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Orthogonality Relation in the Analysis of Variance II

By

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Let $\mathfrak{C}, \mathfrak{A}_1, \cdots, \mathfrak{A}_p\ (p>1)$ be non trivial classifications. We shall say that $\mathfrak{C}$ is decomposed into $\mathfrak{A}_1, \cdots, \mathfrak{A}_p$ if the relation

(1) $$P_{\mathfrak{C}} = \sum_{k=1}^{p} P_{\mathfrak{A}_k}$$

holds. Here we may always assume that the $\mathfrak{A}$ are labeled so that $d(\mathfrak{A}_1) \geqq d(\mathfrak{A}_2) \geqq \cdots \geqq d(\mathfrak{A}_p)$. By the result[1] previously established p must be $\geqq 3$. We show (§ 1) that (1) implies

(2) $$d(\mathfrak{A}_1)\, d(\mathfrak{A}_2) \leqq \sum_{k=3}^{p} d(\mathfrak{A}_k),$$

and that if the equality holds in (2), we must obtain

(3) $$P_{\mathfrak{A}_1 \mathfrak{A}_2} = \sum_{k=3}^{p} P_{\mathfrak{A}_k}$$

that is, the interaction $\mathfrak{A}_1 \mathfrak{A}_2$ is decomposed into classifications $\mathfrak{A}_3, \cdots, \mathfrak{A}_p$. Conversely (3) will imply (1) with $\mathfrak{C} = \mathfrak{A}_1 \wedge \mathfrak{A}_2$. Thus a decomposition of an interaction into classifications is regarded as a special case of a decomposition of a classification into classifications. It is our purpose to investigate the structure of such decompositions. The case $p=3$ was considered in our previous paper[2] and we proved that $P_{\mathfrak{C}} = P_{\mathfrak{A}_1} + P_{\mathfrak{A}_2} + P_{\mathfrak{A}_3}$ holds if and only if $\mathfrak{C} = \mathfrak{A}_1 \wedge \mathfrak{A}_2$, and $\mathfrak{A}_1 \mathfrak{A}_2 = \mathfrak{A}_3$. Here the latter relation implies that $\mathfrak{A}_1$, $\mathfrak{A}_2$ are regularly orthogonal and $d(\mathfrak{A}_1) = d(\mathfrak{A}_2) = d(\mathfrak{A}_3) = 1$[3]. After some preliminary researches (§ 1) we investigate the structure of decompositions (1) for the cases $d(\mathfrak{A}_1) = p-2,\ p-3,\ p-4$. It is to be noted that (2) yields $d(\mathfrak{A}_1) \leqq p-2$. Then we apply the results thus obtained to determine the structure of decompositions (1) for $p=4, 5, 6$ (§ 3). § 4 treats the decompositions of interactions. We show there that $P_{\mathfrak{B}\mathfrak{C}\mathfrak{D}} = P_{\mathfrak{A}_1} + P_{\mathfrak{A}_2} + P_{\mathfrak{A}_3} + P_{\mathfrak{A}_4}$ is characterized as the

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1) Cf. [2]. Theorem 22. The numbers in square brackets refer to the list of references at the end of this paper.

2) Cf. [2]. Theorem 23.

3) Cf. [2]. Theorem 9.

decomposition of the interaction of 2nd order, as indicated by R. A. Fisher[4], in the $3\times3\times3$ factorial experiment.

§ 1. Some lemmas on decompositions.

In the sequel we are only concerned with non trivial classifications. Let us consider a decomposition:

$$P_{\mathfrak{C}}=\sum_{k=1}^{p}P_{\mathfrak{A}_k}. \tag{1}$$

Lemma 1. (1) *implies*

$$d(\mathfrak{A}_1)\,d(\mathfrak{A}_2)\leqq\sum_{k=3}^{p}d(\mathfrak{A}_k), \tag{2}$$

$$d(\mathfrak{C})=\sum_{k=1}^{p}d(\mathfrak{A}_k). \tag{2'}$$

Proof. (1) implies $P_{\mathfrak{C}}\geqq P_{\mathfrak{A}_1}, P_{\mathfrak{A}_2}$, whence we obtain $P_{\mathfrak{C}}\geqq P_{\mathfrak{A}_1\wedge\mathfrak{A}_2}$. By using this relation we get

$$P_{\mathfrak{A}_1\,\mathfrak{A}_2}=P_{\mathfrak{A}_1\wedge\mathfrak{A}_2}-P_{\mathfrak{A}_1}-P_{\mathfrak{A}_2}\leqq P_{\mathfrak{C}}-P_{\mathfrak{A}_1}-P_{\mathfrak{A}_2}=\sum_{k=3}^{p}P_{\mathfrak{A}_k}.$$

Thus (2), (2′) are direct consequences of the definition of degrees of freedom, completing the proof.

We call $\epsilon=d(\mathfrak{C})-d(\mathfrak{A}_1\wedge\mathfrak{A}_2)$ the excess of the decomposition (1). Then (2′) yields

$$\epsilon=\sum_{k=1}^{p}d(\mathfrak{A}_k)-\Big\{\big(d(\mathfrak{A}_1)+1\big)\big(d(\mathfrak{A}_2)+1\big)-1\Big\}=\sum_{k=3}^{p}d(\mathfrak{A}_k)-d(\mathfrak{A}_1)\,d(\mathfrak{A}_2).$$

If $d(\mathfrak{A}_1)\,d(\mathfrak{A}_2)=\sum_{k=3}^{p}d(\mathfrak{A}_k)$ holds, that is, $\epsilon=0$, then we say that (1) is of regular type. From the proof of the above lemma we obtain

Lemma 2. *If* (1) *is of regular type, then*

$$P_{\mathfrak{A}_1\,\mathfrak{A}_2}=\sum_{k=3}^{p}P_{\mathfrak{A}_k} \tag{3}$$

holds. Conversely (3) *implies* (1) *with* $\mathfrak{C}=\mathfrak{A}_1\wedge\mathfrak{A}_2$.

Let C_1, C_2 be two different $\mathfrak{C}$- classes. Put $n_{\mathfrak{A}_k}(C_1, C_2)=n^{(k)}_{A_j}$ or ∞ according as there exists an $\mathfrak{A}_k$-class $A_j^{(k)}$ containing both C_1, C_2 or not. In the sequel the following lemma 3 will play a fundamental rôle to investigate the structure of decompositions.

4) Cf. [1].
5) Cf. [2]. Theorem 2.

Lemma 3.

$$p-1=\sum_{k=1}^{p}\frac{N}{n_{\mathfrak{A}_k}(C_1,C_2)} \tag{4}$$

Proof. Let e_1, e_2 be unit vectors associated with an index taken from C_1, C_2 respectively. Then $(e_1, P_{\mathfrak{C}}\, e_2)=\sum_{k=1}^{p}(e_1, P_{\mathfrak{A}_k} e_2)$ will yield the relation (4).

For example suppose that $C_1, C_2 \subset A_1^{(1)}$. If $n_{A_1^{(1)}}=\frac{N}{p-1}$ holds, there exists besides $A_1^{(1)}$ no $\mathfrak{A}$-class containing both C_1, C_2. If $\frac{N}{p-3}\geqq n_{A_1^{(1)}}>\frac{N}{p-1}$ holds, there exists besides $A_1^{(1)}$ precisely one $\mathfrak{A}$-class containing both C_1, C_2.

