\eta_1\neq 0$, $H_4\neq 0$. Now, from the form of $\tilde{T}$, it is readily seen that, $[r]_{3+\frac{1}{M}}$ in (6.6) does not contain the terms of the order less than 4 with regard to ε. Thus, by the remark after Theorem 4, it follows that

$$d<r \qquad \text{except for} \qquad (\varepsilon_1=0,\ \eta_1=0)\,.$$

Now, when $\varepsilon_1=0$ and $\eta_1=0$, $\theta_1=\pi$ and $\rho_1=1$, namely the point is the origin.

Thus we see that, *in E_{2n} of x^ν's, there exists a domain D with the properties of Theorem 4, the boundary of which is determined by the equation as follows:*

$$|\,(x^1)^2-r\,|^2+\sum_\omega|\,x^\omega\,|^4=r^2\,.$$

§ 7. Condition II.

For the transformation (6.1), putting $\tilde{x}^\nu=(x^\nu)^{N-1}$, we seek for a cylindrical domain of Theorem 3 in the space $\tilde{E}_{2n}$ of $\tilde{x}^\nu$'s. For this purpose, as in § 4, we put as follows:

$$(7.1)\quad\begin{cases} \tilde{\alpha}^\nu=rr_\nu\,e^{i\omega_\nu}\,, \quad \text{where} \quad 0\leqq r_\nu\leqq 1\,; \\ \tilde{x}^\nu=2rr_\nu\cos\theta_\nu\,e^{i(\omega_\nu+\theta_\nu)},\ \tilde{x}^\nu-\tilde{\alpha}^\nu=rr_\nu\,e^{i(\omega_\nu+2\theta_\nu)},\ \text{where}\ -\frac{\pi}{2}\leqq\theta_\nu\leqq\frac{\pi}{2}\,; \\ a^\nu_{\lambda_1\cdots\lambda_N}=R^\nu_{\lambda_1\cdots\lambda_N}\,e^{i\Omega^\nu_{\lambda_1\cdots\lambda_N}}\,. \end{cases}$$

1) However, as in Ex. 2 in § 1, for ρ_1 we put $\rho_1=1+\varepsilon^2\eta_1$.

In $\tilde{x}^\nu$-plane, we consider the distance d_ν from the center $\tilde{\alpha}^\nu = re^{i\omega_\nu}$ of $\overset{\circ}{C}_\nu$ to the transform $'\tilde{x}^\nu$ of $\tilde{x}^\nu$ on $\overset{\circ}{C}_\nu$. Then, by means of (6.5), we have:

$$
\begin{aligned}
(7.2)\qquad d_\nu^2 &= ('\tilde{x}^\nu - \tilde{\alpha}^\nu)('\bar{\tilde{x}}^\nu - \bar{\tilde{\alpha}}^\nu) \\
&= (\tilde{x}^\nu - \tilde{\alpha}^\nu)(\bar{\tilde{x}}^\nu - \bar{\tilde{\alpha}}^\nu) + M\{a^\nu_{\lambda_1\cdots\lambda_N} x^{\lambda_1}\cdots x^{\lambda_N}(x^\nu)^{M-1}(\bar{\tilde{x}}^\nu - \bar{\tilde{\alpha}}^\nu) \\
&\qquad + \bar{a}^\nu_{\lambda_1\cdots\lambda_N} \bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_N}(\bar{x}^\nu)^{M-1}(\tilde{x}^\nu - \tilde{\alpha}^\nu)\} \\
&\quad + M\{a^\nu_{\lambda_1\cdots\lambda_{N+1}} x^{\lambda_1}\cdots x^{\lambda_{N+1}}(x^\nu)^{M-1}(\bar{\tilde{x}}^\nu - \bar{\tilde{\alpha}}^\nu) \\
&\qquad + \bar{a}^\nu_{\lambda_1\cdots\lambda_{N+1}} \bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_{N+1}}(\bar{x}^\nu)^{M-1}(\tilde{x}^\nu - \tilde{\alpha}^\nu)\} \\
&\quad + \cdots\cdots \\
&= r^2 + 8\,M r^3 R_\nu + [r]_{3+\frac{1}{M}},
\end{aligned}
$$

where $M = N-1$ and R_ν is the real part of $\dfrac{1}{4r^3} a^\nu_{\lambda_1\cdots\lambda_N} x^{\lambda_1}\cdots x^{\lambda_N}(x^\nu)^{M-1}(\bar{\tilde{x}}^\nu - \bar{\tilde{\alpha}}^\nu)$.

Then, in order that there exists a cylindrical domain of Theorem 3, it must be $R_\nu \leqq 0$. R_ν is calculated as follows:

$$
\begin{aligned}
R_\nu = \sum_{\lambda_1,\cdots,\lambda_N} R^\nu_{\lambda_1\cdots\lambda_N}(r_{\lambda_1} r_{\lambda_2}\cdots r_{\lambda_N})^{\frac{1}{M}} (\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_N}\cos^{M-1}\theta_\nu)^{\frac{1}{M}} \times \\
\times \cos\Big[{}'\Omega^\nu_{\lambda_1\cdots\lambda_N} + \frac{1}{M}\Big\{\omega_{\lambda_1} + \cdots + \omega_{\lambda_N} + (M-1)\,\omega_\nu\Big\} - \omega_\nu \\
+ \frac{1}{M}\Big\{\theta_{\lambda_1} + \cdots + \theta_{\lambda_N} + (M-1)\,\theta_\nu\Big\} - 2\theta_\nu\Big].
\end{aligned}
$$

Put

$$
(7.3)\qquad {}'\Omega^\nu_{\lambda_1\cdots\lambda_N} + \frac{1}{M}(\omega_{\lambda_1} + \cdots + \omega_{\lambda_N} - \omega_\nu) = \Omega^\nu_{\lambda_1\cdots\lambda_N},
$$

then R_ν is written as follows:

$$
\begin{aligned}
(7.4)\qquad R_\nu = \sum_{\lambda_1,\cdots,\lambda_N} R^\nu_{\lambda_1\cdots\lambda_N}(r_{\lambda_1}\cdots r_{\lambda_N})^{\frac{1}{M}} (\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_N}\cos^{M-1}\theta_\nu)^{\frac{1}{M}} \times \\
\times \cos\Big\{\Omega^\nu_{\lambda_1\cdots\lambda_N} + \frac{1}{M}(\theta_{\lambda_1} + \cdots + \theta_{\lambda_N} - N\,\theta_\nu)\Big\} = (\cos\theta_\nu)^{1-\frac{1}{M}} R'_\nu,
\end{aligned}
$$

where

$$
\begin{aligned}
(7.5)\qquad R'_\nu &= (\cos\theta_\nu)^{1+\frac{1}{M}} R^\nu_{\nu\cdots\nu}\cos\Omega^\nu_{\nu\cdots\nu} \\
&\quad + N\cos\theta_\nu \sum_{\lambda\neq\nu} R^\nu_{\nu\cdots\nu\lambda}\, r_\lambda^{1/M}(\cos\theta_\lambda)^{1/M}\cos\Big\{\Omega^\nu_{\nu\cdots\nu\lambda} + \frac{1}{M}(\theta_\lambda - \theta_\nu)\Big\} \\
&\quad + \cdots\cdots
\end{aligned}
$$

$$+\binom{N}{p}(\cos\theta_\nu)^{\frac{N-p}{M}}\sum_{\lambda_1,\cdots,\lambda_p\neq\nu} R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}\,(r_{\lambda_1}\cdots r_{\lambda_p})^{1/M}\,(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_p})^{1/M}\times$$

$$\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_p}-p\,\theta_\nu)\right\}$$

$$+\cdots\cdots$$

$$+\sum_{\lambda_1,\cdots,\lambda_N\neq\nu} R^\nu_{\lambda_1\cdots\lambda_N}\,(r_{\lambda_1}\cdots r_{\lambda_N})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_N})^{1/M}\times$$

$$\times\cos\left\{\Omega^\nu_{\lambda_1\cdots\lambda_N}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_N}-N\theta_\nu)\right\}.$$

We consider the general term

$$S_p=R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}(r_{\lambda_1}\cdots r_{\lambda_p})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_p})^{1/M}\times$$

$$\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_p}-p\,\theta_\nu)\right\}.$$

Put $\theta_{\lambda_1}=\cdots=\theta_{\lambda_p}=\theta$, then S_p becomes as follows:

$$S_p=R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}\,(r_{\lambda_1}\cdots r_{\lambda_p})^{1/M}(\cos\theta)^{p/M}\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{p}{M}\theta-\frac{p}{M}\theta_\nu\right\}.$$

We consider the case where $2p>M$. Then, by the lemma in § 4, we see that, when $R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}\neq 0$, for any θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$, there exists a set of $(\theta_{\lambda_1},\cdots,\theta_{\lambda_p})$ such that $\cos\theta_{\lambda_1},\cdots,\cos\theta_{\lambda_p}\neq 0$ and $S_p\geqq\eta>0$ for a suitable small constant η.