Lemma 4. $d(\mathfrak{A}_1)\leqq p-2$. *The equality implies that* $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_p)$, *and* (1) *is of regular type.*

Proof. Since $d(\mathfrak{A}_1)\geqq d(\mathfrak{A}_2)\geqq\cdots\geqq d(\mathfrak{A}_p)$ holds, $d(\mathfrak{A}_1)\, d(\mathfrak{A}_2)\leqq\sum_{k=3}^{p}d(\mathfrak{A}_k)\leqq(p-2)\, d(\mathfrak{A}_2)$, and consequently $d(\mathfrak{A}_1)\leqq p-2$. Then it is clear that $d(\mathfrak{A}_1)=p-2$ implies $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_p)$, and (1) is of regular type.

§ 2. Structure of decomposition (1) for the cases $d(\mathfrak{A}_1)=p-2, p-3, p-4$.

Theorem 1. *Suppose that* $d(\mathfrak{A}_1)=p-2$. *The decomposition* (1) *is possible only if* $\mathfrak{C}, \mathfrak{A}_1,\cdots,\mathfrak{A}_p$ *are regular and* $p-1$ *is divisible by* $d(\mathfrak{A}_2)+1$.

Proof. Suppose that the decomposition (1) is possible. It follows by Lemma 4 that (1) is of regular type. Hence any $\mathfrak{A}_1$-class intersects each $\mathfrak{A}_k$-class $(k>1)$ in precisely one $\mathfrak{C}$-class. Then Lemma 3 shows that $n_{A_j^{(1)}}=\frac{N}{p-1}$, that is, $\mathfrak{A}_1$ is regular. To prove that $\mathfrak{C}$ is regular we take arbitrary but fixed two different $\mathfrak{C}$-classes C_1, C_2 contained in $\mathfrak{A}_1$-classes $A_i^{(1)}$, $A_j^{(1)}$ respectively. Lemma 3 shows that there exists an $\mathfrak{A}_k$-class $A_l^{(k)}$ $(k>1)$ containing both C_1, C_2. Since $\epsilon=0$ implies that $C_1=A_i^{(1)}A_l^{(k)}$, $C_2=A_j^{(1)}A_l^{(k)}$, it follows in view of the orthogonality condition[6] that $\frac{n_{C_1}}{n_{C_2}}=\frac{n_{A_i^{(1)}A_l^{(k)}}}{n_{A_j^{(1)}A_l^{(k)}}}=\frac{n_{A_i^{(1)}}}{n_{A_j^{(1)}}}=1$, that is, $n_{C_1}=n_{C_2}$, from which we conclude that $n_C=\text{const}$. This means that $\mathfrak{C}$ is regular. Then it is easy to see that $n_{A_i^{(k)}}=\frac{N}{d(\mathfrak{A}_2)+1}$ $(k>1)$. Hence $\mathfrak{A}_k$ $(k>1)$ is also regular. Apply Lemma 3 for C_1, C_2 both contained in an $\mathfrak{A}_k$-class to obtain that the right side of (4) is a multiple of $d(\mathfrak{A}_2)+1$. This completes the proof.

6) Cf. [2].

Two Latin squares of side $p-1$ are orthogonal to each other if, when they are superposed, every letter of one square occurs once and once only with every letter of the other. If $d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_p)=p-2$, $d(\mathfrak{C})=p(p-2)$, then (1) is possible if and only if a complete set of $p-2$ orthogonal Latin squares of side $p-1$ exists, where $\mathfrak{A}_1,\cdots,\mathfrak{A}_p$ correspond to classifications determined by rows, by columns, and by letters of each of the $p-2$ squares. Complete sets of orthogonal squares of side $p-1$ are known to exist for all prime numbers and for $p-1=4, 8$ and 9 [7]. It seems very difficult to determine the condition under which (1) is possible when $d(\mathfrak{A}_1)=p-2$, $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_p)<p-2$.[8] If $d(\mathfrak{A}_1)=p-2$, $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_p)=1$, we can show that $p-1$ is divisible by 4, so that for $p=7$ the decomposition is impossible (proof omitted).

Next we consider the case $d(\mathfrak{A}_1)=p-3$.

Theorem 2. *Suppose that $d(\mathfrak{A}_1)=p-3\ (p>3)$. The decomposition* (1) *is possible only if $p=4$, $d(\mathfrak{A}_1)=d(\mathfrak{A}_2)=d(\mathfrak{A}_3)=d(\mathfrak{A}_4)=1$.*

Proof. First we show that $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_p)$. To this end we suppose that $d(\mathfrak{A}_2)>d(\mathfrak{A}_k)$ for some $k\geqq 3$. Since any $\mathfrak{A}_1$-class contains at least $d(\mathfrak{A}_2)+1$ $\mathfrak{C}$-classes, it follows that any $\mathfrak{A}_1$-class $A_j^{(1)}$ intersects some $\mathfrak{A}_k$-class at least in two $\mathfrak{C}$-classes, whence Lemma 3 shows that $n_{A_j^{(1)}}>\dfrac{N}{p-2}$. This is a contradiction since we obtain that $N=\sum_{j=1}^{p-2} n_{A_j^{(1)}}>\sum_{j=1}^{p-2}\dfrac{N}{p-2}=N$. Consider the excess of the decomposition (1): $\sum_{k=3}^{p} d(\mathfrak{A}_k)-d(\mathfrak{A}_1)\,d(\mathfrak{A}_2)=(p-2)\,d(\mathfrak{A}_2)-(p-3)\,d(\mathfrak{A}_2)=d(\mathfrak{A}_2)$. Then there exists an $\mathfrak{A}_1$-class $A_1^{(1)}$ and an $\mathfrak{A}_2$-class $A_1^{(2)}$ such that $A_1^{(1)}$ contains just $d(\mathfrak{A}_2)+1$ $\mathfrak{C}$-classes and $A_1^{(2)}$ just $d(\mathfrak{A}_1)+1$ $\mathfrak{C}$-classes. By an analogous argument as in the proof of Theorem 1 we may conclude that $A_i^{(1)}A_j^{(2)}\ (i,j>1)$ contains the same number of $\mathfrak{C}$-classes. Therefore we must have $d(\mathfrak{A}_1)\,d(\mathfrak{A}_2)\leqq\epsilon=d(\mathfrak{A}_2)$, and consequently $d(\mathfrak{A}_1)=1$. This

7) Cf. [1]. Page 81.

8) e. g. If $d(\mathfrak{A}_1)=5$, $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_7)=2$, the structure of $\mathfrak{C},\mathfrak{A}_1,\cdots,\mathfrak{A}_7$ is given (except for the order of $\mathfrak{A}_i$) as follows:

$\mathfrak{C}=(C_{11}, C_{12}, C_{13}, C_{21}, C_{22}, C_{23}, C_{31}, C_{32}, C_{33}, C_{41}, C_{42}, C_{43}, C_{51}, C_{52}, C_{53}, C_{61}, C_{62}, C_{63})$,

$\mathfrak{A}_1=(C_{11}+C_{12}+C_{13},\ C_{21}+C_{22}+C_{23},\ C_{31}+C_{32}+C_{33},\ C_{41}+C_{42}+C_{43},\ C_{51}+C_{52}+C_{53},\ C_{61}+C_{62}+C_{63})$,

$\mathfrak{A}_2=(C_{11}+C_{21}+C_{31}+C_{41}+C_{51}+C_{61},\ C_{12}+C_{22}+C_{32}+C_{42}+C_{52}+C_{62},\ C_{13}+C_{23}+C_{33}+C_{43}+C_{53}+C_{63})$,

$\mathfrak{A}_3=(C_{11}+C_{21}+C_{33}+C_{43}+C_{52}+C_{62},\ C_{12}+C_{22}+C_{31}+C_{41}+C_{53}+C_{63},\ C_{13}+C_{23}+C_{32}+C_{42}+C_{51}+C_{61})$,