Now, for θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$, from $R'_\nu\leqq 0$, it follows that $\sum S_\nu\leqq 0$. We take arbitrarily at most N of r_λ's and put other r_λ's zero, then, as in § 4, from the preceding results on S_p, we see that $R^\nu_{\lambda_1\cdots\lambda_N}=0$ for $\lambda_1\neq\nu,\cdots,\lambda_N\neq\nu$. Then R_ν can be written as follows:

$$R_\nu=\cos\theta_\nu\cdot R''_\nu,$$

where R''_ν is of the analogous form as R'_ν. Then, by the analogous reasonings, we see that $R^\nu_{\nu\lambda_1\cdots\lambda_{N-1}}=0$ for $\lambda_1\neq\nu,\cdots,\lambda_{N-1}\neq\nu$. Repeating this process, ultimately we see that $R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}=0$ $(\lambda_1\neq\nu,\cdots,\lambda_p\neq\nu)$ for any $p>M/2$. Thus we have:

$$R_\nu=(\cos\theta_\nu)^{2-\frac{P}{M}}\mathring{R}_\nu, \tag{7.6}$$

where

$$\mathring{R}_\nu=(\cos\theta_\nu)^{P/M}R^\nu_{\nu\nu\cdots\nu}\cos\Omega^\nu_{\nu\nu\cdots\nu}+\cdots$$

$$+\binom{N}{P}\sum_{\lambda_1,\cdots,\lambda_P\neq\nu} R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_P}(r_{\lambda_1}\cdots r_{\lambda_P})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_P})^{1/M}\times$$

$$\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_P}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_P}-P\theta_\nu)\right\},$$

where $P=[M/2]$[1].

Now, for θ_ν such that $\cos\theta_\nu\neq 0$, from our condition, it must be $\mathring{R}_\nu\leqq 0$. Put $r_\lambda=t^M r'_\lambda$, where $0\leqq r'_\lambda\leqq 1$, then, for fixed θ_ν, taking t sufficiently small, we see that $R^{(p)}_\nu\leqq 0$ for every $p\leqq P$, where

$$R^{(p)}_\nu=(\cos\theta_\nu)^{P/M}R^\nu_{\nu\cdots\nu}\cos\Omega^\nu_{\nu\cdots\nu}$$

$$+tN(\cos\theta_\nu)^{\frac{P-1}{M}}\sum_{\lambda\neq\nu}R^\nu_{\nu\cdots\nu\lambda}r'^{1/M}_\lambda(\cos\theta_\lambda)^{1/M}\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda}+\frac{1}{M}(\theta_\lambda-\theta_\nu)\right\}$$

$$+\cdots\cdots$$

$$+t^p\binom{N}{p}(\cos\theta_\nu)^{\frac{P-p}{M}}\sum_{\lambda_1,\cdots,\lambda_p\neq\nu}R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}(r'_{\lambda_1}\cdots r'_{\lambda_p})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_p})^{1/M}\times$$

$$\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_p}-p\,\theta_\nu)\right\}.$$

In $R^{(p)}_\nu$, we take arbitrarily at most p of r'_λ's and put other r'_λ's zero. Taking θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$, from $R^{(p)}_\nu\leqq 0$, for any set $(r_{\sigma_1},\cdots,r_{\sigma_q})$ of r_λ's where $q\leqq p$, it must be that

$$\sum_{\lambda_1\cdots\lambda_p\neq\nu}R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}(r_{\lambda_1}\cdots r_{\lambda_p})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_p})^{1/M}\times$$
$$\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_p}-p\,\theta_\nu)\right\}\leqq 0 \tag{7.7}$$

for $p\leqq P$, where $(r_{\lambda_1},\cdots,r_{\lambda_p})$ are permutations of $(r_{\sigma_1},\cdots,r_{\sigma_p})$ inclusive of multiple elements.

Next we consider the sum $a^\nu_{\lambda_1\cdots\lambda_L}x^{\lambda_1}\cdots x^{\lambda_L}(x^\nu)^{M-1}(\bar{\bar{x}}^\nu-\bar{\bar{\alpha}}^\nu)$, where $L\geqq N$.

Let the real part of this sum be $2^{1-\frac{1}{M}+\frac{L}{M}}r^{2-\frac{1}{M}+\frac{L}{M}}K_L$. Then, putting

$$a^\nu_{\lambda_1\cdots\lambda_L}=R^\nu_{\lambda_1\cdots\lambda_L}e^{i\,'\Omega^\nu_{\lambda_1\cdots\lambda_L}}\quad\text{and}\quad '\Omega^\nu_{\lambda_1\cdots\lambda_L}+\frac{1}{M}(\omega_{\lambda_1}+\cdots+\omega_{\lambda_L}-\omega_\nu)=\Omega^\nu_{\lambda_1\cdots\lambda_L},$$

we have:

$$K_L=(\cos\theta_\nu)^{1-\frac{1}{M}}\sum_{\lambda_1,\cdots,\lambda_L}R^\nu_{\lambda_1\cdots\lambda_L}(r_{\lambda_1}\cdots r_{\lambda_L})^{\frac{1}{M}}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_L})^{\frac{1}{M}}\times$$

1) The symbol [ ] is a symbol of Gauss, namely, for positive a, $[a]$ expresses the greatest integer not greater than a.

$$\times \cos\left\{\Omega^{\nu}_{\lambda_1\cdots\lambda_L}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_L}-N\theta_\nu)\right\}=(\cos\theta_\nu)^{1-\frac{1}{M}}K'_L,$$

where

$$K'_L=(\cos\theta_\nu)^{\frac{L}{M}}R^{\nu}_{\nu\cdots\nu}\cos\left(\Omega^{\nu}_{\nu\cdots\nu}+\frac{L-N}{M}\theta_\nu\right) \tag{7.8}$$

$$+L(\cos\theta_\nu)^{\frac{L-1}{M}}\sum_{\lambda\neq\nu}R^{\nu}_{\nu\cdots\nu\lambda}\,r_\lambda^{\frac{1}{M}}(\cos\theta_\lambda)^{\frac{1}{M}}\times$$

$$\times\cos\left\{\Omega^{\nu}_{\nu\cdots\nu\lambda}+\frac{1}{M}\theta_\lambda+\frac{L-N-1}{M}\theta_\nu\right\}$$

$$+\cdots\cdots$$

$$+\binom{L}{p}(\cos\theta_\nu)^{\frac{L-p}{M}}\sum_{\lambda_1,\cdots,\lambda_p\neq\nu}R^{\nu}_{\nu\cdots\nu\lambda_1\cdots\lambda_p}(r_{\lambda_1}\cdots r_{\lambda_p})^{\frac{1}{M}}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_p})^{\frac{1}{M}}\times$$

$$\times\cos\left\{\Omega^{\nu}_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_p})+\frac{L-p-N}{M}\theta_\nu\right\}$$

$$+\cdots\cdots$$

$$+\sum_{\lambda_1,\cdots,\lambda_L\neq\nu}R^{\nu}_{\lambda_1\cdots\lambda_L}(r_{\lambda_1}\cdots r_{\lambda_L})^{\frac{1}{M}}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_L})^{\frac{1}{M}}\times$$

$$\times\cos\left\{\Omega^{\nu}_{\lambda_1\cdots\lambda_L}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_L})-\frac{N}{M}\theta_\nu\right\}.$$

We write the general term of the above expression as $\binom{L}{p}(\cos\theta_\nu)^{\frac{L-p}{M}}S_p$. In S_p, we put $\theta_{\lambda_1}=\cdots=\theta_{\lambda_p}=\theta$, then S_p is written as follows:

$$S_p=R^{\nu}_{\nu\cdots\nu\lambda_1\cdots\lambda_p}(r_{\lambda_1}\cdots r_{\lambda_p})^{\frac{1}{M}}(\cos\theta)^{\frac{p}{M}}\times \tag{7.9}$$

$$\times\cos\left(\Omega^{\nu}_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{p}{M}\theta-\frac{p-(L-N)}{M}\theta_\nu\right).$$

Now we consider the case where $p>L-N+P$. Then

$$\frac{p}{M}+\frac{p-(L-N)}{M}>\frac{p+P}{M}\geqq\frac{2P+1}{M}\geqq 1,$$

and evidently $\frac{p}{M}\geqq\frac{p-(L-N)}{M}>0$. Therefore, by the lemma in § 4, we see that, when $p>L-N+P$ and $R^{\nu}_{\nu\cdots\nu\lambda_1\cdots\lambda_p}\neq 0$, for θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$, there exists a set of $\theta_{\lambda_1},\cdots,\theta_{\lambda_p}$ such that $\cos\theta_{\lambda_1},\cdots,\cos\theta_{\lambda_p}\neq 0$ and $S_p\geqq\eta>0$ for a suitable small constant η.