$\mathfrak{A}_4=(C_{11}+C_{23}+C_{31}+C_{42}+C_{53}+C_{62},\ C_{12}+C_{21}+C_{32}+C_{43}+C_{51}+C_{63},\ C_{13}+C_{22}+C_{33}+C_{41}+C_{52}+C_{61})$,

$\mathfrak{A}_5=(C_{11}+C_{23}+C_{32}+C_{41}+C_{52}+C_{63},\ C_{12}+C_{21}+C_{33}+C_{42}+C_{53}+C_{61},\ C_{13}+C_{22}+C_{31}+C_{43}+C_{51}+C_{62})$,

$\mathfrak{A}_6=(C_{11}+C_{22}+C_{33}+C_{42}+C_{51}+C_{63},\ C_{13}+C_{21}+C_{32}+C_{41}+C_{53}+C_{62},\ C_{12}+C_{23}+C_{31}+C_{43}+C_{52}+C_{61})$,

$\mathfrak{A}_7=(C_{11}+C_{22}+C_{32}+C_{43}+C_{53}+C_{61},\ C_{13}+C_{21}+C_{31}+C_{42}+C_{52}+C_{63},\ C_{12}+C_{23}+C_{33}+C_{41}+C_{51}+C_{62})$.

implies that $p=4$, $d(\mathfrak{A}_1)=d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_4)=1$, completing the proof.

From the proof of the above theorem we see that if $d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_4)=1$, the decomposition $P_{\mathfrak{C}}=P_{\mathfrak{A}_1}+\cdots+P_{\mathfrak{A}_4}$ is possible if and only if $\mathfrak{C}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$ are represented as follows;

$$(5_4)\qquad \begin{aligned} \mathfrak{C}&=(C_0, C_1, C_2, C_3, C_4),\\ \mathfrak{A}_1&=(C_0+C_1,\ C_2+C_3+C_4),\\ \mathfrak{A}_2&=(C_0+C_2,\ C_1+C_3+C_4),\\ \mathfrak{A}_3&=(C_0+C_3,\ C_1+C_2+C_4),\\ \mathfrak{A}_4&=(C_0+C_4,\ C_1+C_2+C_3),\end{aligned}$$

where $n_{C_0}:n_{C_1}:n_{C_2}:n_{C_3}:n_{C_4}=1:2:2:2:2$.

Theorem 3. *Suppose that $d(\mathfrak{A}_1)=p-4$ $(p>4)$. Then the decomposition* (1) *is possible only if (I) $p=5$, $d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_5)=1$, or (II) $d(\mathfrak{A}_2)=d(\mathfrak{A}_3)=\cdots=d(\mathfrak{A}_{p-3})=3$, $d(\mathfrak{A}_{p-2})=d(\mathfrak{A}_{p-1})=d(\mathfrak{A}_p)=1$, and $p-3$ is divisible by* 4.

Proof. *Case I.* $d(\mathfrak{A}_p)=1$.

First we show that if $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_p)=1$, then p must be 5.

Suppose that $d(\mathfrak{A}_2)=d(\mathfrak{A}_p)=1$ holds for $p>5$. Since $\epsilon=\sum_{k=3}^{p} d(\mathfrak{A}_k)-d(\mathfrak{A}_1)$ $d(\mathfrak{A}_2)=2$ and $p-3>2$, therefore there exists an $\mathfrak{A}_1$-class $A_1^{(1)}$ containing precisely two $\mathfrak{C}$-classes. By taking Lemma 3 into account we may conclude by a similar argument as in the proof of Theorem 1 that $\mathfrak{A}_2, \mathfrak{A}_3, \cdots, \mathfrak{A}_p$ are regular and each of $A_j^{(1)}$ $(j<p-3)$ contains precisely two $\mathfrak{C}$-classes and $A_{p-3}^{(1)}$ intersects any $A_j^{(k)}$ $(k>1)$ in precisely two $\mathfrak{C}$-classes. Since $\mathfrak{A}_2, \cdots, \mathfrak{A}_p$ are mutually independent, it is not difficult to see that $p-2$ is even and any $\mathfrak{A}_2$-class intersects some $\mathfrak{A}_j$-class $(j>2)$ in $\frac{p-2}{2}$ $\mathfrak{C}$-classes. On the other hand, using Lemma 3 for any two $\mathfrak{C}$-classes contained in an $\mathfrak{A}_2$-class but not in $A_{p-3}^{(1)}$, we see that $p-1$ is even. Thus we obtain a contradiction. Therefore if $d(\mathfrak{A}_2)=d(\mathfrak{A}_p)=1$ holds, then p must be 5.

Next suppose that $d(\mathfrak{A}_2)>1$. We shall show that $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_{p-3})=3$, $d(\mathfrak{A}_{p-2})=d(\mathfrak{A}_{p-1})=d(\mathfrak{A}_p)=1$. Since each $\mathfrak{A}_1$-class intersects every $\mathfrak{A}_2$-class, therefore any $\mathfrak{A}_1$-class containing at least 3 $\mathfrak{C}$-classes. Let $A_1^{(1)}$ be any $\mathfrak{A}_1$-class such that $n_{A_1^{(1)}}\leqq\frac{N}{p-3}$. Put $A_1^{(1)}A_1^{(p)}=C_0+C_1+\cdots+C_l$, $A_1^{(1)}A_2^{(p)}=C_{l+1}+\cdots+C_{l+m+1}$ $(l\geqq m)$, where C_i are $\mathfrak{C}$-classes. Owing to Lemma 3, besides $A_1^{(1)}$ there exists only one $\mathfrak{A}$-class containing 2 $\mathfrak{C}$-classes contained in $A_1^{(1)}A_1^{(p)}$. It follows that (i) either $l=1$, $m=0$, and $n_{A_1^{(1)}}<\frac{N}{p-3}$, or (ii) $l=m=1$ and