Now, if we write the transformation (6.1) briefly as follows:

$$'x = x + T_N + \cdots + T_L + \cdots ,$$

where T_L denotes the sum of the terms of L-th order, then, by making M-th power of both sides, we have:

$$\text{(7.10)} \qquad '\tilde{x} = \tilde{x} + M x^{M-1} (T_N + \cdots + T_L + \cdots)$$

$$\cdots\cdots\cdots\cdots$$

$$+ \binom{M}{p} x^{M-p} (T_N + \cdots + T_L + \cdots)^p$$

$$\cdots\cdots\cdots\cdots$$

$$+ (T_N + \cdots + T_L + \cdots)^M ,$$

consequently, if we put $'\tilde{x} = \tilde{x} + \tilde{T}_{N+M-1} + \cdots + \tilde{T}_{L+M-1} + \cdots$, then

$$\text{(7.11)} \qquad \tilde{T}_{L+M-1} = M x^{M-1} T_L + \mathfrak{P}_{L-1} (T_N, \ldots\ldots, T_{L-1}) ,$$

where $\mathfrak{P}_{L-1}$ denotes the polynomial of the arguments with coefficients of the polynomials of x. Then (7.2) is written as follows:

$$\text{(7.12)} \quad d_\nu^2$$

$$= \left\{ (\tilde{x} - \tilde{\alpha}) + \tilde{T}_{N+M-1} + \cdots + \tilde{T}_{L+M-1} + \cdots \right\} \left\{ (\bar{\tilde{x}} - \bar{\tilde{\alpha}}) + \bar{\tilde{T}}_{N+M-1} + \cdots + \bar{\tilde{T}}_{L+M-1} + \cdots \right\}$$

$$= (\tilde{x} - \tilde{\alpha}) (\bar{\tilde{x}} - \bar{\tilde{\alpha}}) + \left\{ \tilde{T}_{N+M-1} (\bar{\tilde{x}} - \bar{\tilde{\alpha}}) + \bar{\tilde{T}}_{N+M-1} (\tilde{x} - \tilde{\alpha}) \right\}$$

$$+ \cdots$$

$$+ \left\{ \tilde{T}_{L+M-1} (\bar{\tilde{x}} - \bar{\tilde{\alpha}}) + \bar{\tilde{T}}_{L+M-1} (\tilde{x} - \tilde{\alpha}) + \mathfrak{P} (\tilde{T}_{L+M-2}, \bar{\tilde{T}}_{L+M-2}, \cdots, \tilde{T}_{N+M-1}, \bar{\tilde{T}}_{N+M-1}) \right\}$$

$$+ \cdots ,$$

consequently the general term is of the form as follows:

$$\text{(7.13)} \quad \left\{ M x^{M-1} T_L (\bar{\tilde{x}} - \bar{\tilde{\alpha}}) + M \bar{x}^{M-1} \bar{T}_L (\tilde{x} - \tilde{\alpha}) \right\} + \left\{ \mathfrak{P}_{L-1} \cdot (\bar{\tilde{x}} - \bar{\tilde{\alpha}}) + \bar{\mathfrak{P}}_{L-1} \cdot (\tilde{x} - \tilde{\alpha}) \right\}$$

$$+ \mathfrak{P} (\tilde{T}_{L+M-2}, \bar{\tilde{T}}_{L+M-2}, \cdots) .$$

We shall prove that $R^\nu_{\nu \cdots \nu \lambda_1 \cdots \lambda_p} = 0$ for $\lambda_1 \neq \nu, \cdots, \lambda_p \neq \nu$ when $p > L - N + P$, namely $a^\nu_{\lambda_1 \cdots \lambda_L} x^{\lambda_1} \cdots x^{\lambda_L}$ contains at least $(N-P)$-th power of x^ν. We have seen already that, for $L = N$, this holds. We assume that this holds for $L = N$, $N+1, \cdots, L'-1$. Then, from (7.10) and (7.11), since

$$M - p + p(N-P) = M + p(M-P) \geqq M + (M-P) = (M-1) + (N-P) ,$$

we see that, for $L \leqq L'$, $\mathfrak{P}_{L-1} (T_N, \cdots, T_{L-1}) = x^{M-1+(N-P)} \mathfrak{P}'(x)$, where $\mathfrak{P}'(x)$ is a polynomial of x^λ. Consequently, it follows that, for $L \leqq L'-1$, $\tilde{T}_{L+M-1}$

$=x^{2M-P}\mathfrak{P}''(x)$, where $\mathfrak{P}''(x)$ is a polynomial of x^λ. Then, for $L\leqq L'-1$, the expression (7.13) contains at least $(2M-P)$-th power of x^ν or $\bar{x}^\nu$ and, for $L=L'$, the same expression becomes that of the form as follows:

$$M\Big\{a^\nu_{\lambda_1\cdots\lambda_L}\,x^{\lambda_1}\cdots x^{\lambda_L}\,(x^\nu)^{M-1}(\tilde{\bar{x}}^\nu-\tilde{\bar{\alpha}}^\nu)+\bar{a}^\nu_{\lambda_1\cdots\lambda_L}\,\bar{x}^{\lambda_1}\cdots\bar{x}^{\lambda_L}\,(\bar{x}^\nu)^{M-1}(\tilde{x}^\nu-\tilde{\alpha}^\nu)\Big\}$$
$$+(x^\nu)^{2M-P}\,\mathfrak{P}'''(x)+(\bar{x}^\nu)^{2M-P}\,\mathfrak{P}^{(IV)}(x),$$

where $\mathfrak{P}'''(x)$ and $\mathfrak{P}^{(IV)}(x)$ are the polynomials of x^λ and $\bar{x}^\lambda$. Then (7.12) is written as follows:

$$\begin{aligned}(7.14)\qquad d^2_\nu=&\,r^2+8Mr^3(\cos\theta_\nu)^{2-\frac{P}{M}}\,\overset{\circ}{R}_\nu\\ &+2^{3+\frac{1}{M}}\,Mr^{2+\frac{N}{M}}(\cos\theta_\nu)^{2-\frac{P}{M}}\,K''_{N+1}\\ &+\cdots\\ &+2^{2+\frac{L'-2}{M}}\,Mr^{2+\frac{L'-2}{M}}(\cos\theta_\nu)^{2-\frac{P}{M}}\,K''_{L'-1}\\ &+2^{2+\frac{L'-1}{M}}\,Mr^{2+\frac{L'-1}{M}}\Big[(\cos\theta_\nu)^{1-\frac{1}{M}}\,K'_{L'}+(\cos\theta_\nu)^{2-\frac{P}{M}}\,K''_{L'}\Big]\\ &+\cdots,\end{aligned}$$

where $K'_{L'}=(\cos\theta_\nu)^{L'/M}\,S_0+L'(\cos\theta_\nu)^{\frac{L'-1}{M}}\sum S_1+\cdots+\binom{L'}{p}(\cos\theta_\nu)^{\frac{L'-p}{M}}\sum S_p+\cdots+\sum S_{L'}$. Then, for θ_ν sufficiently near to $\pi/2$ or to $-\pi/2$, by similar reasonings as in § 4, by means of the properties of $\sum S_p$, we see that, for $L=L'$, it must be $R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}=0$ for $\lambda_1\neq\nu,\cdots,\lambda_p\neq\nu$ when $p>L'-N+P$. Thus, by induction, we know the validity of our assertion.

As in § 6, to the cylindrical domain in $\tilde{E}_{2n}$, there correspond $(N-1)^n$ cylindrical domains in E_{2n}. Thus, summarizing the results, we have:

Theorem 5. *Given the transformation*

$$T:\quad 'x^\nu=\varphi^\nu(x)=x^\nu+a^\nu_{\lambda_1\cdots\lambda_N}\,x^{\lambda_1}\cdots x^{\lambda_N}+\cdots,\ (N\geqq 3),$$

where $a^\nu_{\lambda_1\cdots\lambda_N}$'s do not all vanish. In order that, in the space E_{2n} of x^ν's, there exist $(N-1)^n$ cylindrical domains determined by $|(x^\nu)^{N-1}-\alpha^\nu|=r$ for $\alpha^\nu=re^{i\omega_\nu}$, such that all the points of the boundary of each domain are transformed to the inner points of that domain except for the origin, it is necessary that the transformation T is of the form as follows:

$$(T):\quad 'x^\nu=\varphi^\nu(x)=x^\nu+(x^\nu)^{N-P}(a^\nu_{\lambda_1\cdots\lambda_P}\,x^{\lambda_1}\cdots x^{\lambda_P}+\cdots),$$

where $P=\left[\frac{N-1}{2}\right]$ *and, that there exists a set of* ω_ν *such that, for any set* $(r_{\sigma_1},\cdots,r_{\sigma_q})$, *and for* θ_ν *sufficiently near to* $\pi/2$ *or to* $-\pi/2$, *there hold*

$$(\Omega)\quad \sum_{\lambda_1,\cdots,\lambda_p\neq\nu} R^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}(r_{\lambda_1}\cdots r_{\lambda_p})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_p})^{1/M}\times$$
$$\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda_1\cdots\lambda_p}+\frac{1}{N-1}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_p}-p\,\theta_\nu)\right\}\leqq 0$$

for all r_λ *and* θ_λ *such that* $0\leqq r_\lambda\leqq 1$ *and* $-\pi/2\leqq\theta_\lambda\leqq\pi/2$, *where* $(\lambda_1,\cdots,\lambda_p)$ *are permutations of* $(r_{\sigma_1},\cdots,r_{\sigma_q})$ *inclusive of multiple elements and*

$$a^\nu_{\lambda_1\cdots\lambda_P}=R^\nu_{\lambda_1\cdots\lambda_P}\,e^{i'\Omega^\nu_{\lambda_1\cdots\lambda_P}},\ ^{(1)}$$
$$\Omega^\nu_{\lambda_1\cdots\lambda_P}={}'\Omega^\nu_{\lambda_1\cdots\lambda_P}+\frac{1}{N-1}\left\{\omega_{\lambda_1}+\cdots+\omega_{\lambda_P}+(N-P-1)\,\omega_\nu\right\}.$$

Next we consider the converse of this theorem. We assume that,

(i) *T is of the form* (T) *of the theorem;*

(ii) *there exists a set of* ω_ν *such that the relations* (Ω) *of the theorem are valid for any* r_λ, θ_λ *and* θ_ν *and moreover, for* $p=P$, *the left-hand sides of* (Ω) *do not vanish at the same time for any* r_λ, θ_λ *and* θ_ν.

In the $\tilde{x}^\nu$-plane, we take a point $\tilde{x}^\nu$ arbitrarily on the boundary or in the interior of $\overset{\circ}{C}_\nu$. Then there exists a circle C_ν passing through the point $\tilde{x}^\nu$. For such $\tilde{x}^\nu$, from (7.12), the square of the distance from the center $\tilde{\alpha}^\nu=rr_\nu\,e^{i\omega_\nu}$ to the transform $'\tilde{x}^\nu$ of $\tilde{x}^\nu$ becomes

$$d_\nu^2=r^2\,r_\nu^2+8M\,r^3\,r_\nu^{3-\frac{P}{M}}(\cos\theta_\nu)^{2-\frac{P}{M}}\{R'_\nu+[r]_{1/M}\},\tag{7.15}$$

where

$$\begin{aligned}R'_\nu=&(\cos\theta_\nu)^{P/M}\,r_\nu^{P/M}\,R^\nu_{\nu\cdots\nu}\cos\Omega^\nu_{\nu\cdots\nu}\\&+P(\cos\theta_\nu)^{\frac{P-1}{M}}\,r_\nu^{\frac{P-1}{M}}\sum_{\lambda\neq\nu}R^\nu_{\nu\cdots\nu\lambda}(r_\lambda)^{1/M}(\cos\theta_\lambda)^{\frac{1}{M}}\times\\&\qquad\times\cos\left\{\Omega^\nu_{\nu\cdots\nu\lambda}+\frac{1}{M}(\theta_\lambda-\theta_\nu)\right\}\\&+\cdots\\&+\sum_{\lambda_1,\cdots,\lambda_P\neq\nu}R^\nu_{\lambda_1\cdots\lambda_P}(r_{\lambda_1}\cdots r_{\lambda_P})^{1/M}(\cos\theta_{\lambda_1}\cdots\cos\theta_{\lambda_P})^{1/M}\times\\&\qquad\times\cos\left\{\Omega^\nu_{\lambda_1\cdots\lambda_P}+\frac{1}{M}(\theta_{\lambda_1}+\cdots+\theta_{\lambda_P}-P\,\theta_\nu)\right\}.\end{aligned}\tag{7.16}$$

---

1) In the following, we use this expression for $a^\nu_{\lambda_1\cdots\lambda_P}$.

From the assumption (ii), for any r_λ, θ_λ and θ_ν, it follows that $R''_\nu < 0$. Then there exists sufficiently small r_0 such that, for $0 \leqq r \leqq r_0$, $R'_\nu + [r]_{1,M} < 0$. For such r, from (7.15), $d_\nu < r\, r_\nu$ except for $\theta_\nu = \pi/2$ or $-\pi/2$. Then, by the reasonings in the outset of § 1, we see that all the points of C_ν converge to the origin remaining in it when T is infinitely iterated.[1] Thus we have

Theorem 6. *For the transformation T of the form (T) of Theorem 5, we assume that there exists a set of ω_ν such that the relations (Ω) of Theorem 5 are valid for any r_λ, θ_λ and θ_ν and moreover, for $p=P$, the left-hand sides of (Ω) do not vanish at the same time for any r_λ, θ_λ and θ_ν. Then, in each plane, there exist $N-1$ domains D_ν such that all the points of each D_ν converge to the origin remaining in it when T is infinitely iterated on these points. The boundaries of the domains D_ν are determined by the equations as follows:*

$$|(x^\nu)^{N-1} - \alpha^\nu| = r$$

where $\alpha^\nu = r\, e^{i\omega_\nu}$.

Chapter III. Application to the differential equations.

§ 8. Setting of the problem.

We consider the differential equations as follows:

$$\frac{dx^1}{\xi^1} = \frac{dx^2}{\xi^2} = \cdots\cdots = \frac{dx^n}{\xi^n}, \tag{8.1}$$

where

$$\xi^\nu = c^\nu_\lambda x^\lambda + c^\nu_{\lambda_1\lambda_2} x^{\lambda_1} x^{\lambda_2} + \cdots. \tag{8.2}$$

In this chapter, we consider the case where all the eigen values μ_ν's of $\| c^\nu_\lambda \|$ lie on a straight line passing through the origin. When all the eigen values μ_ν's lie in the same side with regard to the origin, all the integrals of (8.1) are already determined.[2] In this chapter, we consider the general case where some μ_ν's lie in one side and some other μ_ν's lie in the other side. By means of a suitable linear transformation of the variables x^ν's, without loss of generality, we may assume that $\| c^\nu_\lambda \|$ is of Jordan's form. We consider the group

1) Of course, for our purpose, it is sufficient if we assume the weaker condition that $R''_\nu < 0$ for any r_λ, θ_λ and θ_ν. But here, confering the necessary conditions (Ω), as a simple example, we have adopted the assumption (ii).

2) M. H. Dulac, Bull. Soc. Math. France (1912).
M. Urabe, Jour. Sci. Hiroshima Univ. (1951), pp. 39-43.

$\mathfrak{G}$ of transformations having $\xi^\nu(x)$ as the operator functions. Then the finite transformations of $\mathfrak{G}$ are given by the integrals $'x^\nu=\varphi^\nu(x,t)$ of the differential equations as follows:

$$\frac{d\,'x^\nu}{dt}=\xi^\nu('x)\,. \tag{8.3}$$

If we put $X\equiv\xi^\nu\frac{\partial}{\partial x^\nu}$, then, by the initial condition that $\varphi^\nu(x,0)=x^\nu$, we have:

$$'x^\nu=\varphi^\nu(x,t)=e^{tX}(x^\nu)=e^{\mu_\nu t}\,x^\nu+te^{\mu_\nu t}\,x^{\nu-1}+\cdots+a^\nu_{\lambda\mu}(t)\,x^\lambda\,x^\mu+\cdots\,. \tag{8.4}$$

These are parametric forms of the integrals of (8.1), which pass through the point x^ν. Since $\varphi^\nu(x,t)$ is one-valued with regard to t, therefore we can study the properties of the integrals, investigating the mode of $\varphi^\nu(x,t)$ for radial variation of t.