$n_{A_1^{(1)}}=\frac{N}{p-3}$. Consider the case (i). Since there exists besides $A_1^{(1)}$ precisely one $\mathfrak{A}$-class containing C_0, C_2 and C_1, C_2 respectively, therefore we must obtain that $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_{p-3})=2$, $d(\mathfrak{A}_{p-2})=d(\mathfrak{A}_{p-1})=d(\mathfrak{A}_p)=1$. It is easy to see that $n_{C_0}=n_{C_1}=n_{C_2}$, so that $\mathfrak{A}_2, \cdots, \mathfrak{A}_{p-3}$ are regular. Since $\epsilon=\sum_{k=3}^{p} d(\mathfrak{A}_k)-d(\mathfrak{A}_1)$ $d(\mathfrak{A}_2)=2(p-5)+3-2(p-4)=1$, we may assume that $A_j^{(1)}$ $(j<p-3)$ contains precisely 3 $\mathfrak{C}$-classes and $A_{p-3}^{(1)}$ precisely 4 $\mathfrak{C}$-classes. We may put $A_{p-3}^{(1)}A_1^{(2)}=C_1'$, $A_{p-3}^{(1)}A_2^{(2)}=C_2'$, $A_{p-3}^{(1)}A_3^{(2)}=C_3'+C_4'$. Then $n_{C_1'}=n_{C_2'}=n_{C_3'+C_4'}$ since $\mathfrak{A}_2$ is regular. There exists an $\mathfrak{A}$-class containing C_3' but not C_4'. Let $A_1^{(k)}$ be such a class. k must be $<p-2$ because of $n_{C_1'}<\frac{1}{3}n_{A_{p-3}^{(1)}}$. We may assume that $A_{p-3}^{(1)}A_1^{(k)}=C_3'$, $A_{p-3}^{(1)}A_2^{(k)}=C_4'$ and $A_{p-3}^{(1)}A_3^{(k)}=C_1'+C_2'$. Therefore $n_{C_1'}+n_{C_2'}=n_{C_3'}=n_{C_4'}$. This together with the relation $n_{C_1'}=n_{C_2'}=n_{C_3'}+n_{C_4'}$ gives $n_{C_1}=0$. This is a contradiction. Suppose that (ii) is the case. Then $\mathfrak{A}_1$ is regular. Indeed, $n_{A_j^{(1)}}\leqq\frac{N}{p-3}$ implies $n_{A_j^{(1)}}=\frac{N}{p-3}$ since (i) is impossible. By the argument as above we conclude that $\mathfrak{C}$ is regular and any $A_j^{(1)}$ $(j=1,\cdots,p-3)$ contains precisely 4 $\mathfrak{C}$-classes. We may put $A_j^{(1)}A_1^{(p)}=C_{j1}+C_{j2}$, $A_j^{(1)}A_2^{(p)}=C_{j3}+C_{j4}$. $d(\mathfrak{A}_k)$ $(2\leqq k<p)$ will be 2 or 3. If it were $d(\mathfrak{A}_k)=2$, we may put $A_1^{(1)}A_1^{(k)}=C_{11}+C_{13}$, $A_1^{(1)}A_2^{(k)}=C_{12}$, $A_1^{(1)}A_3^{(1)}=C_{13}$. Consider an $\mathfrak{A}_l$-class $A_1^{(l)}$ such that $A_1^{(1)}A_1^{(l)}=C_{12}+C_{14}$. Since $n_{A_1^{(k)}}=n_{A_1^{(l)}}=\frac{N}{2}$, it follows from the orthogonality condition that $A_1^{(k)}$ intersects $A_1^{(l)}$ in precisely $p-3$ $\mathfrak{C}$-classes and $A_j^{(1)}A_1^{(k)}$ can not be disjoint to $A_j^{(1)}A_1^{(l)}$. This is a contradiction. Therefore $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_{p-3})=3$, $d(\mathfrak{A}_{p-2})=d(\mathfrak{A}_{p-1})=d(\mathfrak{A}_p)=1$. Since an $\mathfrak{A}_2$-class intersects $A_1^{(p)}$, $A_2^{(p)}$ in the same number of $\mathfrak{C}$-classes, $p-3$ is divisible by 2 and consequently $p>6$. Hence $d(\mathfrak{A}_3)=3$. Consider the intersection of an $\mathfrak{A}_2$-class and an $\mathfrak{A}_3$-class, then the orthogonality condition shows that $p-3$ is divisible by 4.[9]

9) We remark that $P_{\mathfrak{A}_{p-2}}+P_{\mathfrak{A}_{p-1}}+P_{\mathfrak{A}_p}$ must be an classification operator, so that Case (II) is a special case considered in Theorem I. Indeed, it follows from Lemma 3 that there exists no pair of $\mathfrak{C}$-classes C_{ji}, $C_{j'i'}$ $(j\neq j')$ both contained in only two of $\mathfrak{A}_k$-classes $(k=p, p-1, p-2)$. Hence we may put $A_j^{(1)}A_1^{(p-1)}=C_{j1}+C_{j3}$, $A_j^{(1)}A_2^{(p-1)}=C_{j2}+C_{j4}$, $A_j^{(1)}A_1^{(p-2)}=C_{j1}+C_{j4}$, $A_j^{(1)}A_2^{(p-2)}=C_{j2}+C_{j3}$. Therefore $P_{\mathfrak{A}_{p-2}}+P_{\mathfrak{A}_{p-1}}+P_{\mathfrak{A}_p}$ is an classification operator $P_{\mathfrak{B}}$, and consequently $P_{\mathfrak{C}}=\sum_{k=1}^{p-3}P_{\mathfrak{A}_k}+P_{\mathfrak{B}}$, where $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{B})=3$.

In order to prove the theorem it is sufficient to show that the following case II is impossible.

Case II. $d(\mathfrak{A}_p)>1$.

Consider any $\mathfrak{A}_1$-class $A_1^{(1)}$ such that $n_{A_1^{(1)}}\leqq\frac{N}{p-3}$. Let $d+1$ stand for the number of $\mathfrak{C}$-classes C_i contained in $A_1^{(1)}$. First suppose that there exists an $\mathfrak{A}_k$ $(k>1)$ such that $d_k=d(\mathfrak{A}_k)<d$. We shall show that this is impossible. Taking into account of Lemma 3 we may put $A_1^{(1)}A_1^{(k)}=C_0+C_1+\cdots+C_q$, $A_1^{(1)}A_2^{(k)}=C_{q+1},\cdots, A_1^{(1)}A_{d_k+1}^{(k)}=C_d$, where $d=q+d_k$. Since there exists besides $A_1^{(1)}$ precisely one $\mathfrak{A}$-class containing two C_i, it follows that we can group $C_{q+1},\cdots, C_d$ into r sets such that each such set augmented by C_0 generates some $A_1^{(1)}A_1^{(l)}$, where $A_1^{(l)}$ has the like properties as $A_1^{(k)}$. Let $A_1^{(m)}$ $(m\neq k)$ be any $\mathfrak{A}$-class having the like properties as $A_1^{(k)}$, then we can show that $A_1^{(1)}A_1^{(m)}$ intersects $A_1^{(1)}A_1^{(k)}$ in precisely one $\mathfrak{C}$-class. Indeed, $A_1^{(1)}A_1^{(m)}$ intersects $A_1^{(1)}A_1^{(k)}$ in at most one $\mathfrak{C}$-class since otherwise $n_{A_1^{(1)}}>\frac{N}{p-3}$ will hold by Lemma 3. If $A_1^{(1)}A_1^{(m)}$ is disjoint to $A_1^{(1)}A_1^{(k)}$, $A_1^{(1)}A_1^{(m)}$ will be $C_{q+1}+\cdots+C_d$ since $n_{A_1^{(k)}}=n_{A_1^{(m)}}\geqq\frac{N}{2}$ yields $n_{A_1^{(1)}A_1^{(k)}}=n_{A_1^{(1)}A_1^{(m)}}=\frac{1}{2}n_{A_1^{(1)}}$. For any C_i $(i\leqq q)$ and any C_j $(j>q)$ there exists an $\mathfrak{A}$-class different from $A_1^{(1)}$ which contains C_i, C_j. This class intersects $A_1^{(1)}$ in just these two $\mathfrak{C}$-classes. Hence $n_{C_i}+n_{C_j}=\frac{1}{2}n_{A_1^{(1)}}$. Then $n_{C_0}=n_{C_1}=\cdots=n_{C_d}=\frac{1}{4}n_{A_1^{(1)}}$ and consequently $q=1$, $d=3$. Then two possible cases will occur: (i) $p=7$, $d(\mathfrak{A}_2)=d(\mathfrak{A}_3)=\cdots=d(\mathfrak{A}_7)=2$. (ii) $d(\mathfrak{A}_2)=\cdots=d(\mathfrak{A}_{p-6})=3$, $d(\mathfrak{A}_{p-5})=\cdots=d(\mathfrak{A}_p)=2$. In the first case (i) we may assume that $A_j^{(1)}A_1^{(n)}$ $(2\leqq n<7)$ contains just two $\mathfrak{C}$-classes (proof omitted). It is easy to see that $\mathfrak{C}$ is regular and consequenty $n_{A_j^{(1)}}=\frac{N}{4}$. Let $A_1^{(1)}A_1^{(6)}=C_0+C_1$, $A_1^{(1)}A_1^{(5)}=C_2+C_3$. Then $A_1^{(5)}$ intersects $A_1^{(6)}$ in less than two $\mathfrak{C}$-classes, contradicting the orthogonality condition. In the second case (ii) $\epsilon=\sum_{k=3}^{p}d(\mathfrak{A}_k)-d(\mathfrak{A}_1)\,d(\mathfrak{A}_2)=0$. This implies that every $A_j^{(1)}$ contains just 4 $\mathfrak{C}$-classes. It is easy to see that $\mathfrak{C}$ is regular and we may assume that $A_j^{(1)}A_1^{(n)}$ $(p-5\leqq n\leqq p)$ contains just two $\mathfrak{C}$-classes. Then by the same argument as in the first case we reach a contradiction. Now we return to the proof. Consider grouping of $C_{q+1},\cdots, C_d$ corresponding to C_i $(0\leqq i\leqq q)$. We obtain that $\sum_{j=2}^{p}(d-d_j)=\sum_{j\neq k}(d-d_j)+(d-d_k)=(q+1)\,d_k+d-d_k=d+(d-d_k)\,d$.