We classify the eigen values μ_ν's as follows:

$$\mu_a=r_a\,e^{i\omega},\quad \mu_l=r_l\,e^{i(\omega+\pi)}\quad\text{and}\quad \mu_z=0\,.\qquad (r_a,r_l>0) \tag{8.5}$$

Put $t=\rho\,e^{-i\theta}$ and vary ρ, θ being fixed. Then we see that,

$$\begin{cases}\text{for } \theta \text{ such that } -\pi/2<\omega-\theta<\pi/2: & |e^{\mu_a t}|>1 \text{ and } |e^{\mu_l t}|<1;\\ \text{for } \theta \text{ such that } \pi/2<\omega-\theta<3\pi/2: & |e^{\mu_a t}|<1 \text{ and } |e^{\mu_l t}|>1;\end{cases} \tag{8.6}$$

and, for any θ, $e^{\mu_z t}=1$.

For the eigen values of (8.5), it is known already[1] that there exist two invariant varieties for $\mathfrak{G}$ defined by the equations as follows:

$$A:\begin{cases}x^l=f^l(x^a)\,,\\ x^z=f^z(x^a)\,;\end{cases}\qquad\qquad L:\begin{cases}x^a=g^a(x^l)\,,\\ x^z=g^z(x^l)\,.\end{cases}$$

Here f and g are sums of the second and higher orders of the arguments and they are the solutions of the equations as follows:

$$\begin{cases}\xi^a(x,f)\dfrac{\partial f^l}{\partial x^a}=\xi^l(x,f)\,,\\[2ex] \xi^a(x,f)\dfrac{\partial f^z}{\partial x^a}=\xi^z(x,f)\,;\end{cases}\qquad\qquad \begin{cases}\xi^l(x,g)\dfrac{\partial g^a}{\partial x^l}=\xi^a(x,g)\,,\\[2ex] \xi^l(x,g)\dfrac{\partial g^z}{\partial x^l}=\xi^z(x,g)\,.\end{cases}$$

Here $\xi(x,f)$ and $\xi(x,g)$ denote the functions obtained by substituting f for

1) M. H. Dulac, Bull. Soc. Math. France (1912).
M. Urabe, Jour. Sci. Hiroshima Univ. (1950), pp. 195-207.

x^l, x^z and g for x^a, x^z, respectively. Then, in each invariant variety, the transformation of $\mathfrak{G}$ is expressed respestively as follows:

in A: $'x^a=\varphi^a(x,t)\equiv e^{\mu_a t}x^a+te^{\mu_a t}x^{a-1}+\cdots+a^a_{bc}(t)\,x^b x^c+\cdots,$

$'x^l=f^l('x^a),\ 'x^z=f^z('x^a);$

in L: $'x^l=\varphi^l(x,t)\equiv e^{\mu_l t}x^l+te^{\mu_l t}x^{l-1}+\cdots+a^l_{pq}(t)\,x^p x^q+\cdots,$

$'x^a=g^a('x^l),\ 'x^z=g^z('x^l).$

For example, consider the transformation: $'x^a=\varphi^a(x,t)$. We take a positive number ρ_0 arbitrarily and put $\rho_0 e^{-i\theta}=t_0$. By suitable linear transformation of the variables x^a, the transformation can be reduced to that of the form as follows:

$$'x^a=\varphi^a(x,t_0)=e^{\mu_a t_0}x^a+\delta\, x^{a-1}+a^a_{bc}(t_0)\,x^b x^c+\cdots,$$

where δ is an arbitrary positive number. We take θ such that $\pi/2<\omega-\theta<3\pi/2$, then $|e^{\mu_a t_0}|=\lambda_a<1$. We take δ so small that $\lambda_a+\delta<1$. If we take r sufficiently small, then, for $|x^a|\leqq r$, there exists a positive number K such that

$$|a^a_{bc}(t_0)\,x^b x^c+\cdots|\leqq K|x|^2$$

where $|x|=\max.|x^a|$. Then it follows that

$$|'x^a|\leqq(\lambda_a+\delta+K|x|)|x|.$$

We take $|x|$ so small that there exists Λ such that

$$\lambda_a+\delta+K|x|<\Lambda<1.$$

Then $|'x^a|<\Lambda|x|$, consequently, when the transformation $'x^a=\varphi^a(x,t_0)$ is infinitely iterated, x^a tend to zero. Returning to the initial variables, we have the same conclusion. When $x^a\to 0$, evidently x^l and x^z tend to zero. Thus we see that, *in the invariant variety A, for θ such that $\pi/2<\omega-\theta<3\pi/2$, all the points sufficiently near to the origin converge to the origin when the transformation is infinitely iterated, namely when t radially removes indefinitely from the origin in the direction determined by θ*[1]. Similarly we see that, *in the invariant variety L, for θ such that $-\pi/2<\omega-\theta<\pi/2$, the same conclusion is valid.*

Saying the above results in other words, we see that, *in the sufficiently small neighborhood of the origin, there exists an integral curve passing through any point in the invariant variety A or L, which approches indefinitely*

1) For, $\varphi^\nu(x,kt_0)=\varphi^\nu[\varphi\{x,(k-1)t_0\},t_0]$.

the origin. The integral curves correspond to radial variation of t for θ such that $\cos(\omega-\theta)\neq 0$.

Thus it becomes a problem to determine the domain of x^ν in the neighborhood of the origin such that all the points of that domain converge to the origin corresponding to radial variation of t for θ such that $\cos(\omega-\theta)=0$. If such domain exists and moreover it is of $2n$ dimensions, then there exist integrals of (8.1) approaching to the origin and moreover containing n arbitrary constants. In this chapter, making use of the results of Chapter I and II, we study this problem.

§ 9. Conditions for the domains.

In this paragraph, we seek for the domains discussed in the preceding chapters such that the integrals of (8.1) passing through any point of these domains approach indefinitely the origin corresponding to the radial variation of t for θ such that $\cos(\omega-\theta)=0$.

We assume that $\cos(\omega-\theta)=0$. Then it follows that $\mu_\nu t=\pm i\rho r_\nu$. In this case, by attaching the sign to r_ν, without loss of generality, we may assume that

$$\mu_\nu e^{-i\theta}=i r_\nu . \tag{9.1}$$

We assume that $r_1, r_2, \cdots, r_n$ *are mutually commensurable.* Then there exists a positive number ρ_0 such that $\rho_0 r_\nu=2\pi p_\nu$ where p_ν are integers. Put $t_0=\rho_0 e^{-i\theta}$, where $\theta\equiv\omega+\frac{\pi}{2}$ (mod π) by our assumption. Then evidently $e^{t_0\mu_\nu}=1$, consequently, for this value of t, the transformation of $\mathfrak{G}$ becomes

$$'x^\nu=\varphi^\nu(x, t_0)=x^\nu+t_0 x^{\nu-1}+\frac{t_0^2}{2!}x^{\nu-2}+\cdots+a^\nu_{\lambda\mu}(t_0)x^\lambda x^\mu+\cdots . \tag{9.2}$$

From the properties of $\varphi^\nu(x, t)$, the radial variation of t corresponds to the iteration of the above transformation. Consequently the problem is to seek for the domains, of which all the points converge to the origin by infinite iteration of the transformation. The transformation of (9.2) is transformed to the transformation of type A_2 by suitable linear transformation of the variables x^ν's, when the linear terms $t_0 x^{\nu-1}$, $\frac{t_0^2}{2!}x^{\nu-2}, \cdots$ exist. By §§ 3 and 5, for the transformation of type A_2, there exists neither a hypersphere nor a cylindrical domain of the desired property. Thus we have

Theorem 7. *When the Jordan's form of* $\| c^{\nu}_{\lambda} \|$ *is not of diagonal form, there exists neither a hypersphere nor a cylindrical domain such that the integrals passing through any point of these domains approach indefinitely the origin by radial removing of t from the origin in the direction of* θ *such that* $\cos(\omega-\theta)$.