Extreme members of this relation imply that q and r are independent for any $d_k < d$. It is easy to see that $n_{C_0}=n_{C_1}=\cdots=n_{d}$. We can conclude in view of $n_{A_1^{(k)}} \geqq \frac{N}{2}$ that $q=1$ and $d_k=2$. We must obtain $d=3$. This is impossible since (i), (ii) can not occur. Therefore $d=d_1=\cdots=d_p$ must hold. This is equivalent to say that $n_{A_1^{(1)}}=\frac{N}{p-1}$. Two cases are possible, that is, (iii) $d=p-4$ and (iv) $d<p-4$. We shall show that these two cases can not occur.

Suppose that we are in the case (iii). Then there exists an $A_1^{(2)}$ such that $n_{A_1^{(2)}}=\frac{N}{p-1}$. We may put $A_1^{(1)} A_1^{(2)}=C_1$, $A_1^{(1)} A_i^{(2)}=C_i$, $A_i^{(1)} A_1^{(2)}=C_i'$ $(i=2,3,\cdots,p)$. Let C be any $\mathfrak{C}$-class contained in $A_i^{(1)} A_j^{(1)}$ $(i,j>1)$. Consider any $\mathfrak{A}$-class A containing C, C_1. It follows by Lemma 3 that $n_{A_1^{(1)}}=\frac{N}{p-1}$ implies $A_1^{(1)} A = C_1$. $n_A=\frac{n_{A_1^{(1)}A}}{n_{A_1^{(1)}}}N=\frac{n_{C_1}}{n_{A_1^{(1)}}}N=\frac{n_{A_1^{(1)}A_1^{(2)}}}{n_{A_1^{(1)}}}N=n_{A_1^{(2)}}=\frac{N}{p-1}$. The same Lemma implies also $A A_i^{(1)}=C$, so that $n_c=n_{AA_i^{(1)}}=\frac{1}{N}n_A\, n_{A_i^{(1)}}=\frac{1}{N}n_{A_1^{(2)}}n_{A_i^{(1)}}=n_{c_i'}$, and $n_C=n_{C_i'}=\frac{n_{A_i^{(1)}}}{p-1}$. Similarly we have $n_C=n_{C_j}=\frac{n_{A_j^{(2)}}}{p-1}$. Hence $n_{A_i^{(1)}}=n_{A_j^{(2)}}$ $(i, j>1)$. Let $k+1$ be the number of $\mathfrak{C}$-classes contained in $A_i^{(1)} A_j^{(2)}$ $(i,j>1)$. Then $(k+1)n_C=n_{A_i^{(1)}A_j^{(2)}}=\frac{1}{N}n_{A_i^{(1)}}\,n_{A_j^{(2)}}=\frac{1}{N}(p-1)^2 N$. Extreme members of this relation show that $k+1$ is independent of i, j. $d(\mathfrak{C})=(d+1)^2+kd^2-1$. On the other hand $d(\mathfrak{C})=\sum_{k=1}^{p} d(\mathfrak{A}_k)=pd=(d+1)^2+2d-1$. Hence we must have $2d=kd^2$. Since $d>1$ by our assumption, it follows that $p=6$, $d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_6)=2$. But we can show that this is impossible (proof omitted). Next consider the case (iv). Since $\epsilon=\sum_{k=3}^{p} d(\mathfrak{A}_k)-d(\mathfrak{A}_1)\,d(\mathfrak{A}_2)=2d$ holds, there exists an $A_1^{(1)}$ containing just $d+1$ $\mathfrak{C}$-classes. This implies that $n_{A_1^{(1)}}=\frac{N}{p-1}$. Suppose that there exists another such $\mathfrak{A}_1$-classes $A_2^{(1)}$, we can conclude in view of the orthogonality condition that $\mathfrak{A}_2,\cdots,\mathfrak{A}_p$ are regular, that is, $n_{A_j^{(k)}}=\frac{N}{d+1}$ $(k\geqq 2)$. Indeed, put $A_1^{(1)} A_j^{(2)}=C_i$, $A_2^{(1)} A_j^{(2)}=C_i'$. By the similar argument as in the case $d=p-4$ above considered it can be shown that $n_{C_i}=n_{C_j}$. This implies that $n_{A_i^{(2)}}=n_{A_j^{(2)}}$, consequently $\mathfrak{A}_2$ is regular. The same is true for any $\mathfrak{A}_k\ (k\geqq 2)$. If there exists an $A_j^{(1)}\ (j>2)$ such that $A_j^{(1)}$ intersects some $\mathfrak{A}_2$-class in precisely one $\mathfrak{C}$-class, we can conclude by the similar argument as above that $A_j^{(1)}$ consists of just $d+1$ $\mathfrak{C}$-classes, that is, $n_{A_j^{(1)}}=\dfrac{N}{p-1}$. If we take into account of $2d<2(d+1)$, we may assume that each of $A_1^{(1)}, A_2^{(1)},\cdots, A_{p-4}^{(1)}$ consists of just $d+1$ $\mathfrak{C}$-classes, and $A_{p-3}^{(1)}$ $3d+1$ $\mathfrak{C}$-classes. Since $3(d+1)>3d+1$, there exist at least two $\mathfrak{A}_2$-class, $A_1^{(2)}$, $A_2^{(2)}$ such that each of $A_{p-3}^{(1)}A_1^{(2)}$, $A_{p-3}^{(1)}A_2^{(2)}$ consists of just two $\mathfrak{C}$-classes, so that we may put $A_{p-3}^{(1)}A_1^{(2)}=C_1+C_2$, $A_{p-3}^{(1)}A_2^{(2)}=C_3+C_4$. $n_{C_1}+n_{C_2}=n_{C_3}+n_{C_4}=\dfrac{1}{d+1}\,n_{A_{p-3}^{(1)}}=\dfrac{3N}{(p-1)(d+1)}$ since $n_{A_{p-3}^{(1)}}=\dfrac{N}{p-3}$ holds. Let $n_{C_1}\geqq n_{C_2}$ and $n_{C_3}\geqq n_{C_4}$. Then by considering an $\mathfrak{A}$-class different from $A_{p-3}^{(1)}$ which contains C_1, C_3, it follows that $n_{C_1}+n_{C_3}=\dfrac{1}{d+1}\,n_{A_{p-3}^{(1)}}$. Therefore $n_{C_1}=n_{C_2}=n_{C_3}=n_{C_4}=\dfrac{3N}{2(p-1)(d+1)}$. If C is any $\mathfrak{C}$-class not in $A_{p-3}^{(1)}$, then orthogonality condition shows that $n_C=\dfrac{N}{p-1}\,\dfrac{N}{d+1}\,\dfrac{1}{N}=\dfrac{N}{(p-1)(d+1)}$. Let A be an $\mathfrak{A}$-class containing C_1, C', where C' is any $\mathfrak{C}$-class contained in $A_{p-3}^{(1)}$ and is different from C_1, C_2, C_3, C_4. Then A can not contain C_2, C_3, C_4. $AA_2^{(2)}$ is disjoint to $A_{p-3}^{(1)}$. Let s be the number of $\mathfrak{C}$-classes contained in $AA_2^{(2)}$, then $n_{AA_2^{(2)}}=\dfrac{sN}{(p-1)(d+1)}$. $AA_1^{(2)}$ intersects $A_{p-3}^{(1)}$ in C_1. Let t be the number of $\mathfrak{C}$-classes contained in $AA_1^{(2)}$ but not in $A_{p-3}^{(1)}$. Then $n_{AA_1^{(2)}}=\dfrac{tN}{(p-1)(d+1)}+\dfrac{3N}{2(p-1)(d+1)}$. Since $n_{AA_1^{(2)}}=\dfrac{1}{N}\,n_A\,n_{A_1^{(2)}}=\dfrac{1}{N}\,n_A\,n_{A_2^{(2)}}=n_{AA_2^{(2)}}$, we obtain that $s=t+\dfrac{3}{2}$. This is a contradiction since s, t are integers. There remains the case to examine where there exists besides $A_1^{(1)}$ no $\mathfrak{A}_1$-class containing just $d+1$ $\mathfrak{C}$-classes. Then among these $\mathfrak{A}_1$-classes there exists an $\mathfrak{A}_1$-class $A_2^{(1)}$ containing the least number of $\mathfrak{C}$-classes. It follows from our assumption and $p-4>d$ that this number must be $d+2$. We may put $A_2^{(1)}A_1^{(2)}=C_0'+C_1'$, $A_2^{(1)}A_j^{(2)}=C_j'\ (j=2,3,\cdots,d+1)$ and $A_1^{(1)}A_i^{(2)}=C_i\ (i=1,2,\cdots,d+1)$ where $n_{C_2'}\geqq n_{C_3'}\geqq\cdots\geqq n_{C_{d+1}'}$. Since $\dfrac{n_{C_0'}+n_{C_1'}}{n_{C_1}}=\dfrac{n_{C_2'}}{n_{C_2}}=\cdots=\dfrac{n_{C_{d+1}'}}{n_{C_{d+1}}}=\dfrac{n_{A_2^{(1)}}}{n_{A_1^{(1)}}}$ holds, we obtain $n_{C_2}\geqq n_{C_3}\geqq\cdots\geqq$