Thus, in the following, we consider the transformation of the form as follows:

$$(9.3) \qquad T: \ 'x^{\nu}=x^{\nu}+a^{\nu}_{\lambda\mu}(t_0)x^{\lambda}x^{\mu}+\cdots .$$

In this case, it must be $c^{\nu}_{\lambda}=\mu_{\nu}\,\delta^{\nu}_{\lambda}$ [1] (not summed by ν), namely

$$(9.4) \qquad \xi^{\nu}=\mu_{\nu}\,x^{\nu}+c^{\nu}_{\lambda\mu}\,x^{\lambda}\,x^{\mu}+\cdots . \qquad \text{(not summed by } \nu)$$

Now, in our case, by means of $t=\rho\, e^{-i\theta}$, the differential equations (8.3) are transformed to the equations as follows:

$$(9.5) \qquad \frac{d\,'x^{\nu}}{d\rho}=e^{-i\theta}\,\xi\,('x).$$

In the following, we write $e^{-i\theta}\,\xi^{\nu}(x)$ as $\xi^{\nu}(x)$ for brevity. Then, from (9.1) and (9.4), it follows that

$$(9.6) \qquad \xi^{\nu}(x)=i\,r_{\nu}\,x^{\nu}+c^{\nu}_{\lambda\mu}\,x^{\lambda}\,x^{\mu}+\cdots, \qquad \text{(not summed by } \nu)$$

and the differential equations (9.5) are written as follows:

$$(9.7) \qquad \frac{d\,'x^{\nu}}{d\rho}=\xi^{\nu}('x).$$

Then we have the group $\mathfrak{G}$ of transformations with the real parameter ρ, and the transformations of $\mathfrak{G}$ are written as follows:

$$(9.8) \qquad T(\rho): \ 'x^{\nu}=\varphi^{\nu}(x,\rho)=e^{ir_{\nu}\rho}\,x^{\nu}+a^{\nu}_{\lambda\mu}(\rho)\,x^{\lambda}\,x^{\mu}+\cdots,$$

and moreover $T(\rho_0)$ becomes T of (9.3). Substituting (9.8) into (9.7) and comparing the coefficients of the products of x^{ν}'s, we have:

$$\left\{\begin{aligned}
&\frac{d\,a^{\nu}_{\lambda_1\lambda_2}}{d\rho}=i\,r_{\nu}\,a^{\nu}_{\lambda_1\lambda_2}+c^{\nu}_{\lambda_1\lambda_2}\,e^{i\rho(r_{\lambda_1}+r_{\lambda_2})},\\
&\frac{d\,a^{\nu}_{\lambda_1\lambda_2\lambda_3}}{d\rho}=i\,r_{\nu}\,a^{\nu}_{\lambda_1\lambda_2\lambda_3}+c^{\nu}_{\mu_1\mu_2}\left[e^{i\rho r_{\mu_1}}\,\delta^{\mu_1}_{(\lambda_1}\,a^{\mu_2}_{\lambda_2\lambda_3)}+e^{i\rho r_{\mu_2}}\,a^{\mu_1}_{(\lambda_1\lambda_2}\,\delta^{\mu_2}_{\lambda_3)}\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +c^{\nu}_{\lambda_1\lambda_2\lambda_3}\,e^{i\rho(r_{\lambda_1}+r_{\lambda_2}+r_{\lambda_3})},\\
&\cdots\cdots\cdots\cdots,
\end{aligned}\right.$$

1) δ^{ν}_{λ} is Kronecker's delta.

$$(9.9)\quad \begin{cases} \dfrac{d a^{\nu}_{\lambda_1\cdots\lambda_N}}{d\rho} = i r_\nu a^{\nu}_{\lambda_1\cdots\lambda_N} + c^{\nu}_{\mu_1\mu_2}[e^{i\rho r_{\mu_1}} \delta^{\mu_1}_{(\lambda_1} a^{\mu_2}_{\lambda_2\cdots\lambda_N)} + a^{\mu_1}_{(\lambda_1\lambda_2} a^{\mu_2}_{\lambda_3\cdots\lambda_N)} + \\ \qquad \cdots + e^{i\rho r_{\mu_2}} a^{\mu_1}_{(\lambda_1\cdots\lambda_{N-1}} \delta^{\mu_2}_{\lambda_N)}] \\ \quad + c^{\nu}_{\mu_1\mu_2\mu_3} \sum a^{\mu_1}_{(\lambda_1\cdots} a^{\mu_2}_{\cdots\cdots} a^{\mu_3}_{\cdots\lambda_N)} \\ \quad + \cdots\cdots \\ \quad + c^{\nu}_{\lambda_1\cdots\lambda_N} e^{i\rho(r_{\lambda_1}+\cdots+r_{\lambda_N})}, \text{(1)} \\ \cdots\cdots\cdots\cdots. \qquad\qquad (\text{not summed by } \nu \text{ and } \lambda) \end{cases}$$

By these equations, $a^{\nu}_{\lambda_1\lambda_2}(\rho)$, $a^{\nu}_{\lambda_1\lambda_2\lambda_3}(\rho),\cdots$ are successively determined, making use of the initial conditions that $a^{\nu}_{\lambda_1\lambda_2}(0)=a^{\nu}_{\lambda_1\lambda_2\lambda_3}(0)=\cdots=0$. From the first, we have:

$$a^{\nu}_{\lambda_1\lambda_2}(\rho)=c^{\nu}_{\lambda_1\lambda_2} e^{ir_\nu\rho}\int_0^\rho e^{i\rho(r_{\lambda_1}+r_{\lambda_2}-r_\nu)}\,d\rho\,.$$

Consequently we have:

$$(9.10)\quad \begin{cases} \text{for } r_\nu \neq r_{\lambda_1}+r_{\lambda_2}, & a^{\nu}_{\lambda_1\lambda_2}(\rho)=c^{\nu}_{\lambda_1\lambda_2}\dfrac{e^{i\rho(r_{\lambda_1}+r_{\lambda_2})}-e^{i\rho r_\nu}}{(r_{\lambda_1}+r_{\lambda_2}-r_\nu)\,i}, \\ \text{for } r_\nu = r_{\lambda_1}+r_{\lambda_2}, & a^{\nu}_{\lambda_1\lambda_2}(\rho)=c^{\nu}_{\lambda_1\lambda_2}\,\rho\, e^{ir_\nu\rho}. \end{cases}$$

Therefore it follows that

$$(9.11)\quad \begin{cases} \text{for } r_\nu \neq r_{\lambda_1}+r_{\lambda_2}, & a^{\nu}_{\lambda_1\lambda_2}(\rho_0)=0; \\ \text{for } r_\nu = r_{\lambda_1}+r_{\lambda_2}, & a^{\nu}_{\lambda_1\lambda_2}(\rho_0)=c^{\nu}_{\lambda_1\lambda_2}\,\rho_0\,. \end{cases}$$

Substituting (9.10) into the second of (9.9) and again substituting the obtained results into the third, and so on, we easily get:

(i) when any one of the relations $r_\nu=r_{\lambda_1}+r_{\lambda_2},\cdots,\ r_\nu=r_{\lambda_1}+\cdots+r_{\lambda_N}$ does not hold,

$a^{\nu}_{\lambda_1\lambda_2}(\rho),\cdots,\ a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho)$ are respectively of the forms

$L[e^{i\rho r_\nu},\ e^{i\rho(r_{\lambda_1}+r_{\lambda_2})}],\cdots,L[e^{i\rho r_\nu}, e^{i\rho(r_{\mu_1}+r_{\mu_2})},\cdots, e^{i\rho(r_{\lambda_1}+\cdots+r_{\lambda_N})}]$ (2);

(ii) when any one of the relations $r_\nu=r_{\lambda_1}+r_{\lambda_2},\cdots,r_\nu=r_{\lambda_1}+\cdots+r_{\lambda_{N-1}}$ does not hold and $r_\nu=r_{\lambda_1}+\cdots+r_{\lambda_N}$ holds, $a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho)$ is of the form as follows:

$$a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho)=\kappa^{\nu}_{\lambda_1\cdots\lambda_N}\,\rho\, e^{i\rho r_\nu}+L[e^{i\rho r_\nu},\cdots, e^{i\rho(r_{\mu_1}+\cdots+r_{\mu_{N-1}})}],$$

where $\quad \kappa^{\nu}_{\lambda_1\cdots\lambda_N}=c^{\nu}_{\lambda_1\cdots\lambda_N}+$ polynomial of $[c^{\nu}_{\mu_1\mu_2},\cdots,c^{\nu}_{\mu_1\cdots\mu_{N-1}}]$.

1) The term having indices $\lambda_1, \lambda_2, \cdots, \lambda_N$ enclosed by round brackets expresses the mean of all the terms having indices of permutations of $\lambda_1, \lambda_2, \cdots, \lambda_N$.

2) $L[\cdots]$ denotes a linear combination of the arguments with constant coefficients.

Consequently, when $\kappa^{\nu}_{\lambda_1\cdots\lambda_N}=0$ for any ν and $\lambda_1,\cdots,\lambda_N$, $a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho)$ is of the form $L[e^{ir_\nu\rho},\cdots,e^{i\rho(r_{\lambda_1}+\cdots+r_{\lambda_N})}]$, consequently $a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho_0)=a^{\nu}_{\lambda_1\cdots\lambda_N}(0)=0$, namely $T(\rho_0)$ becomes an identity. In this case, the group $\mathfrak{G}$ is majorized.[1] When $\kappa^{\nu}_{\lambda_1\cdots\lambda_N}\neq 0$, $a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho_0)=\kappa^{\nu}_{\lambda_1\cdots\lambda_N}\rho_0\neq 0$.

In the case (ii), it is easily seen that, whether $\kappa^{\nu}_{\lambda_1\cdots\lambda_N}=0$ or not, for $L>N$, $a^{\nu}_{\lambda_1\cdots\lambda_L}(\rho)$ is of the form as follows:

$$a^{\nu}_{\lambda_1\cdots\lambda_L}(\rho)=L'[e^{i\rho r_\nu},e^{i\rho(r_{\mu_1}+r_{\mu_2})},\cdots,e^{i\rho(r_{\lambda_1}+\cdots+r_{\lambda_L})}],$$

where $L'[\cdots]$ is a linear combination of the arguments with the coefficients which are polynomials of ρ. Consequently, in general $a^{\nu}_{\lambda_1\cdots\lambda_L}(\rho_0)\neq 0$. Now, by our assumption, $r_1,\cdots,r_n$ are all mutually commensurable and moreover some are positive and some are negative, therefore the relations of the form $r_\nu=r_{\lambda_1}+\cdots+r_{\lambda_N}$ arise certainly infinite times. Thus, *the necessary and sufficient condition that $T(\rho_0)$ becomes an identity, namely, $\mathfrak{G}$ is majorized, is that $\kappa^{\nu}_{\lambda_1\cdots\lambda_N}=0$ for ν and $\lambda_1,\cdots,\lambda_N$ such that $r_\nu=r_{\lambda_1}+\cdots+r_{\lambda_N}$.* When $\mathfrak{G}$ is majorized, the integral curves are completely determined.[2] Thus we exclude this case, consequently, for certain ν and $\lambda_1,\cdots,\lambda_N$, $\kappa^{\nu}_{\lambda_1\cdots\lambda_N}\neq 0$, therefore there exists non-vanishing $a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho_0)$. Then, *the discrimination whether the desired domains—specially hyperspheres or cylindrical domains—exist or not, is got by applying the results of the preceding chapters to $a^{\nu}_{\lambda_1\cdots\lambda_N}(\rho_0)$ determined in the above way.*

§ 10. Examples.

In this paragraph, we give two simple examples, where the desired domains exist.

Example 1. *One of the eigen values μ_1 is zero and $c^{\nu}_{\lambda_1\lambda_2}$, $c^{\nu}_{\lambda_1\lambda_2\lambda_3}$ are as follows:*

(i) *$c^{\nu}_{\lambda_1\lambda_2}$'s are all zero except for $c^{\nu}_{1\nu}$;*

(ii) *$c^{\nu}_{\lambda_1\lambda_2\lambda_3}$'s are all zero except for $c^{\nu}_{11\nu}$.*

Since $r_1=0$, it is valid that $r_\nu=r_1+r_\nu$ for any ν. Thus, by (9.10), for $a^{\nu}_{\lambda_1\lambda_2}(\rho)$, we have:

1) M. Urabe, *Application of Majorized Group of Transformations to Functional Equations,* this journal, 16 (1952), pp. 267-283.

2) M. Urabe, do.

(10.1) $a^{\nu}_{1\nu}(\rho)=c^{\nu}_{1\nu}\,\rho\, e^{ir_{\nu}\rho}$ and other $a^{\nu}_{\lambda_1\lambda_2}(\rho)$'s vanish.

The second of (9.9) is written as follows:

$$(10.2)\quad \begin{cases} \text{for } \nu=1: & \dfrac{da^1_{\lambda_1\lambda_2\lambda_3}}{d\rho}=2c^1_{11}\,\delta^1_{(\lambda_1}\,a^1_{\lambda_2\lambda_3)}+c^1_{\lambda_1\lambda_2\lambda_3}\,e^{i\rho(r_{\lambda_1}+r_{\lambda_2}+r_{\lambda_3})}; \\ \text{for } \nu\neq 1: & \dfrac{da^{\nu}_{\lambda_1\lambda_2\lambda_3}}{d\rho}=ir_{\nu}\,a^{\nu}_{\lambda_1\lambda_2\lambda_3}+2c^{\nu}_{1\nu}\,[\delta^1_{(\lambda_1}\,a^{\nu}_{\lambda_2\lambda_3)}+\delta^{\nu}_{(\lambda_1}\,a^1_{\lambda_2\lambda_3)}\,e^{ir_{\nu}\rho}] \\ & \qquad +c^{\nu}_{\lambda_1\lambda_2\lambda_3}\,e^{i\rho(r_{\lambda_1}+r_{\lambda_2}+r_{\lambda_3})}. \end{cases}$$

In the first of (10.2), if at least one of $(\lambda_1, \lambda_2, \lambda_3)$ is not 1, then the right-hand side vanishes, consequently, for such $(\lambda_1, \lambda_2, \lambda_3)$, $a^1_{\lambda_1\lambda_2\lambda_3}(\rho)=0$. If $\lambda_1=\lambda_2=\lambda_3=1$, then it follows that

$$\frac{da^1_{111}}{d\rho}=2\,(c^1_{11})^2\,\rho+c^1_{111},$$

consequently we have

$$a^1_{111}(\rho)=(c^1_{11})^2\,\rho^2+c^1_{111}\,\rho. \tag{10.3}$$

We consider the second of (10.2). When at least one of $(\lambda_1, \lambda_2, \lambda_3)$ is neither 1 nor ν, we have $\dfrac{da^{\nu}_{\lambda_1\lambda_2\lambda_3}}{d\rho}=ir_{\nu}\,a^{\nu}_{\lambda_1\lambda_2\lambda_3}$, consequently $a^{\nu}_{\lambda_1\lambda_2\lambda_3}(\rho)=0$. When the set $(\lambda_1, \lambda_2, \lambda_3)$ is one of $(1,1,1)$, $(1,\nu,\nu)$ and (ν,ν,ν), we have the same equation for $a^{\nu}_{\lambda_1\lambda_2\lambda_3}(\rho)$, consequently $a^{\nu}_{\lambda_1\lambda_2\lambda_3}(\rho)=0$. When $(\lambda_1, \lambda_2, \lambda_3)$ is $(1,1,\nu)$, we have:

$$\frac{da^{\nu}_{11\nu}}{d\rho}=ir_{\nu}\,a^{\nu}_{11\nu}+\frac{2}{3}\,c^{\nu}_{1\nu}\,[2a^{\nu}_{1\nu}+a^1_{11}\,e^{ir_{\nu}\rho}]+c^{\nu}_{11\nu}\,e^{i\rho r_{\nu}}$$

$$=ir_{\nu}\,a^{\nu}_{11\nu}+\frac{2}{3}\,c^{\nu}_{1\nu}\,[2c^{\nu}_{1\nu}+c^1_{11}]\,\rho\,e^{ir_{\nu}\rho}+c^{\nu}_{11\nu}\,e^{i\rho r_{\nu}},$$

consequently it follows that

$$a^{\nu}_{11\nu}(\rho)=\left[\frac{1}{3}\,c^{\nu}_{1\nu}\,(2c^{\nu}_{1\nu}+c^1_{11})\,\rho^2+c^{\nu}_{11\nu}\,\rho\right]e^{ir_{\nu}\rho}. \tag{10.4}$$

We determine c^1_{111} and $c^{\nu}_{11\nu}$ so that

$$(c^1_{11})^2\,\rho_0+c^1_{111}=0,\qquad \frac{1}{3}\,c^{\nu}_{1\nu}\,(2c^{\nu}_{1\nu}+c^1_{11})\,\rho_0+c^{\nu}_{11\nu}=0. \tag{10.5}$$

Then, from (10.3) and (10.4), it follows that

$$a^1_{111}(\rho_0)=a^{\nu}_{11\nu}(\rho_0)=0.$$

Thus, we see that $a^{\nu}_{\lambda_1\lambda_2\lambda_3}(\rho_0)=0$ for any ν and $\lambda_1, \lambda_2, \lambda_3$. Then, when $c^1_{11}=c^2_{12}=\cdots=c^n_{1n}=c$ is real and not zero, we see that, $T(\rho_0)$ of (9.8) becomes the transformation of Example 2 in § 1, for $\rho_0 > 0$ when $c < 0$ and for $\rho_0 < 0$ when $c > 0$. Thus we see that, when $c^{\nu}_{11\nu}$ satisfy (10.5), there exists the desired hypersphere.