$n_{C_{d+1}}$. Consider an $\mathfrak{A}$-class A different from $A_2^{(1)}$ which contains two $\mathfrak{C}$-classes C_2', C_k' $(k \neq 2)$. $\dfrac{n_{A_2^{(1)}A}}{n_{A_1^{(1)}A}} = \dfrac{n_{A_2^{(1)}}}{n_{A_1^{(1)}}}$ must hold. $n_{A_2^{(1)}A} = n_{C_2'} + n_{C_k'} > n_{C_2'}$. Therefore $A_1^{(1)} A = C_1$. Consequently $n_{C_0'} + n_{C_1'} = n_{C_2'} + n_{C_k'}$. It follows from this relation that $n_{C_0'} = n_{C_1'} = \cdots = n_{C_{d+1}'}$. Therefore $n_{A_1^{(2)}} = \dfrac{2N}{d+2}$, $n_{A_2^{(2)}} = \cdots = n_{A_{d+1}^{(2)}} = \dfrac{N}{d+2}$. If we apply Lemma 3 for C_2, C_2', then $p-1$ will be a multiple of $d+2$. Hence $p-1 \geqq 2(d+1)$ in view of $p-1 > d+1$, and consequently $p-4 > 2d$. On the other hand $p-4 \leqq \epsilon = 2d$. This is a contradiction. Thus we complete the proof of Theorem 3.

From the proof of this theorem the structure of the decomposition $P_{\mathfrak{C}} = \sum_{k=1}^{5} P_{\mathfrak{C}_k}$ with $d(\mathfrak{A}_k)=1$ $(k=1, 2, \cdots, 5)$ is given as follows:

$$
(5_5) \qquad
\begin{aligned}
\mathfrak{C} &= (C_0, C_1, C_2, C_3, C_4, C_5),\\
\mathfrak{A}_1 &= (C_0 + C_1,\ C_2 + C_3 + C_4 + C_5),\\
\mathfrak{A}_2 &= (C_0 + C_2,\ C_1 + C_3 + C_4 + C_5),\\
\mathfrak{A}_3 &= (C_0 + C_3,\ C_1 + C_2 + C_4 + C_5),\\
\mathfrak{A}_4 &= (C_0 + C_4,\ C_1 + C_2 + C_3 + C_5),\\
\mathfrak{A}_5 &= (C_0 + C_5,\ C_1 + C_2 + C_3 + C_4),
\end{aligned}
$$

where $n_{C_0} : n_{C_1} : n_{C_2} : n_{C_3} : n_{C_4} : n_{C_5} = 1:3:3:3:3:3$.

§ 3. Cases $p=4$, 5, 6.

We shall apply the results established in § 2 to determine the structure of decompositions (1) for $p=4$, 5, 6.

Case $p=4$.

The decomposition $P_{\mathfrak{C}} = \sum_{k=1}^{4} P_{\mathfrak{A}_k}$ is possible only in the following two cases.

	$\mathfrak{C}$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{A}_4$	Case
D. F.	8	2	2	2	2	I
	4	1	1	1	1	II

Case I. $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$ correspond to classifications determined by rows, by columns, and by a complete set of orthogonal Latin squares of side 3.

Case II. The structure is given by (5_4).

Case $p=5$.

The decomposition $P_{\mathfrak{C}}=\sum_{k=1}^{5} P_{\mathfrak{A}_k}$ is possible only in the following three cases.

	$\mathfrak{C}$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{A}_4$	$\mathfrak{A}_5$	Case
	15	3	3	3	3	3	I
D. F.	7	3	1	1	1	1	II
	5	1	1	1	1	1	III

Case I. $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$ correspond to classifications determined by rows, by columns, and by a complete set of orthogonal Latin squares of side 4.

Case II. The structure is given as follows:

$$\mathfrak{C}=(C_{11}, C_{12}, C_{21}, C_{22}, C_{31}, C_{32}, C_{41}, C_{42}),$$

$$\mathfrak{A}_1=(C_{11}+C_{12},\ C_{21}+C_{22},\ C_{31}+C_{32},\ C_{41}+C_{42}),$$

$$\mathfrak{A}_2=(C_{11}+C_{21}+C_{31}+C_{41},\ C_{12}+C_{22}+C_{32}+C_{42}),$$

$$\mathfrak{A}_3=(C_{11}+C_{22}+C_{32}+C_{41},\ C_{12}+C_{21}+C_{31}+C_{42}),$$

$$\mathfrak{A}_4=(C_{11}+C_{21}+C_{32}+C_{42},\ C_{12}+C_{22}+C_{31}+C_{41}),$$

$$\mathfrak{A}_5=(C_{11}+C_{22}+C_{31}+C_{42},\ C_{12}+C_{21}+C_{32}+C_{41}),$$

where $n_{C_{ij}}=\text{const.}$

Case III. The structure is given by (5_5).

Case $p=6$. The decomposition $P_{\mathfrak{C}}=\sum_{k=1}^{6} P_{\mathfrak{A}_k}$ is possible only in the following two cases.

	$\mathfrak{C}$	$\mathfrak{A}_1$	$\mathfrak{A}_2$	$\mathfrak{A}_3$	$\mathfrak{A}_4$	$\mathfrak{A}_5$	$\mathfrak{A}_6$	Case
D. F.	24	4	4	4	4	4	4	I
	6	1	1	1	1	1	1	II

Case I. $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$, $\mathfrak{A}_6$ correspond to classifications determined by rows, by columns, and by a complete set of orthogonal Latin squares of side 5.

Case II. The structure is given as follows:

$$
\begin{aligned}
\mathfrak{C} &= (C_0, C_1, C_2, C_3, C_4, C_5, C_6)\,,\\
\mathfrak{A}_1 &= (C_0+C_1,\ C_2+C_3+C_4+C_5+C_6)\,,\\
\mathfrak{A}_2 &= (C_0+C_2,\ C_1+C_3+C_4+C_5+C_6)\,,\\
\mathfrak{A}_3 &= (C_0+C_3,\ C_1+C_2+C_4+C_5+C_6)\,,\\
\mathfrak{A}_4 &= (C_0+C_4,\ C_1+C_2+C_3+C_5+C_6)\,,\\
\mathfrak{A}_5 &= (C_0+C_5,\ C_1+C_2+C_3+C_4+C_6)\,,\\
\mathfrak{A}_6 &= (C_0+C_6,\ C_1+C_2+C_3+C_4+C_5)\,,
\end{aligned} \tag{5_6}
$$

where $n_{C_0} : n_{C_1} : n_{C_2} : n_{C_3} : n_{C_4} : n_{C_5} : n_{C_6} = 14:4:4:4:4:4:4$.

§ 4. Structure of decompositions of interactions.

Theorem 4. (i) $P_{\mathfrak{A}_1\mathfrak{A}_2} = P_{\mathfrak{A}_3} + P_{\mathfrak{A}_4}$ *holds if and only if* $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4$ *correspond to classifications determined by rows, by columns, and by a complete set of orthogonal Latin squares of side 3.*

(ii) *The interaction* $\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}_3\cdots\mathfrak{B}_n$ $(n \geqq 3)$ *can not be decomposed into two classifications.*

Proof.

ad (i). It follows from Lemma 2 and § 3 (Case $p=4$).

ad (ii). Suppose that $P_{\mathfrak{B}_1\mathfrak{B}_2\cdots\mathfrak{B}_n} = P_{\mathfrak{A}_1} + P_{\mathfrak{A}_2}$ holds. Then $d(\mathfrak{B}_1)\,d(\mathfrak{B}_2)\cdots d(\mathfrak{B}_n) = d(\mathfrak{A}_1) + d(\mathfrak{A}_2)$. We may assume that $d(\mathfrak{B}_1) \geqq d(\mathfrak{B}_2) \geqq \cdots \geqq d(\mathfrak{B}_n)$. It follows from $d(\mathfrak{B}_1), \cdots, d(\mathfrak{B}_n) \geqq d(\mathfrak{A}_1),\ d(\mathfrak{A}_2)$ that $d(\mathfrak{B}_1)=2,\ d(\mathfrak{B}_2)=d(\mathfrak{B}_3)=\cdots = d(\mathfrak{B}_n)=1$. Choose $B_j \in \mathfrak{B}_j$ $(j=1,2,\cdots,n)$ such that $n_{B_1} \leqq \dfrac{N}{3},\ n_{B_2}, \cdots, n_{B_n} \leqq \dfrac{N}{2}$. If we take $A_i \in \mathfrak{A}_i$ $(i=1,2)$ such that $B_1B_2\cdots B_nA_1A_2 \neq 0$, then $P_{\mathfrak{B}_1\mathfrak{B}_2\cdots\mathfrak{B}_n} = P_{\mathfrak{A}_1} + P_{\mathfrak{A}_2}$ implies that

$$
\begin{aligned}
&(\xi, e_{B_1B_2\cdots B_n})\Big(\frac{N}{n_{B_1}}-1\Big)\Big(\frac{N}{n_{B_2}}-1\Big)\cdots\Big(\frac{N}{n_{B_n}}-1\Big)\\
&-(\xi, e_{B_1^cB_2\cdots B_n})\Big(\frac{N}{n_{B_2}}-1\Big)\cdots\Big(\frac{N}{n_{B_n}}-1\Big)-(\xi, e_{B_1B_2^cB_3\cdots B_n})\Big(\frac{N}{n_{B_1}}-1\Big)\Big(\frac{N}{n_{B_3}}-1\Big)\\
&\cdots\Big(\frac{N}{n_{B_n}}-1\Big)-\cdots=(\xi, e_{A_1A_2})\Big\{\Big(\frac{N}{n_{A_1}}-1\Big)+\Big(\frac{N}{n_{A_2}}-1\Big)\Big\}\\
&+(\xi, e_{A_1A_2^c})\Big\{\Big(\frac{N}{n_{A_1}}-1\Big)-1\Big\}+(\xi, e_{A_1^cA_2})\Big\{\Big(\frac{N}{n_{A_2}}-1\Big)-1\Big\}-2\,(\xi, e_{A_1^cA_2^c})\,.
\end{aligned}
$$

---

10) Cf. [2]. Lemma 9.

Let ξ be variable fundamental unit vectors. Comparing the values taken by both sides of this equation, we obtain $\left(\frac{N}{n_{B_1}}-1\right)\left(\frac{N}{n_{B_2}}-1\right)\cdots\left(\frac{N}{n_{B_{n-1}}}-1\right)=2$. Hence $\frac{N}{n_{B_1}}=3$, $\frac{N}{n_{B_2}}=2,\cdots,$ $\frac{N}{n_{B_{n-1}}}=2$. Similar argument shows that $\frac{N}{n_{B_n}}$ $=2$. Therefore $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n$ are regularly orthogonal. Then there exists a classification $\mathfrak{B}$ [11] such that $P_{\mathfrak{B}_1\mathfrak{B}_2\cdots\mathfrak{B}_n}=P_{\mathfrak{B}_1\mathfrak{B}}$, where $d(\mathfrak{B})=1$. Then we obtain $P_{\mathfrak{B}_1\mathfrak{B}}=P_{\mathfrak{A}_1}+P_{\mathfrak{A}_2}$, which is impossible by (i). Thus we complete the proof.

Theorem 5. (i) $P_{\mathfrak{A}_1\mathfrak{A}_2}=P_{\mathfrak{A}_3}+P_{\mathfrak{A}_4}+P_{\mathfrak{A}_5}$ *holds if and only if it corresponds to the case I, II given in* § 3 (Case $p=5$).

(ii) *The interaction* $\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}_3\cdots\mathfrak{B}_n$ $(n\geqq 3)$ *with* $d(\mathfrak{B}_1)\geqq\cdots\geqq d(\mathfrak{B}_n)$ *is decomposed into three classifications if and only if* $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_n$ *are regularly orthogonal and* $d(\mathfrak{B}_1)=3$, $d(\mathfrak{B}_2)=\cdots=d(\mathfrak{B}_n)=1$.

Proof.

ad (i). It follows from Lemma 2 and § 3 (Case $p=5$).

ad (ii). Necessity. By the similar argument as in the proof of Theorem 4 (ii) we may conclude that $\mathfrak{B}_1, \cdots, \mathfrak{B}_n$ are regularly orthogonal and $d(\mathfrak{B}_1)=3$, $d(\mathfrak{B}_2)=\cdots=d(\mathfrak{B}_n)=1$. Sufficiency. Since there is a classification $\mathfrak{B}$ [11] such that $\mathfrak{B}=\mathfrak{B}_2\mathfrak{B}_3\cdots\mathfrak{B}_n$, we can write $P_{\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_n}=P_{\mathfrak{C}\mathfrak{C}}$. Then (ii) follows from § 3 (Case $p=5$).