Example 2. $\xi^{\nu} = (x^{\nu})^2 [c^{\nu} + c^{\nu}_{\lambda} x^{\lambda} + \cdots]$, *where* $c^{\nu} \neq 0$.

In this case, all the eigen values are zero, and the necessary properties for our purpose are obtained by the following way rather than by means of (9.9). The functions $\varphi^{\nu}(x, \rho)$ of (9.8) are formally determined by $\varphi^{\nu}(x, \rho) = e^{\rho X}(x^{\nu})$ where $X \equiv \xi^{\mu} \dfrac{\partial}{\partial x^{\mu}}$. Then it follows that

$$X(x^{\nu}) = \xi^{\nu} = (x^{\nu})^2 [c^{\nu} + c^{\nu}_{\lambda} x^{\lambda} + \cdots].$$

We assume that $X^p(x^{\nu})$ contains the factor $(x^{\nu})^2$ for positive integer p. Put $X^p(x^{\nu}) = (x^{\nu})^2 \psi^{\nu}(x)$, then

$$\begin{aligned} X^{p+1}(x^{\nu}) &= X[(x^{\nu})^2 \psi^{\nu}(x)] = \xi^{\nu}\, 2x^{\nu} \psi^{\nu} + (x^{\nu})^2 X \psi^{\nu} \\ &= (x^{\nu})^2 [2x^{\nu} (c^{\nu} + c^{\nu}_{\lambda} x^{\lambda} + \cdots) \psi^{\nu} + X \psi^{\nu}]. \end{aligned}$$

Consequently $X^{p+1}(x^{\nu})$ also contains the factor $(x^{\nu})^2$. Thus we see that $\varphi^{\nu}(x, \rho) - x^{\nu}$ contains the factor $(x^{\nu})^2$. From (9.11), we see, that $a^{\nu}_{\nu\nu}(\rho_0) = c^{\nu} \rho_0$ and other $a^{\nu}_{\lambda_1 \lambda_2}(\rho_0)$ vanish. Then $T(\rho_0)$ of (9.8) becomes the transformation of the form in Theorem 3. From $c^{\nu} \neq 0$, $a^{\nu}_{\nu\nu}(\rho_0) \neq 0$. Thus, by Theorem 3, we see that there exists the desired cylindrical domain.

§ 11. Integral curves in the case where the desired domains exist.

In §§ 9 and 10, we have considered the iteration of $T(\rho_0)$ corresponding to the radial variation of t in the direction of t_0. In this paragraph, when there exists the domain considered in the preceding paragraphs—namely the domains, of which all the points converge to the origin remaining in it when $T(\rho_0)$ is infinitely iterated on these points—, we investigate the properties of the integral curves corresponding to the radial variation of t.

The integral curve passing through the point x^{ν} is determined by (9.8). The functions $\varphi^{\nu}(x, \rho)$ of (9.8) are also expressed as

(11.1) $$\varphi^{\nu}(x, \rho) = e^{\rho X}(x^{\nu}), \quad \text{where} \quad X \equiv \xi^{\mu} \frac{\partial}{\partial x^{\mu}}.$$

We take the functions Ξ^{ν} such that $\Xi^{\nu} \gg \xi^{\nu}$, and put $\Xi^{\mu} \dfrac{\partial}{\partial x^{\mu}} \equiv X'$. Then it is evident that, when $F \gg f$, $X' F \gg X f$. Consequently $X'^p(x^\nu) \gg X^p(x^\nu)$. Therefore, for positive ρ, $\rho^p X'^p(x^\nu) \gg \rho^p X^p(x^\nu)$. Put

$$e^{\rho X'}(x^\nu) = \Phi^\nu(x, \rho) = A^\nu_\lambda(\rho)\, x^\lambda + A^\nu_{\lambda_1\lambda_2}(\rho)\, x^{\lambda_1} x^{\lambda_2} + \cdots ,$$

then we have:

$$| a^\nu_{\lambda_1\lambda_2}(\rho) | \leqq A^\nu_{\lambda_1\lambda_2}(\rho), \cdots, | a^\nu_{\lambda_1\cdots\lambda_N}(\rho) | \leqq A^\nu_{\lambda_1\cdots\lambda_N}(\rho), \cdots .$$

Now $A^\nu_{\lambda_1\lambda_2}(\rho), \cdots, A^\nu_{\lambda_1\cdots\lambda_N}(\rho), \cdots$ are the power series with positive coefficients with regard to ρ, consequently, for ρ such that $0 \leqq \rho \leqq \rho_0$, we have:

$$A^\nu_{\lambda_1\lambda_2}(\rho) \leqq A^\nu_{\lambda_1\lambda_2}(\rho_0), \cdots, A^\nu_{\lambda_1\cdots\lambda_N}(\rho) \leqq A^\nu_{\lambda_1\cdots\lambda_N}(\rho_0), \cdots .$$

Thus we see that, for ρ such that $0 \leqq \rho \leqq \rho_0$,

$$(11.2) \qquad | a^\nu_{\lambda_1\lambda_2}(\rho) | \leqq A^\nu_{\lambda_1\lambda_2}(\rho_0), \cdots, \ | a^\nu_{\lambda_1\cdots\lambda_N}(\rho) | \leqq A^\nu_{\lambda_1\cdots\lambda_N}(\rho_0), \cdots .$$

Now, taking δ_0 sufficiently small, for $| x | \leqq \delta_0$, there exists a positive number K such that

$$A^\nu_{\lambda_1\lambda_2}(\rho_0)\, | x^{\lambda_1} x^{\lambda_2} | + \cdots + A^\nu_{\lambda_1\cdots\lambda_N}(\rho_0)\, | x^{\lambda_1} \cdots x^{\lambda_N} | + \cdots \leqq K | x |^2 ,$$

where $| x | = \max. | x^\nu |$. Consequently, from (11.2), for $| x | \leqq \delta_0$, it follows that

$$| a^\nu_{\lambda_1\lambda_2}(\rho)\, x^{\lambda_1} x^{\lambda_2} + \cdots + a^\nu_{\lambda_1\cdots\lambda_N}(\rho)\, x^{\lambda_1} \cdots \lambda^{\lambda_N} + \cdots | \leqq K | x |^2 .$$

Then, for ρ such that $0 \leqq \rho \leqq \rho_0$, we have:

$$(11.3) \qquad | 'x^\nu | \leqq | x^\nu | + K | x |^2 \leqq | x | (1 + K | x |) .$$

We take δ such that $\delta < \min. (1/K, \delta_0)$, then, for x^ν such that $| x | < \delta/2$, from (11.3), we have:

$$(11.4) \qquad | 'x^\nu | < \delta .$$

Then, in the space E_{2n} of x^ν's, we consider the hypersphere V, of which the center is the origin and the radius is δ. When there exists the domain explained in the outset of this paragraph—specially the domain of the preceding chapters—, by the reasonings in these chapters, we see that it can be made so small as we desire. We take such a domain which lies in the interior of the hypersphere V' with radius $\delta/2$ and having the origin as the center. Let this domain be D. Then, from (11.4), when $x^\nu \in D$, $'x^\nu \in V$, namely the integral curve C passing through the point x^ν, lies in V for ρ such that $0 \leqq \rho \leqq \rho_0$. Now, from the properties of D, $T(\rho_0)\, x^\nu = \underset{1}{x^\nu} \in D$. Then, for ρ such that

$\rho_0 \leqq \rho \leqq 2\rho_0$, $T(\rho)\, x^\nu = T(\sigma+\rho_0)\, x^\nu = T(\sigma)\, \underset{1}{x}^\nu$ and $0 \leqq \sigma \leqq \rho_0$. Consequently, again from (11.4), we see that $T(\rho)\, x^\nu \in V$. Repeating this proccss, we see that, for any ρ, $T(\rho)\, x^\nu \in V$. Thus, we have

Theorem 8. *When there exists a domain D of the preceding chapters, of which all the points converge to the origin remaining in it when $T(\rho_0)$ is infinitely iterated on these points, the integral curve passing through any point of D approches indefinitely the origin remaining in the certain neighborhood of the origin.*

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