Theorem 6. $P_{\mathfrak{A}_1\mathfrak{A}_2}=P_{\mathfrak{A}_3}+P_{\mathfrak{A}_4}+P_{\mathfrak{A}_5}+P_{\mathfrak{A}_6}$ *holds if and only if* $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{A}_6$ *correspond to classifications determined by rows, by columns, and by a complete set of orthogonal Latin squares of side 5.*

Theorem 7. (i) *The interaction* $\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}_3$ *is decomposed into 4 classifications if and only if* $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ *are regularly orthogonal and* $d(\mathfrak{B}_1)=d(\mathfrak{B}_2)=d(\mathfrak{B}_3)=2$. *In this case the decomposition is unique.*

(ii) *The interaction* $\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}_3\cdots\mathfrak{B}_n$ $(n\geqq 4)$ *can not be decomposed into 4 classifications.*

Proof.

ad (i). Suppose that $P_{\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}_3}=P_{\mathfrak{A}_1}+P_{\mathfrak{A}_2}+P_{\mathfrak{A}_3}+P_{\mathfrak{A}_4}$ holds. We may assume that $d(\mathfrak{B}_1)\geqq d(\mathfrak{B}_2)\geqq d(\mathfrak{B}_3)\geqq d(\mathfrak{A}_1)\geqq(\mathfrak{A}_2)\geqq d(\mathfrak{A}_3)\geqq d(\mathfrak{A}_4)$. Since $d(\mathfrak{B}_1)$ $d(\mathfrak{B}_2)\,d(\mathfrak{B}_3)=d(\mathfrak{A}_1)+d(\mathfrak{A}_2)+d(\mathfrak{A}_3)+d(\mathfrak{A}_4)$ holds, we can conclude that the decomposition is possible only if

(I) $d(\mathfrak{B}_1)=4$, $d(\mathfrak{B}_2)=d(\mathfrak{B}_3)=d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_4)=1$,

11) Cf. [2]. Theorem 9.

(II) $d(\mathfrak{B}_1)=d(\mathfrak{B}_2)=2,\ d(\mathfrak{B}_3)=d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_4)=1$.

(III) $d(\mathfrak{B}_1)=d(\mathfrak{B}_2)=d(\mathfrak{B}_3)=d(\mathfrak{A}_1)=\cdots=d(\mathfrak{A}_4)=2$.

Let B_i be a $\mathfrak{B}_i$-class and A_j be an $\mathfrak{A}_j$-class. If $B_1B_2B_3A_1A_2A_3A_4\neq 0$, we have

$$\Big(\frac{N}{n_{B_1}}-1\Big)\Big(\frac{N}{n_{B_2}}-1\Big)\Big(\frac{N}{n_{B_3}}-1\Big)(\xi, e_{B_1B_2B_3}-\Big(\frac{N}{n_{B_1}}-1\Big)\Big(\frac{N}{n_{B_2}}-1\Big)$$

$$\times\ (\xi, e_{B_1B_2B_3^c})-\cdots+\Big(\frac{N}{n_{B_1}}-1\Big)(\xi, e_{B_1B_2^cB_3^c})+\cdots-(\xi, e_{B_1^cB_2^cB_3^c})$$

$$(\alpha)\quad \begin{aligned}&=\Big\{\Big(\frac{N}{n_{A_1}}-1\Big)+\Big(\frac{N}{n_{A_2}}-1\Big)+\Big(\frac{N}{n_{A_3}}\Big)+\Big(\frac{N}{n_{A_4}}-1\Big)\Big\}(\xi, e_{A_1A_2A_3A_4})\\ &+\Big\{-1+\Big(\frac{N}{n_{A_2}}-1\Big)+\Big(\frac{N}{n_{A_3}}-1\Big)+\Big(\frac{N}{n_{A_3}}-1\Big)\Big\}(\xi, e_{A_1^cA_2A_3A_4})+\cdots\\ &+\Big\{-2+\Big(\frac{N}{n_{A_3}}-1\Big)+\Big(\frac{N}{n_{A_4}}\Big)\Big\}(\xi, e_{A_1^cA_2^cA_3A_4})+\cdots\\ &+\Big\{-3+\Big(\frac{N}{n_{A_4}}-1\Big\}(\xi, e_{A_1^cA_2^cA_3^cA_4})+\cdots-4(\xi, e_{A_1^cA_2^cA_3^cA_4^c}).\end{aligned}$$

Suppose that B_i are chosen such that $\frac{N}{n_{B_i}}\geqq d(\mathfrak{B}_i)+1$. It follows from (I), (II), (III) that (d) implies $\Big(\frac{N}{n_{B_1}}-1\Big)\Big(\frac{n}{n_{B_2}}-1\Big)=4$. Hence $\frac{N}{n_{B_1}}=d(\mathfrak{B}_1)+1$, $\frac{N}{n_{B_2}}=d(\mathfrak{B}_2)+1$ must hold. Therefore $\mathfrak{B}_1$, $\mathfrak{B}_2$ are regular. If $d(\mathfrak{B}_2)=d(\mathfrak{B}_3)$ holds, we see also that $\mathfrak{B}_3$ is regular.

Case I. Since $\mathfrak{B}_2$, $\mathfrak{B}_3$ are regularly orthogonal and $d(\mathfrak{B}_2)=d(\mathfrak{B}_3)=1$, the interaction $\mathfrak{B}_2\mathfrak{B}_3$ is regarded as classification $\mathfrak{B}$, and we have $P_{\mathfrak{B}_1\mathfrak{B}}=P_{\mathfrak{A}_1}+\cdots P_{\mathfrak{A}_4}$. This is impossible by Lemma 2 and § 3 (Case $p=6$).

Case II. Let $\mathfrak{C}=\mathfrak{B}_1\wedge\mathfrak{B}_2\wedge\mathfrak{B}_3$. (α) implies that $(\xi, e_{B_1B_2B_3})=(\xi, e_{A_1A_2A_3A_4})$, that is, A_i intersect in precisely one $\mathfrak{C}$-class. We remark that this statement for any A_i such that $A_1A_2A_3A_4\neq 0$. Since $\mathfrak{C}$-consists of $3\times3\times2(=18)$ $\mathfrak{C}$-classes. But $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$ generate at most $2\times2\times2\times2(=16)$ $\mathfrak{C}$-classes. This is a contradiction.

Case III. $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ are regularly orthogonal and $d(\mathfrak{B}_1)=d(\mathfrak{B}_2)=d(\mathfrak{B}_3)=2$ holds. Thus the case III only is possible. That the converse is also true follows from the known fact from $3\times3\times3$ factorial experiment. We can show that in this case the decomposition is unique. Its proof is not so difficult, so we omit it.

ad (ii). Suppose that $P_{\mathfrak{B}_1\mathfrak{B}_2\cdots\mathfrak{B}_n}=\sum_{k=1}^{4}P_{\mathfrak{A}_k}$. Here we may assume that $d(\mathfrak{B}_1)\geqq\cdots\geqq d(\mathfrak{B}_n)$. We can show by an argument used in the proof of Theorem 4 (ii) that $\mathfrak{B}_2,\cdots,\mathfrak{B}_n$ are regularly orthogonal and $d(\mathfrak{B}_3)=\cdots d(\mathfrak{B}_n)=1$. Hence the interaction $\mathfrak{B}_3\mathfrak{B}_4\cdots\mathfrak{B}_n$ is regarded as a classification $\mathfrak{B}$. Then $P_{\mathfrak{B}_1\mathfrak{B}_2\mathfrak{B}}=\sum_{k=1}^{4}P_{\mathfrak{C}_k}$. (i) shows that this is impossible.

References

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